On the Cauchy-problem in effective theories of gravity

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'Connecting the Dots' workshop

AEI, Potsdam

June 14, 2023

Contents

Effective theories of gravity

Motivation

Observational developments: e.g. gravitational-wave astronomy – precision tests of GR in a strong field, highly dynamical regime

- Detection: searching for signals that match with theoretical templates
- Numerical relativity simulations of mergers of compact objects – theoretical templates of gravitational wave signatures
- Problems:
  (1) What theories should we focus on?
  (2) Doing simulations is not so straightforward, there are mathematical obstructions, e.g. theory must possess a well-posed initial value formulation

Effective field theories

EFT provides a framework to parameterize strong field deviations from GR: enumerate all higher derivative terms with the desired field content and symmetry. These higher derivative terms generically arise from UV complete theories by "integrating out" certain degrees of freedom.

Vacuum gravity in d dimensions: diffeomorphism-invariant action, only dynamical field is $g_{\mu\nu}$.

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\Big(\underbrace{R}_{2\partial\text{ theory}} + \underbrace{\ell^2\left(\beta_1 R^2 + \beta_2 R_{\mu\nu}R^{\mu\nu} + \beta_3 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)}_{4\partial\text{ theory}} + \underbrace{\ldots}_{\text{higher }\partial\text{ terms}}\Big)$$

where ℓ is a UV length scale. To study processes at energies $\ll \ell^{-1}$, one can truncate the series at some finite order. Expansion makes sense if

$$|R_{\mu\nu\alpha\beta}| \ll \ell^{-2},$$

i.e. in the *weakly coupled* regime. One can still study strong field phenomena using the weakly coupled EFT, e.g. nonlinear dynamics of black holes of size $L \gg \ell$.

Effective field theories

EFT provides a framework to parameterize strong field deviations from GR: enumerate all higher derivative terms with the desired field content and symmetry.

Vacuum gravity in d dimensions: diffeomorphism-invariant action, only dynamical field is $g_{\mu\nu}$. After field redefinitions one could write

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\Big(\underbrace{R}_{\substack{2\partial \text{ theory} \\ \text{Einstein-Hilbert term}}} + \underbrace{\ell^2\beta\,\mathcal{L}_{GB}}_{\substack{4\partial \text{ theory} \\ \text{Gauss-Bonnet term} \\ \text{(only in } d>4)}} + \underbrace{\ldots}_{\text{higher } \partial \text{ terms}} \Big)$$

with

$$\mathcal{L}_{GB} = \frac{1}{4}\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} R_{\mu_3\mu_4}{}^{\nu_3\nu_4}$$

That is, EFT of vacuum gravity in $d > 4$ dimensions up to 4 derivatives is Einstein-Gauss-Bonnet theory, it has 2nd order equations of motion.

In $d = 4$ the leading order corrections start at 6 derivatives, e.o.m. is higher than 2nd order, discussed later

Scalar-tensor effective field theory

Scalar-tensor theories in 4 dimensions (with parity symmetry) [Weinberg (2008)]:

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\Big(\underbrace{R - X + V(\phi)}_{\substack{2\partial \text{ theory} \\ \text{Einstein-scalar-field} \\ \text{theory}}} + \underbrace{\ell^2\left(\alpha(\phi)X^2 + \beta(\phi)\mathcal{L}_{GB}\right)}_{4\partial \text{ terms}} + \underbrace{\ldots}_{\text{higher } \partial \text{ terms}}\Big)$$

with $X \equiv -\frac{1}{2}(\partial\phi)^2$ and

$$\mathcal{L}_{GB} = \frac{1}{4}\delta^{\mu_1\mu_2\mu_3\mu_4}_{\nu_1\nu_2\nu_3\nu_4} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} R_{\mu_3\mu_4}{}^{\nu_3\nu_4}$$

Theory with $\alpha(\phi) = 0$, $\beta(\phi) \neq 0$: Einstein-scalar-Gauss-Bonnet (EsGB) theory.

Equations of motion are second order in 4 dimensions (the theory is in the Horndeski class)

Phenomenologically interesting: e.g. black holes have scalar hair in most EsGB theories

The Cauchy problem

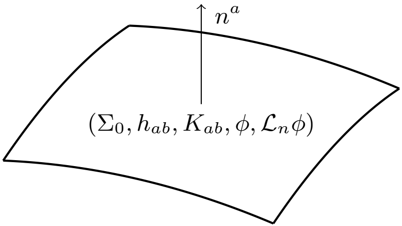
The Cauchy-problem

Given suitable initial data on a (non-characteristic) Cauchy surface Σ_0 that satisfies the constraints, the initial value problem is *well-posed* if

i) there exists a unique solution of the equations of motion,

ii) the solution depends continuously on the initial data (in a suitable norm), e.g.

$$||\Phi||_{H^s}(t) \leq C(t)||\Phi||_{H^s}(0)$$



Local well-posedness: above conditions hold for a finite time $T > 0$.

Solutions to nonlinear PDEs tend to blow up after a finite time, for generic initial data the best one can hope for (usually) is to establish local well-posedness.

Local well-posedness and hyperbolicity

A sufficient condition for the nonlinear equations to admit a locally well-posed IVP is that the gauge-fixed equations of motion are **strongly hyperbolic**.

System is strongly hyperbolic if a certain matrix $M(\xi)$ (constructed out of the principal terms in the linearised PDE around a generic background field configuration) admits a positive definite hermitian matrix $K(\xi)$ called the *symmetrizer* that satisfies

$$K(\xi)M(\xi) = M^\dagger(\xi)K(\xi),$$

depends smoothly on its arguments and is bounded.

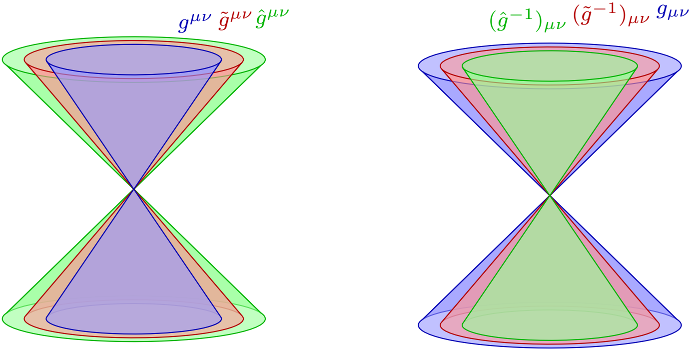
A necessary condition for strong hyperbolicity is that $M(\xi)$ be diagonalisable with real eigenvalues. Eigenvalues $\xi_0(\xi_i)$ of $M(\xi)$ are called *characteristic speeds* (e.g. for wave equation in flat space these would be $\xi_0 = \pm|\xi| = \pm\sqrt{\xi_i \xi^i}$), eigenvectors are *characteristic polarisations*.

Well-posed formulation of theories with second order equations motion

Setting up the modified harmonic gauge: auxiliary metrics

Solution: tricky choice of gauge and gauge-fixing [Kovács & Reall (2020)]

Consider a 4d spacetime (M, g) and introduce two auxiliary (inverse) Lorentzian metrics: $\tilde{g}^{\mu\nu}$ and $\hat{g}^{\mu\nu}$.



(a) Cotangent space

(b) Tangent space

The modified harmonic gauge

Define

$$H^\mu \equiv \tilde{g}^{\nu\rho}\nabla_\nu\nabla_\rho x^\mu = -\tilde{g}^{\nu\rho}\Gamma^\mu_{\nu\rho}[g]$$

The modified harmonic gauge condition is $H^\mu = 0$.

Recall

$$E^{\mu\nu} = -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \qquad\qquad E_\phi = -\frac{16\pi G}{\sqrt{-g}}\frac{\delta S}{\delta \phi}$$

We now define

$$E^{\mu\nu}_{\mathrm{mhg}} = E^{\mu\nu} + \hat{P}_\alpha{}^{\beta\mu\nu}\partial_\beta H^\alpha$$

where $\hat{P}_\alpha{}^{\beta\mu\nu} = \delta_\alpha^{(\mu}\hat{g}^{\nu)\beta} - \frac{1}{2}\delta_\alpha^\beta\hat{g}^{\mu\nu}$.
The modified harmonic gauge equations of motion are then

$$E^{\mu\nu}_{\mathrm{mhg}} = 0 \qquad\qquad E_\phi = 0$$

Setting $\tilde{g}^{\mu\nu} = \hat{g}^{\mu\nu} = g^{\mu\nu}$ recovers the usual harmonic gauge equations of motion.

Statement of the main result

Theorem

The modified harmonic gauge equations of motion

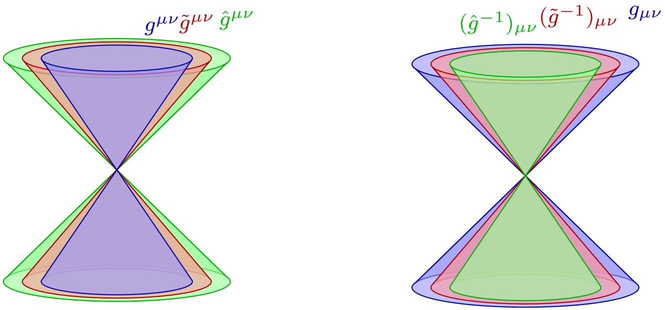
$$E^{\mu\nu}_{\mathrm{mhg}} \equiv E^{\mu\nu} + \hat{P}_{\alpha}{}^{\beta\mu\nu}\partial_{\beta}H^{\alpha} = 0 \qquad\qquad E_{\phi} = 0$$

admit a locally well-posed initial value problem in the following two theories

(i) *Einstein-scalar-field theory (2∂ST) and*

(ii) *the weakly coupled 4-derivative EFT (4∂ST)*

provided that the causal cones of g, $\tilde{g}$ and $\hat{g}$ are related as below.



Similar results apply to any Lovelock and Horndeski theory.

Main idea

Separation of causal cones of different types of mode solutions

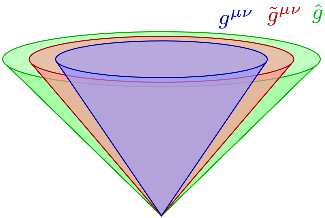
(i) "pure gauge" modes propagate along the null cone of $\tilde{g}^{\mu\nu}$

(ii) "gauge condition violating" modes propagate along the null cone of $\hat{g}^{\mu\nu}$

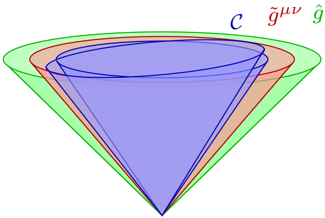
(iii) "physical" polarizations

- propagate along the null cone of $g^{\mu\nu}$ is Einstein-scalar-field theory
- propagate along characteristic hypersurfaces that are "almost null" in weakly coupled 4∂ST and are gauge-invariant [Reall (2021)]

If characteristic polynomial for physical d.o.f is hyperbolic then the theory admits a well-posed formulation (e.g. in a suitable choice of the modified harmonic gauge condition and gauge fixing), otherwise the theory breaks down independently of gauge choice. (See also [Hegade, Ripley, Yunes (2023)])



2∂ST



4∂ST

Application to numerical relativity

There is plenty of freedom in how we choose the auxiliary metrics!

Simplest choice: Let n_μ be the unit normal (w.r.t. g) to $x^0 = \mathrm{const.}$ surfaces. Then we choose

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - a(x) n^\mu n^\nu \qquad \hat{g}^{\mu\nu} = g^{\mu\nu} - b(x) n^\mu n^\nu .$$

The modified harmonic formulation has been used to perform numerical relativity simulations of black hole binaries in EsGB theories [East & Ripley (2021)], [Corman, East & Ripley (2021)].

More recent result: combining idea of the modified harmonic gauge with the CCZ4 formulation used in numerical relativity also yields a well-posed formulation of EsGB theory. Simulations of black hole binaries in EsGB theories [Aresté Saló, Clough & Figueras (2022)]

Theories with higher than second order equations of motion

Higher derivative EFTs with Lorentz symmetry

Theories with a variational principle and higher than 2nd order e.o.m. have pathologies.

- solutions associated with extra d.o.f. may be "pathological" (see however [Deffayet, Held, Mukohyama, Vikman (2023)])
- it may still be possible to write the e.o.m. in a strongly hyperbolic form, guaranteeing local well-posedness (global well-posedness may still be an issue), see e.g. [Noakes (1982)], [Held, Lim (2023)]

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\left\{R + \ell^2\left(\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right)\right\}$$

- but raises the issue of how to extract "physical" solutions

Proposals:

- perturbation theory in couplings, e.g. [Witek, *et al.*], [Okounkova, *et al.*]; more recently [Ghersi & Stein (2021)] dynamical renormalization group approach
- allow only a restricted class of initial data
- modify the equations of motion in some way
  - reduction of order method, e.g. [Flanagan, Wald (1996)]
  - "fixing-the-equations" procedure [Cayuso, Ortiz, Lehner (2017)], [Israel, Stewart (1976)]

Muller-Israel-Stewart proposal: "fixing" the equations

Toy example: fixing the heat equation [Geroch (1994)]

$$\begin{aligned} \partial_t T &= \partial_x q \\ q &= \sigma \partial_x T \end{aligned}$$

Turn the parobolic system to a hyperbolic one (with characteristic velocity v)

$$\begin{aligned} \partial_t T &= \partial_x q \\ \partial_t q &= v^2 \left(\partial_x T - \frac{q}{\sigma} \right) \end{aligned}$$

Justification for the replacement: for any $t > 0$, we have

$$||q - \sigma \partial_x T||_{L_2}(t) \leq \left(\frac{\sigma}{v^2 t} \right)^2 f(t, q(0), T(0), \partial_x q(0), \partial_x T(0), \ldots)$$

If v is a typical sound speed of the material then σ / v^2 is of the order of a mean free time. Assuming that the spatial gradients of the initial data are not too large, RHS is microscopically small, $q - \sigma \partial_x T$ is unmeasurably small.

[Geroch (1994)] shows that for a certain class of fluid theories, a large class of fixing procedures yields the same physics as the original fluid equations – mathematical underpinning of MIS approach widely used in simulations of heavy ion collisions

Fixing the equations of gravitational theories

Validity of the approach depends on whether there is significant energy cascade to UV. There is some evidence that in many situations of interest MIS-type approach might work [Cayuso, Ortiz, Lehner (2017)]

- detected waveforms from BH binaries
- stability of Minkowski space, BH solutions etc.

However, in other situations, cascade could be physical (e.g. singularity theorems)

Gravitational equivalent to Geroch's result is lacking, it is not understood what is the class of fixing procedures that solves the issue of well-posedness for a general theory, while faithfully reproducing the physics of the original theory in the relevant regime.

Some promising results in this direction:

- in quartic (8-derivative) vacuum gravity [Cayuso, Figueras, Franca, Lehner (2022)]
- in Einstein-scalar-Gauss-Bonnet theory [Franchini, Bezares, Barausse, Lehner (2022)]
- in k-essence [Lara, Bezares, Barausse (2022)]
- in cubic Galileon theory [Gerhardinger, Giblin, Tolley, Trodden (2022)]

Lorentz-symmetry-breaking theories

A result in Blas-Pujolas-Sibiryakov-Hořava theory

$$S = \frac{1}{16\pi G} \int dt d^3 x N \sqrt{\gamma} \Big(K_{ij} K^{ij} - \lambda K^2 - \mathcal{V}(R[\gamma], D_i \ln N) \Big)$$

- Lorentz-violating theory in $3+1$ dimensions, there is a preferred foliation
- invariant under foliation-preserving diffeomorphisms

$$t \to \tilde{t}(t), \qquad x^i \to \tilde{x}^i(t, x)$$

- action invariant under the Lifshitz scaling ($b =$ const.)

$$t \to b^d t, \qquad x^i \to b x^i$$

- enumerate operators in the potential $\mathcal{V}$ of dimensions up to 6
- theory is ghost free and power-counting renormalizable, candidate for a simple UV complete theory [Hořava (2009)], [Blas, Pujolas, Sibiryakov (2009)]

Theorem (ÁDK, *forthcoming*)

There exists an open set of coupling constants (consistent with constraints due to unitarity and "stability" of the theory in a flat background) such that BPSH theory admits a locally well-posed initial value formulation.

Conclusion

Conclusions

Theories with second order equations:

- EFTs with 2nd order equations possess a locally well-posed initial value formulation, at least for weak couplings

- if characteristic polynomial for physical d.o.f is hyperbolic then the theory admits a well-posed formulation (e.g. in a suitable choice of the modified harmonic gauge condition and gauge fixing), otherwise the theory breaks down independently of gauge choice

Theories with higher derivatives:

- The problem of well-posedness is more subtle but there are some promising directions

Lorentz-symmetry-breaking theories:

- Local well-posedness result for Blas-Pujolas-Sibiryakov-Hořava-Lifshitz gravity