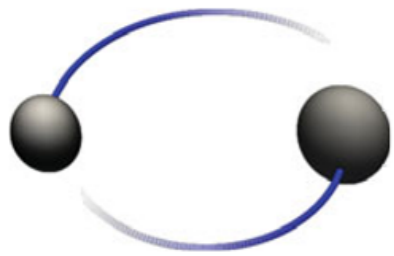


Nonlinearities in BH ringdown

ADRIEN KUNTZ

With B. Bucciotti, S. Yi, E. Trincherini, E. Barausse, E. Berti...

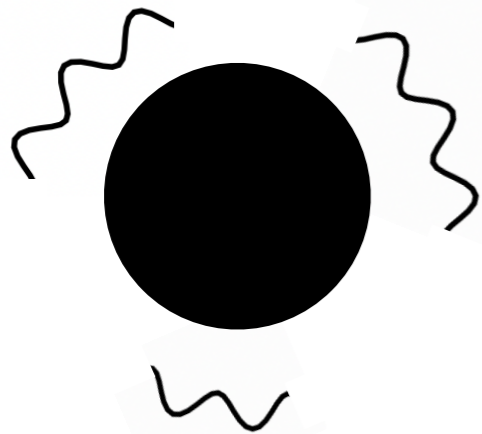


WHAT ARE QNMs?

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}^{(1)}$$

$$\Rightarrow G_{\mu\nu}^{(1)}[\mathbf{h}^{(1)}] = 0$$



Using background symmetries and fixing gauge:

$$h^{(1)} \sim \sum_{\ell, m, \omega, \pm} e^{-i\omega t} \psi_{\pm}^{(1), \ell m}(r) Y_{\pm}^{\ell m}(\theta, \phi) + \text{c.c.}$$

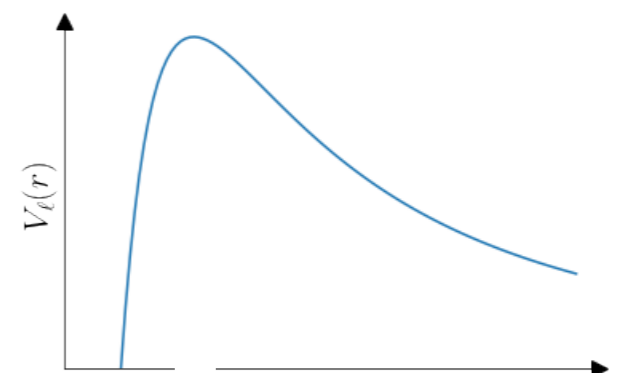
Equatorial (reflection) symmetry:

$$\pm = (-1)^{\ell+m}$$

$$\longrightarrow \frac{d^2 \psi^{(1)}}{dr_*^2} + (\omega^2 - V_\ell(r)) \psi^{(1)} = 0$$

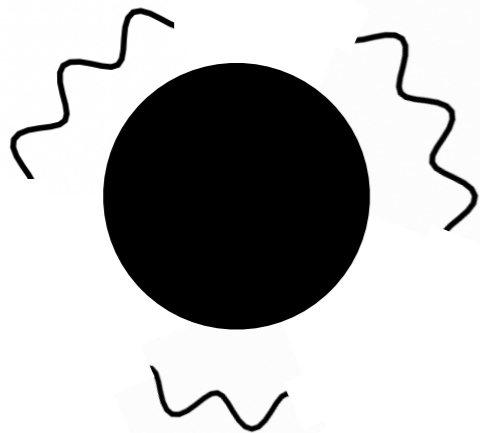
$$r_* = r + 2GM \log(r/(2GM) - 1)$$

$$V_\ell(r) = \left(1 - \frac{2GM}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right)$$



WHAT ARE QNMs?

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}^{(1)}$$



$$\frac{d^2\psi^{(1)}}{dr_*^2} + (\omega^2 - V_\ell(r))\psi^{(1)} = 0$$

A QNM is a solution with:

$$\psi^{(1)}(r) \propto e^{-i\omega r_*}, \quad r_* \rightarrow -\infty$$

$$\psi^{(1)}(r) \propto e^{i\omega r_*}, \quad r_* \rightarrow \infty$$

This is possible only if the frequency is quantised:

$$\omega = \omega_{\ell mn}$$

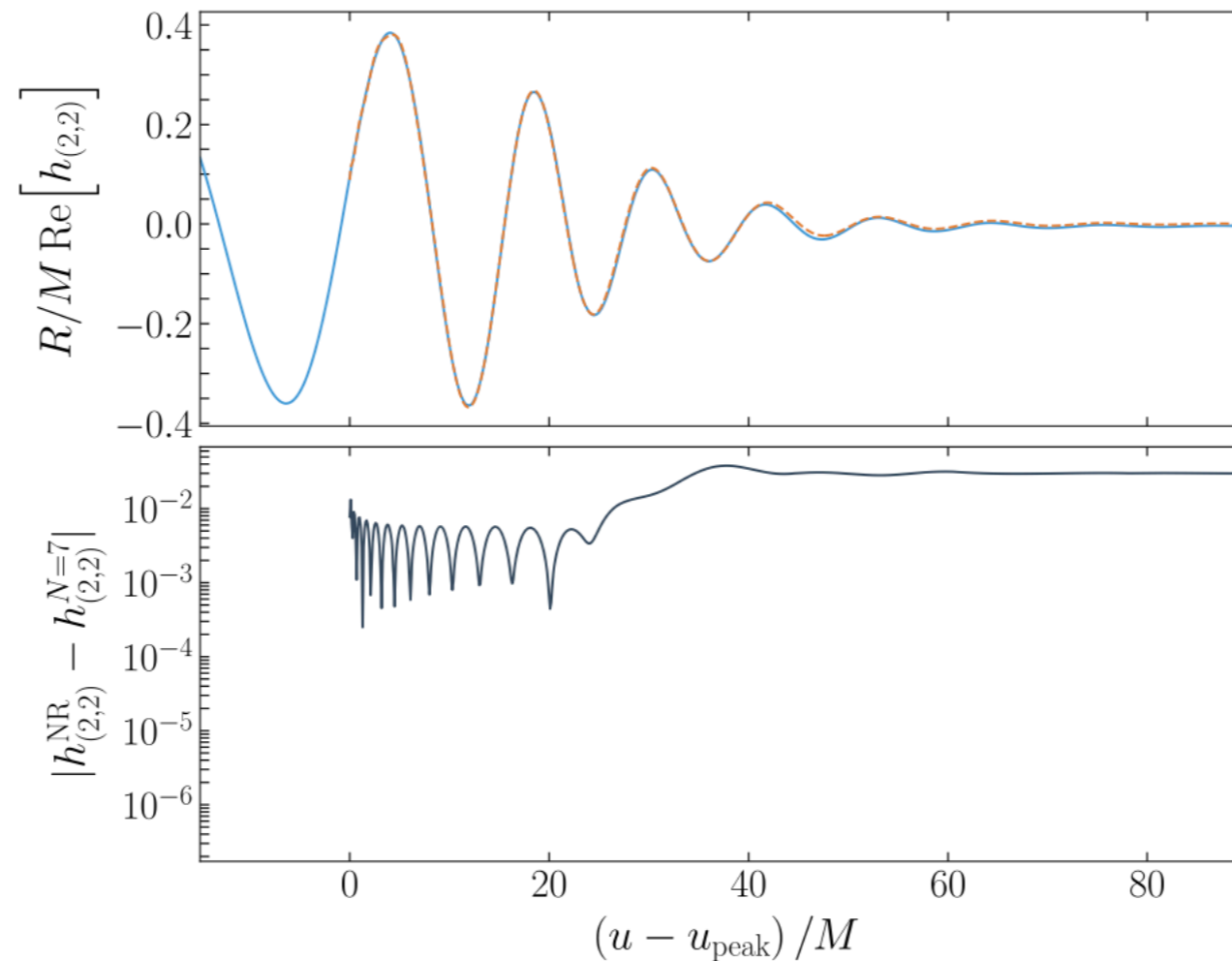
The amplitude is **arbitrary** in this approach

ℓ	n	$\omega_{\ell mn}$
2	0	$0.37367 + 0.08896i$
	1	$0.34671 + 0.27391i$
	2	$0.30105 + 0.47828i$
3	0	$0.59944 + 0.09270i$
	1	$0.58264 + 0.28130i$
	2	$0.55168 + 0.47909i$



It **MUST** be fitted to NR

A REMARKABLE FIT



Zertuche et al. (2022)

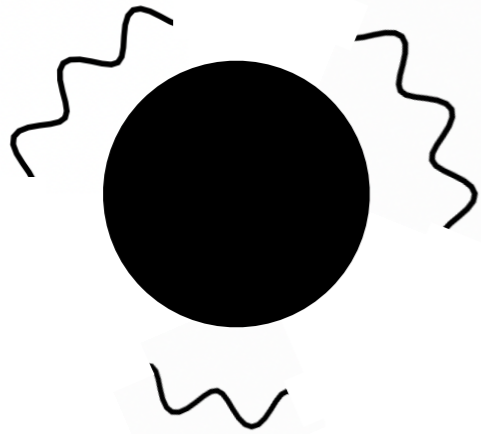
SXS:BBH:0305 \Leftrightarrow GW150914

$$h = \sum_{lmn} \mathcal{A}_{lmn} e^{-i\omega_{lmn}(t-r_*)} Y^{\ell m}(\theta, \phi)$$

$\ll 1$

NONLINEARITIES?

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}^{(1)} + \varepsilon^2 h_{\mu\nu}^{(2)}$$



$$G_{\mu\nu}^{(1)}[\mathbf{h}^{(2)}] = -G_{\mu\nu}^{(2)}[\mathbf{h}^{(1)}, \mathbf{h}^{(1)}]$$

Same operator!

Source term

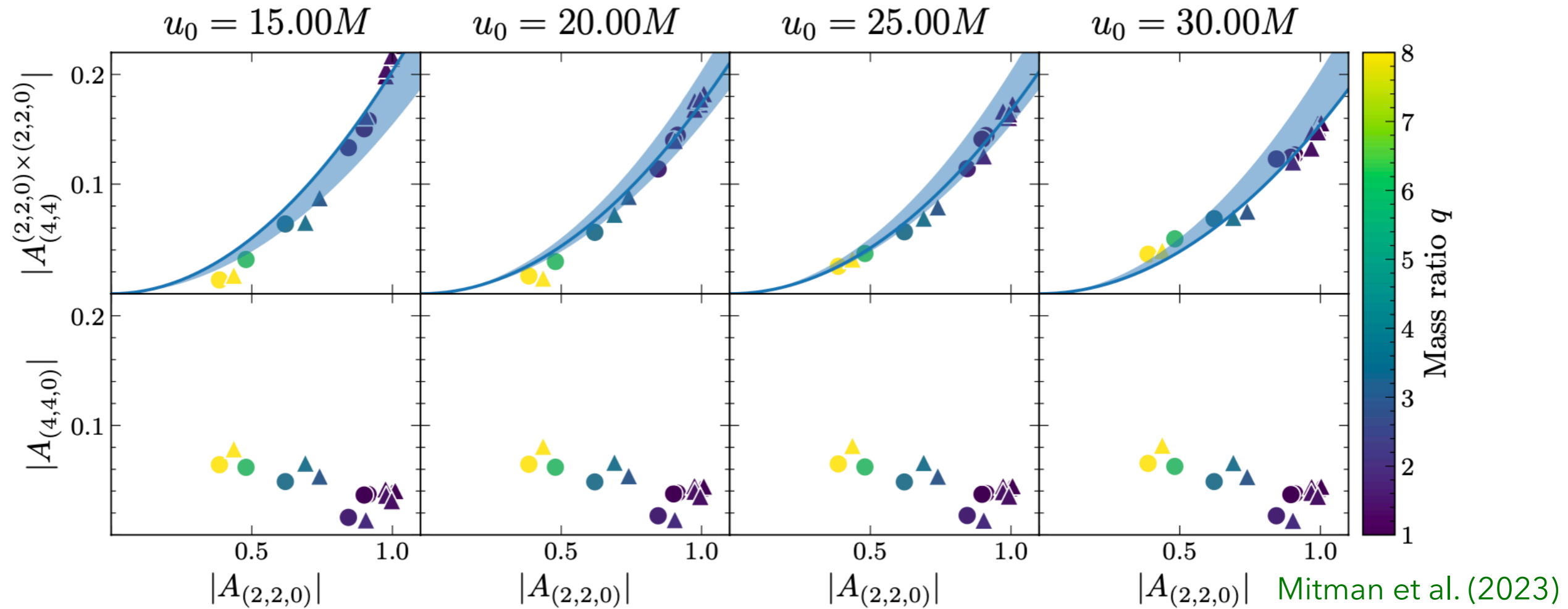
$$h^{(1)} \times h^{(1)} \sim \left(\sum_{\ell_1, m_1, n_1} e^{-i\omega_1 t} \psi^{(1), \ell_1 m_1} Y^{\ell_1 m_1} + \text{c.c.} \right) \times \left(\sum_{\ell_2, m_2, n_2} e^{-i\omega_2 t} \psi^{(1), \ell_2 m_2} Y^{\ell_2 m_2} + \text{c.c.} \right)$$

« QUADRATIC » MODES:

[Nakano&Ioka\(2007\)](#), [London&al\(2017\)](#), [Lagos&Hui\(2023\)](#), [Bucciotti AK&al\(2024\)](#)....

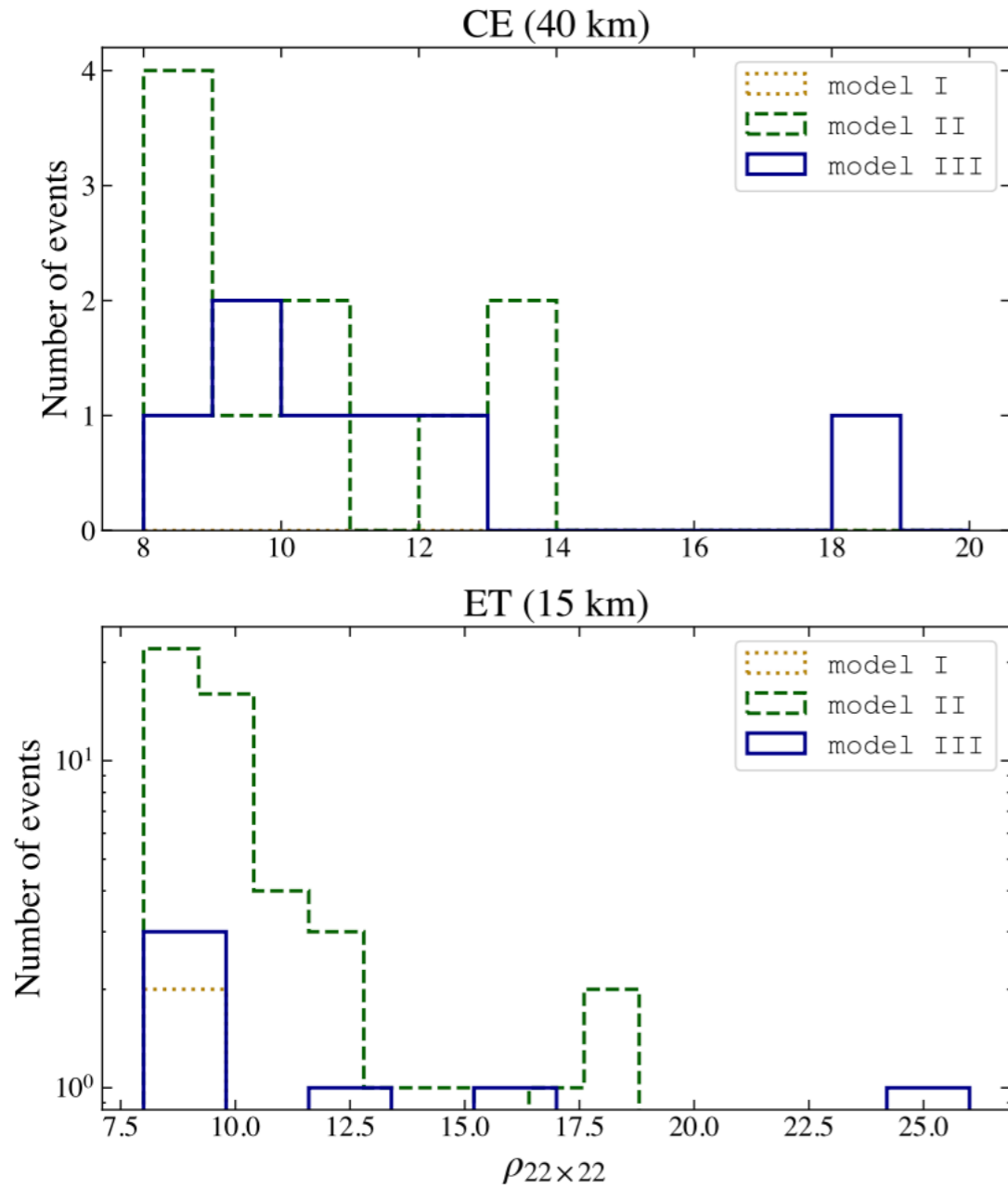
- Frequency $\omega_1 + \omega_2$ or $\omega_1 - (\omega_2)^*$
- Angular momentum $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ and $m = m_1 + m_2$
- Parity $(-1)^{\ell+m}$
- Amplitude is quadratic in linear modes

NONLINEARITIES!

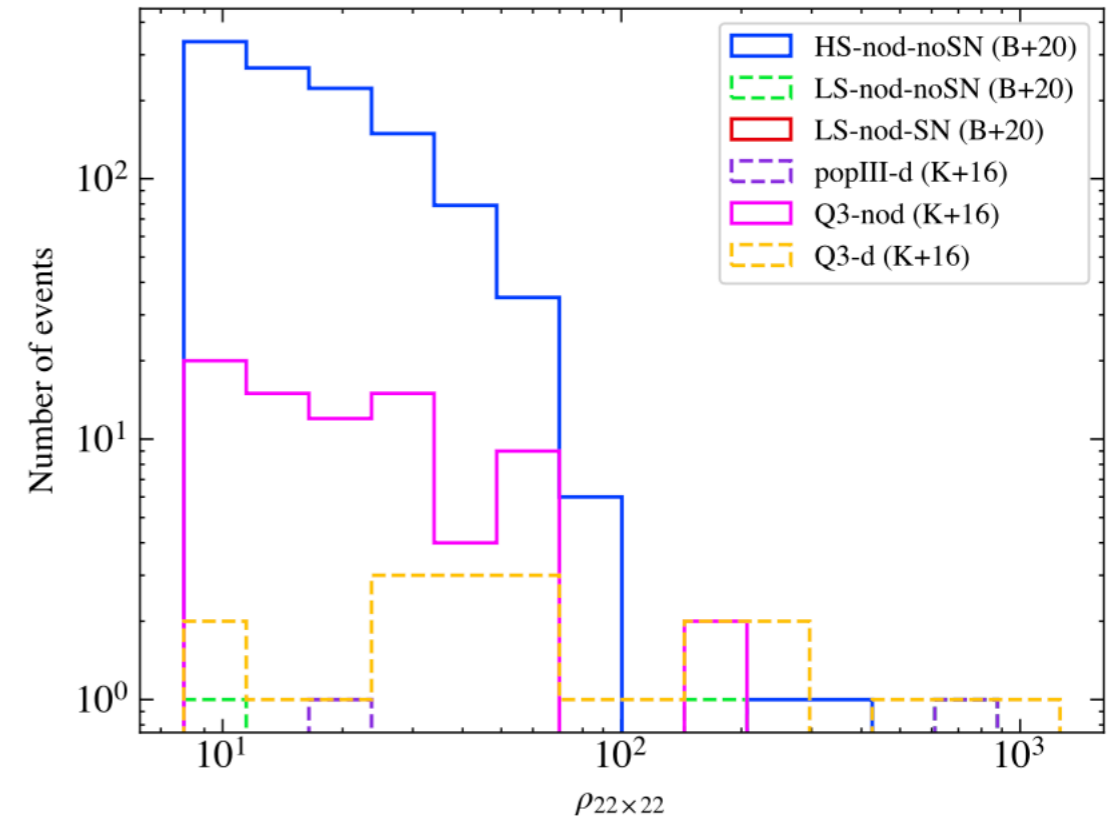


$$h = \sum_{lmn} \mathcal{A}_{lmn} e^{-i\omega_{lmn}(t-r_*)} Y^{\ell m}(\theta, \phi) + \mathcal{A}_{44}^{220 \times 220} e^{-2i\omega_{220}(t-r_*)} Y^{44}(\theta, \phi)$$

OBSERVABILITY



S. Yi, AK & al (2024)



Can we have a better understanding of these nonlinearities?

$$\frac{A_{44}^{220 \times 220}}{(A_{220})^2} = ?$$

SECOND-ORDER RW

$$G_{\mu\nu}^{(1)}[\mathbf{h}^{(2)}] = -G_{\mu\nu}^{(2)}[\mathbf{h}^{(1)}, \mathbf{h}^{(1)}] \longrightarrow \frac{d^2\psi^{(2)}}{dr_*^2} + (\omega^2 - V_\ell(r))\psi^{(2)} = S(r)$$

A toy-model: $S(r) = f(r)\psi_1^{(1)}(r)\psi_2^{(1)}(r)$

HOMOGENEOUS SOLUTION: Renormalization of linear amplitudes

PARTICULAR SOLUTION: Amplitude fixed by $S(r)$!

$$\psi^{(2)}(r) \propto e^{-i\omega r_*}, \quad r_* \rightarrow -\infty$$

A QNM is a solution with:

$$\psi^{(2)}(r) \propto e^{i\omega r_*}, \quad r_* \rightarrow \infty$$

Issue with the asymptotics!

Gleiser&al (95,98,99)

Brizuela&al (06,07,09)

Nakano&Ioka (07)

We want $\frac{d^2\psi^{(2)}}{dr_*^2} + \omega^2\psi^{(2)} = \mathcal{O}\left(\frac{e^{i\omega r_*}}{r^2}\right)$ We have $S = \mathcal{O}(e^{i\omega r_*})$

REGULARIZATION

Bucciotti, Juliano, AK, Trincherini (2024)

$$\Psi^{(2)}(r) = \psi^{(2)}(r) + \Delta(r)\psi_1^{(1)}\psi_2^{(1)}$$

$$\rightarrow \frac{d^2\Psi^{(2)}}{dr_*^2} + (\omega^2 - V_\ell(r))\Psi^{(2)} = \tilde{S}(r)$$

$$\tilde{S}(r) \propto \left(1 - \frac{2M}{r}\right) e^{-i\omega r_*}, \quad r_* \rightarrow -\infty$$

$$\tilde{S}(r) \propto \frac{e^{i\omega r_*}}{r^2}, \quad r_* \rightarrow \infty$$

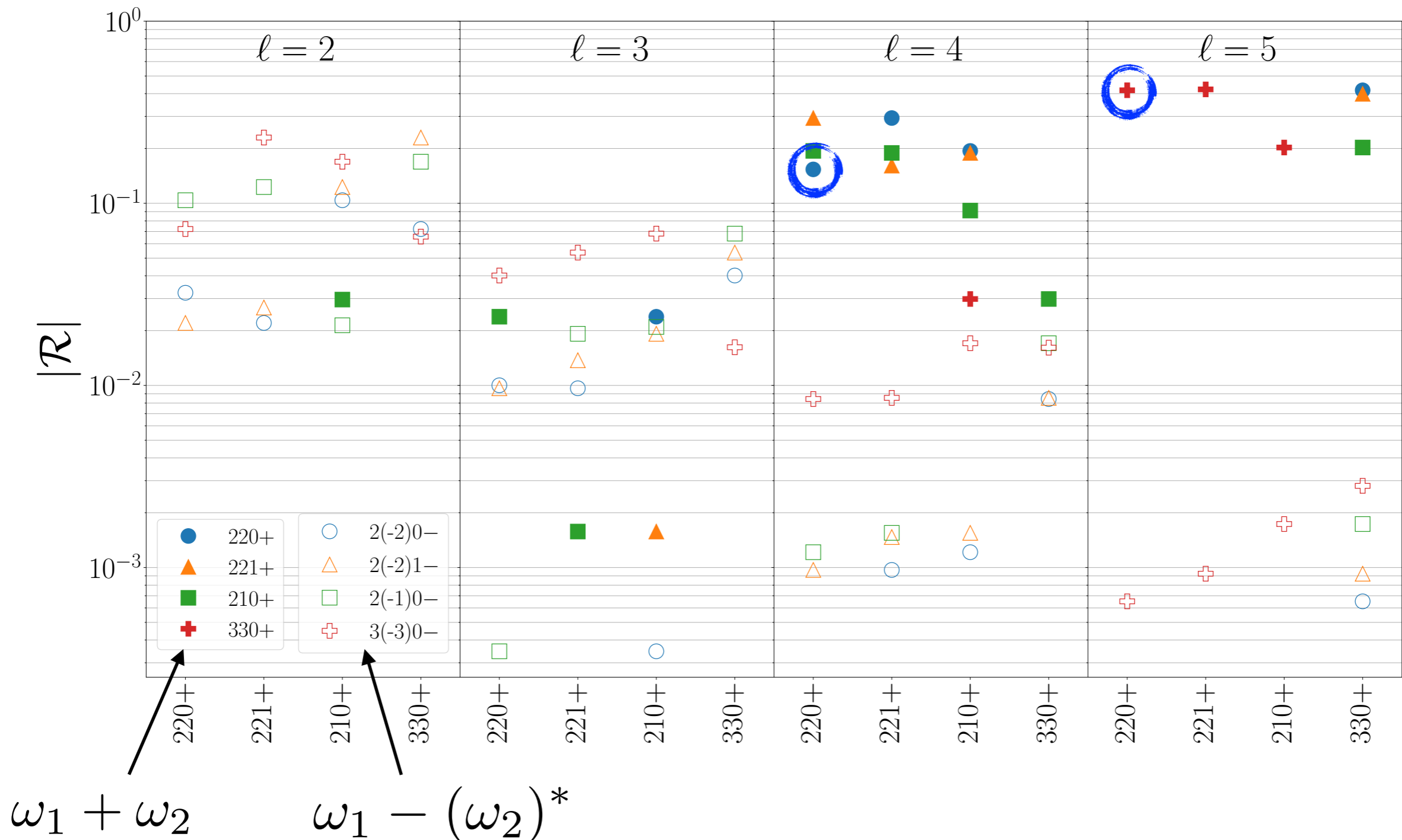
(Similar to $V_\ell(r)$)

The $\Delta(r)$ cancels when going from RW gauge to TT gauge

LEAVER SOLUTION

Bucciotti, Juliano, AK, Trincherini (2024)

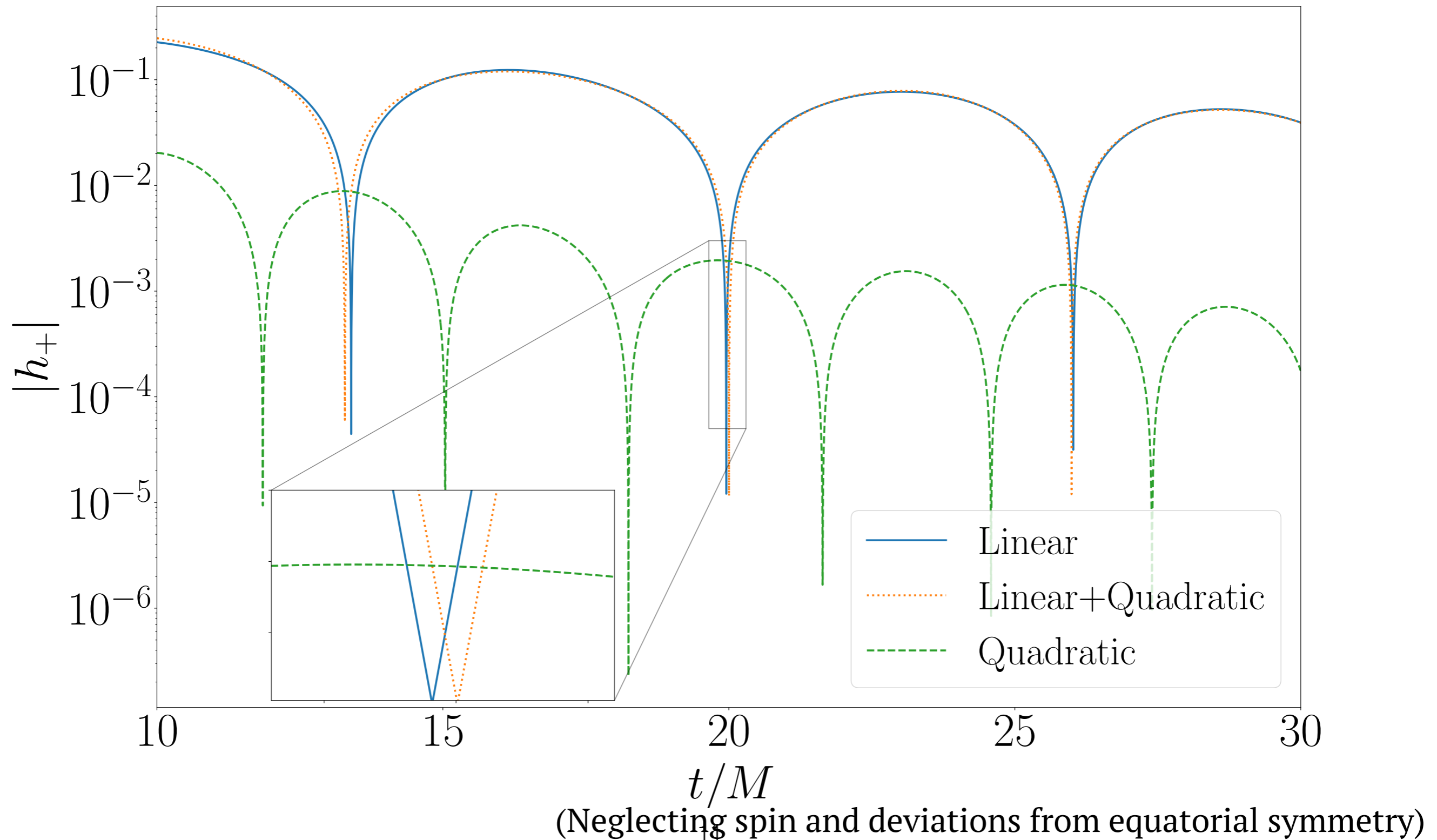
$$\mathcal{R} = \mathcal{A}^{(2)} / (\mathcal{A}_1^{(1)} \mathcal{A}_2^{(1)})$$



NONLINEAR RINGDOWN

Bucciotti, Juliano, AK, Trincherini (2024)

GW190521-LIKE MERGER



CONCLUSIONS

- As precision improves, we should aim to model nonlinearities in ringdown
- Quadratic modes will be observable by ET/CE/LISA
- Amplitude of quadratic modes is fully determined by:
 - Amplitudes of linear modes
 - mass and spin of final BH

Available at:

<https://github.com/akuntz00/QuadraticQNM>

- Applications:
 - Improvement of ringdown waveforms for free
 - Tests of GR

BACKUP

DEVIATIONS TO REFLECTION SYMMETRY

Parity is not given by reflection symmetry

POLARIZATION PARAMETER:

$$\kappa_{lm} = \frac{\mathcal{A}_{+,lm} + i\mathcal{A}_{-,lm}}{\mathcal{A}_{+,lm} - i\mathcal{A}_{-,lm}}$$

TOTAL RATIO:

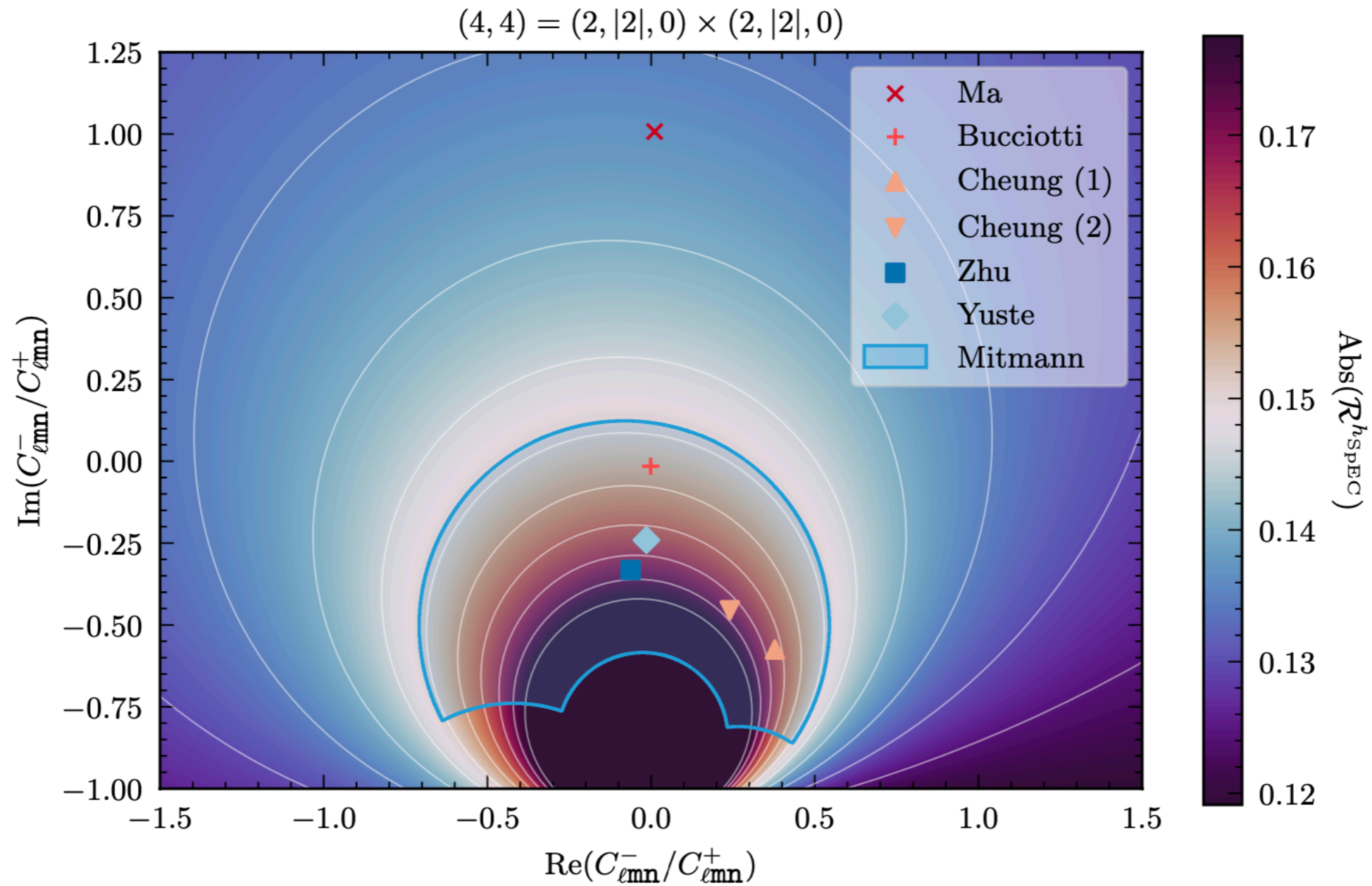
$$\frac{\mathcal{A}^{(2)}}{\mathcal{A}_1^{(1)} \mathcal{A}_2^{(1)}} = \frac{1}{4} [\alpha_+ + \beta_+] , \quad \kappa^{(2)} = \frac{\alpha_+ - \beta_+}{\alpha_+ + \beta_+} ,$$

$$\alpha_+ = \mathcal{R}_{--\rightarrow+} (1 - \kappa_1^{(1)}) (1 - \kappa_2^{(1)}) + \mathcal{R}_{++\rightarrow+} (1 + \kappa_1^{(1)}) (1 + \kappa_2^{(1)}) ,$$

$$\beta_+ = \mathcal{R}_{+-\rightarrow-} (1 + \kappa_1^{(1)}) (1 - \kappa_2^{(1)}) + \mathcal{R}_{-+\rightarrow-} (1 - \kappa_1^{(1)}) (1 + \kappa_2^{(1)})$$

DEVIATIONS TO REFLECTION SYMMETRY

Bourg&al (2024)



SELECTIONS RULES

Bucciotti, Juliano, AK, Trincherini (2024)

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & -m \end{pmatrix} = (-1)^{\ell_1 + \ell_2 + \ell} \begin{pmatrix} \ell_2 & \ell_1 & \ell \\ m_2 & m_1 & -m \end{pmatrix}$$

Rule 1. Quadratic modes vanish if $\ell_1 = \ell_2$, $m_1 = m_2$ and ℓ is odd

Rule 2. Quadratic modes vanish if $\ell_1 = \ell_2$, $n_1 = n_2$, $m_1 = m_2$, $\kappa_1 = \kappa_2$ and ℓ is odd