Introduction to the post-adiabatic expansion for EMRIs

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Fundamental Physics Meets Waveforms With LISA

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What is the post-adiabatic expansion?

These are all basically the same thing:

- post-adiabatic approximation
- two-timescale expansion
- multiscale expansion
- multi-timescale analysis
- method of averaging
- near-identity averaging transformations

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Methods of separating secular from oscillatory effects

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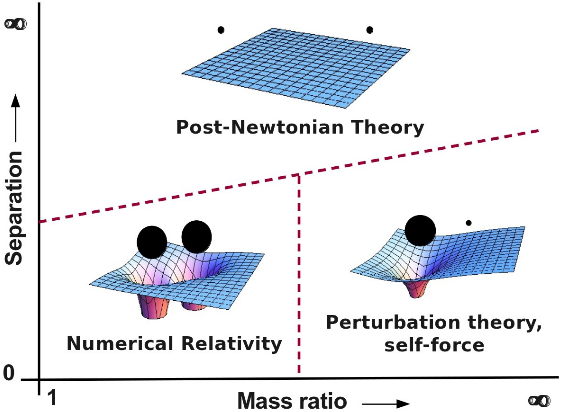
Fast: $10 - 100$ms per full waveform

Modular: readily accommodate non-GR effects

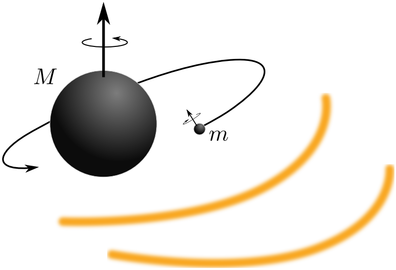
What is the post-adiabatic expansion?

A way of organizing the self-force expansion for extreme mass ratios



[Image credit: Leor Barack]

Gravitational self-force theory



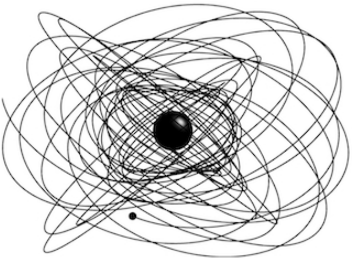
- $\epsilon = 1/q = m/M \ll 1$
- small body perturbs spacetime:

$$\mathsf{g}_{\mu\nu} = g^{\text{Kerr}}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \dots$$

- perturbation affects m's motion:

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)} + \epsilon^2 f^\mu_{(2)} + \dots$$

Zeroth order: test mass on a geodesic in Kerr



[image credit: Steve Drasco]

- constants $J_A = (M, a, E, L_z, Q)$:
  1. energy E
  2. angular momentum L_z
  3. Carter constant Q, related to orbital inclination
- phases $\varphi_A = (\varphi_r, \varphi_\theta, \varphi_\phi)$ with frequencies $\Omega_A(J_B)$

- simple ODEs:

$$\frac{d\varphi_A}{dt} = \Omega_A(J_B)$$
$$\frac{dJ_A}{dt} = 0$$

Post-adiabatic expansion

[Flanagan, Hinderer, Lynch, Miller, Moxon, AP, van de Meent, Warburton, ...]

- full, evolving system:

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}_B)$$
$$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right]$$

- waveform

$$h_{\ell m} = \sum_{k^i}\left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3)\right] e^{-i(m\tilde{\varphi}^\phi + k^i\tilde{\varphi}_i)}$$

- secondary spin: (i) new slow parameters
  (ii) new precession phase

Rapid waveforms

Offline step

- solve field equations for amplitudes $h^{(n)}_{\ell m k^i}$ and forcing functions $F^{(n-1)}_A$ on grid of $\tilde{J}_A$ values

Online step

- solve ODEs for $\tilde{\varphi}_A$ and $\tilde{J}_A$, add up mode amplitudes $h^{(n)}_{\ell m k^i}$
- `FastEMRIWaveforms` (**FEW**) software package:
  waveform in $\sim 10 - 100$ms
  [Chapman-Bird, Chua, Hughes, Katz, Speri, Warburton, ...]

Accuracy and post-adiabatic counting [Hinderer & Flanagan]

- phases have an expansion

$$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\epsilon t) + \epsilon^0\varphi_A^{(1)}(\epsilon t) + O(\epsilon)$$

- a model that gets $\varphi_A^{(0)}$ and $\varphi_A^{(1)}$ right should be enough for precise parameter extraction [Burke et al. 2023]
- also need to account for transient orbital resonances (0.5PA)

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- dissipative piece of $f^\mu_{(2)}$
- conservative piece of $f^\mu_{(1)}$
- linear secondary spin effects

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Current state of play

Goal: 1PA waveforms over full parameter space

Background Spacetime	Orbital Configuration	Adiabatic	Post-1-adiabatic			
		1SF (Dissipative)	1SF (Conservative)	2SF (Dissipative)	Spin Effects (Conservative)	Spin Effects (Dissipative)
Schwarzschild	Circular	✓✓✓	✓✓✓	✓✓✓	✓✓✓	✓✓✓
	Eccentric	✓✓✓	✓✓✓	✗	✓✓ , ✓✓✓*	✓, ✓✓*
Kerr	Circular	✓✓✓	✓✓	✗	✓,✓✓*	✓✓✓*
	Eccentric Equatorial	✓✓✓	✓✓	✗	✓,✓✓*	✓✓*
	Generic	✓✓✓	✓	✗	✓	✓*
	Resonances	✓✓✓	✓	✗	✗	✗

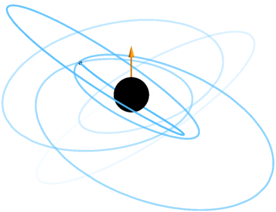
✓✓✓ Evolving Waveform ✓✓ Driven Inspiral ✓ Snapshot Calculation *(Anti-)Aligned Spin Only

[table courtesy of Josh Mathews]

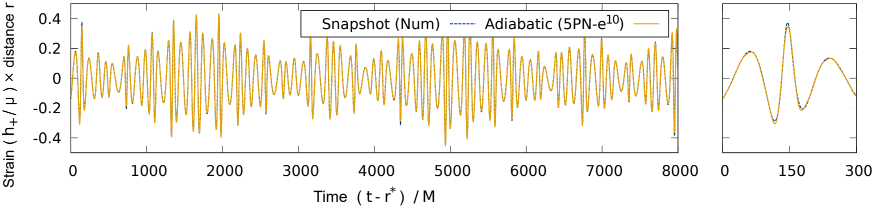
State of the art in parameter space coverage: 0PA waveforms in Kerr [Hughes et al., Kyoto co., & FEW]

In FEW:

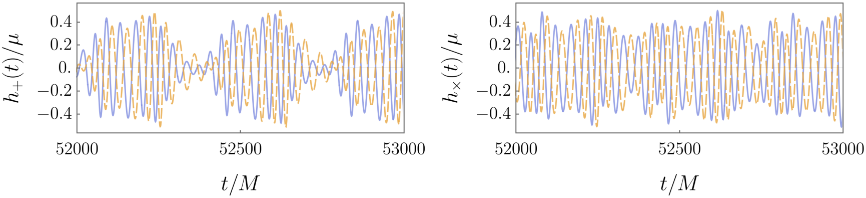
- generic orbits in Kerr: 5.5PN-e^{10} approximation [Isoyama, Kyoto co., Chua, AP]
- equatorial orbits in Kerr: fully relativistic waveforms



[animation credit: P. Lynch]



1PA secondary spin effects [Prague groups; Drummond & Hughes et al.]



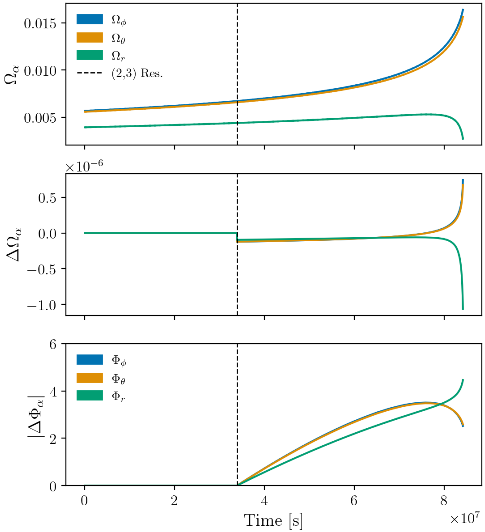
- waveforms including 1PA conservative spin effects "available" for generic configurations in Kerr
- currently missing efficient method of computing non-flux dissipative effects (but see [Grant; Skoupý and Witzany; Mathews et al.])

Resonances [Flanagan & Hinderer, Lynch et al., Isoyama et al., ...]

Ω_r/Ω_θ becomes momentarily rational

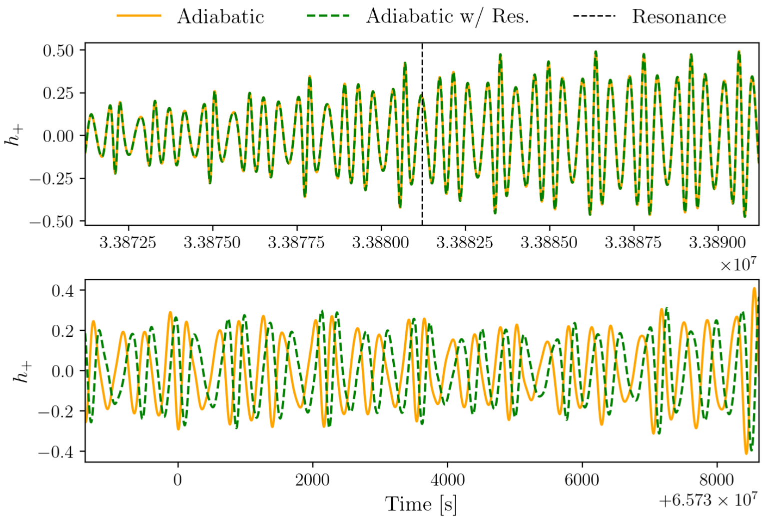
Ω_A jump slightly across the resonance

leads to a significant cumulative change in phase



[Image credit: C. Chapman-Bird]

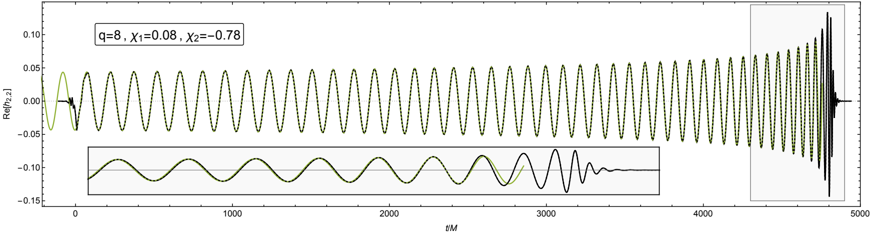
Resonances in FEW [Chapman-Bird & Warburton]



[Image credit: C. Chapman-Bird]

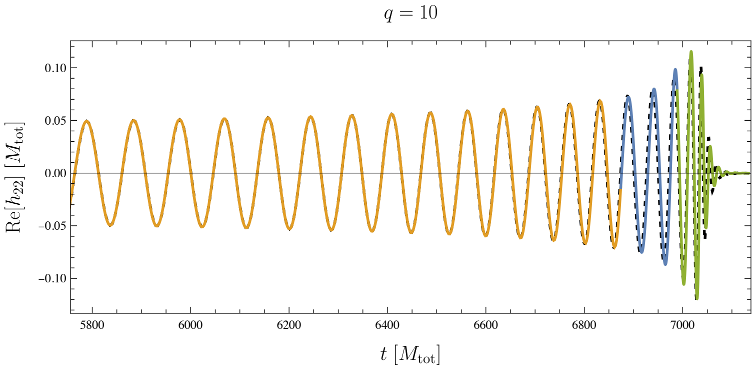
State of the art in accuracy:
Complete 1PA waveforms [Mathews, AP, Warburton, Wardell & MSF group]

Comparison with NR waveform from SXS collaboration



- complete quasicircular 1PA model with generic secondary spin and linear primary spin
- in FEW: version with zero primary spin, aligned secondary spin

Complete inspiral-merger ringdown models
[Kuchler, Compere, Durkan, AP & MSF group]



- 1PA inspiral, next-to-leading transition, leading-order plunge
- see L. Kuchler's talk

Conclusion

Post-adiabatic self-force waveforms

- high accuracy for IMRIs and EMRIs
- native fast waveform generation
  —fast enough for data analysis (?)

Status

- 0PA: equatorial inspirals into Kerr primary
- resonances, secondary spin, precession in the pipeline
- complete 1PA: quasicircular inspiral into slowly spinning primary

Challenge

- coverage of parameter space
- non-GR effects