DFT = O(D, D) completion of GR

: non-Riemannian bonus

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Introduction

– Surely, General Relativity is based on Riemannian geometry, where the only geometric and gravitational field is the Riemannian metric, $g_{\mu\nu}$. Other fields are meant to be extra matter.

– However, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric:

They form the closed string massless (NS-NS) sector, being ubiquitous in all string theories,

$$\int \mathrm{d}^D x \, \sqrt{-g} e^{-2\phi} \left(R_g + 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) \qquad \text{where} \qquad H = \mathrm{d}B \, .$$

This action hides $\mathbf{O}(D, D)$ symmetry of T-duality which transforms g, B, ϕ into one another. Buscher 1987

– T-duality hints at a natural extension of GR, or the $\mathbf{O}(D, D)$ completion of GR , where the entire closed string massless sector, $\{g, B, \phi\}$, constitutes the gravitational multiplet.

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Take-home message would be

- **DFT = O(D, D) completion of GR** : the pure gravitational theory that string theory predicts.

- DFT assumes the whole closed-string massless (NS-NS) sector as the gravitational multiplet.

  The **O**(D, D) Symmetry Principle then fixes its coupling to extra matter unambiguously.

- The previous Lagrangian itself is identified as a scalar curvature in novel differential geometry,

$$R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \quad \Rightarrow \quad S_{(0)} \quad : \quad \text{Pure Gravity}$$

- The EOM of $\{g, B, \phi\}$ are unified into a single master formula,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations}$$

  which is the **O**(D, D) completion of Einstein Field Equations, as A, B are **O**(D, D) indices.

- Further, taking **O**(D, D) covariant field variables as its truely fundamental constituents, DFT can accommodate not only conventional supergravity but also various non-Riemannian gravities where string becomes chiral, *e.g.* Newton–Cartan, Carroll, or Gomis–Ooguri.

- The theory appears to be defined on 'doubled-yet-gauged spacetime': the doubled coordinates are gauged such that a gauge orbit corresponds to a single physical point.

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- The theory appears to be defined on 'doubled-yet-gauged spacetime': the doubled coordinates are gauged such that a gauge orbit corresponds to a single physical point.

Plan

I. Classification of DFT-geometries in terms of two non-negative integers, $(n, \bar{n})$.

II. Doubled-yet-gauged spacetime and sigma models.

III. Review of covariant derivatives, ∇_A, and curvatures, $S_{(0)}$, S_{AB}, G_{AB} in DFT.

IV. Derivation of the Einstein Double Field Equations, $G_{AB} = 8\pi G T_{AB}$,

$$G_{AB} := 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \nabla_A G^{AB} = 0\,,$$

$$T_{AB} := 4V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \nabla_A T^{AB} = 0\,.$$

V. Remarks on cosmology, non-Riemannian gravities, and graded Poisson geometry.

This talk is an overview of speaker's collaborative works over the last decade, thanks to
Stephen Angus (2), Thomas Basile, Kyungho Cho (4), Guilherme Franzmann, Euihun Joung, Shinji Mukohyama, Kevin Morand (2), as well as earlier Imtak Jeon (8), Kanghoon Lee (8) ...

Notation

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots, M, N, \cdots$	$\mathbf{O}(D, D)$ vector	$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$p, q, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ vector	$\eta_{pq} = \mathrm{diag}(- + + \cdots +)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ spinor	$C_{\alpha\beta}, \qquad (\gamma^p)^T = C\gamma^p C^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+ - - \cdots -)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D{-}1, 1)_R$ spinor	$\bar{C}_{\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}\bar{\gamma}^{\bar{p}}\bar{C}^{-1}$

- Further, the $\mathbf{O}(D, D)$ metric, $\mathcal{J}_{AB}$, decomposes the doubled coordinates into two parts,

$$x^A = (\tilde{x}_\mu, x^\nu)\,, \qquad \partial_A = (\tilde{\partial}^\mu, \partial_\nu)\,.$$

  where μ, ν are D-dimensional curved indices.

- The twofold local Lorentz symmetries, *i.e.* spin groups, indicate two distinct locally inertial frames for the left and right moving sectors $\Rightarrow$ Unification of IIA and IIB.

Closed-string massless sector as 'Gravitational Fields'

The gravitational fields consist of the DFT-dilaton, d, and DFT-metric, $\mathcal{H}_{MN}$:

$$\mathcal{H}_{MN}=\mathcal{H}_{NM}\,,\qquad\qquad \mathcal{H}_{K}{}^{L}\mathcal{H}_{M}{}^{N}\mathcal{J}_{LN}=\mathcal{J}_{KM}\,.$$

Combining $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$, we get a pair of symmetric projection matrices,

$$P_{MN}=P_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,\qquad P_{L}{}^{M}P_{M}{}^{N}=P_{L}{}^{N}\,,$$

$$\bar{P}_{MN}=\bar{P}_{NM}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,\qquad \bar{P}_{L}{}^{M}\bar{P}_{M}{}^{N}=\bar{P}_{L}{}^{N}\,,$$

which are orthogonal and complete,

$$P_{L}{}^{M}\bar{P}_{M}{}^{N}=0\,,\qquad\qquad P_{M}{}^{N}+\bar{P}_{M}{}^{N}=\delta_{M}{}^{N}\,.$$

Further, taking the “square roots" of the projectors,

$$P_{MN}=V_{M}{}^{p}V_{N}{}^{q}\eta_{pq}\,,\qquad\qquad \bar{P}_{MN}=\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

we get a pair of DFT-vielbeins satisfying their own defining properties,

$$V_{Mp}V^{M}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{M\bar{p}}\bar{V}^{M}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Mp}\bar{V}^{M}{}_{\bar{q}}=0\,,$$

or equivalently

$$V_{M}{}^{p}V_{Np}+\bar{V}_{M}{}^{\bar{p}}\bar{V}_{N\bar{p}}=\mathcal{J}_{MN}\,.$$

Solution to the defining relation, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_K{}^L \mathcal{H}_M{}^N \mathcal{J}_{LN} = \mathcal{J}_{KM}$?

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \qquad \text{or} \qquad \mathcal{H}_{MN} = \mathcal{J}_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The left one is well-known: it contains a Riemannian metric and reduces DFT to SUGRA.

The right one is a *flat* background which admits no Riemannian nor SUGRA interpretation.

Thus, DFT describes not only Riemannian SUGRA but also non-Riemannian novel geometries.

Classification

Kevin Morand & JHP 1707.03713

The most general form of the DFT-metric, $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_K{}^L\mathcal{H}_M{}^N\mathcal{J}_{LN} = \mathcal{J}_{KM}$, is characterized by two non-negative integers, $(n, \bar{n})$, $0 \leq n + \bar{n} \leq D$:

$$\mathcal{H}_{AB} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda} + Y_i^\mu X_\lambda^i - \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\lambda^{\bar{\imath}} \\ B_{\kappa\rho}H^{\rho\nu} + X_\kappa^i Y_i^\nu - \bar{X}_\kappa^{\bar{\imath}} \bar{Y}_{\bar{\imath}}^\nu & K_{\kappa\lambda} - B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda} + 2X_{(\kappa}^i B_{\lambda)\rho} Y_i^\rho - 2\bar{X}_{(\kappa}^{\bar{\imath}} B_{\lambda)\rho} \bar{Y}_{\bar{\imath}}^\rho \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} H & Y_i(X^i)^T - \bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T - \bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{pmatrix} \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}$$

i) Symmetric and skew-symmetric fields : $H^{\mu\nu} = H^{\nu\mu}$, $K_{\mu\nu} = K_{\nu\mu}$, $B_{\mu\nu} = -B_{\nu\mu}$;

ii) Two kinds of zero eigenvectors: with $i, j = 1, 2, \cdots, n$ & $\bar{\imath}, \bar{\jmath} = 1, 2, \cdots, \bar{n}$,

$$H^{\mu\nu}X_\nu^i = 0\,, \qquad H^{\mu\nu}\bar{X}_\nu^{\bar{\imath}} = 0\,, \qquad K_{\mu\nu}Y_j^\nu = 0\,, \qquad K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^\nu = 0\,;$$

iii) Completeness relation: $H^{\mu\rho}K_{\rho\nu} + Y_i^\mu X_\nu^i + \bar{Y}_{\bar{\imath}}^\mu \bar{X}_\nu^{\bar{\imath}} = \delta^\mu{}_\nu$.

- **The trace is $\mathcal{H}_A{}^A = 2(n - \bar{n})$ which is $\mathbf{O}(D, D)$ invariant.**
- **The coset is $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions $D^2 - (n - \bar{n})^2$ as Nambu–Goldstone moduli.**

Berman-Blair-Otsuki, Cho-JHP 2019

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I. $(n,\bar{n})=(0,0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin :

$$\mathcal{H}_{MN}\equiv\begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g-Bg^{-1}B \end{pmatrix},\qquad e^{-2d}\equiv\sqrt{|g|}e^{-2\phi}$$

Giveon, Rabinovici, Veneziano '89, Duff '90

II. Generically, on worldsheet, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^{i}_{\mu}\,\partial_{+}x^{\mu}(\tau,\sigma)\equiv 0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\,\partial_{-}x^{\mu}(\tau,\sigma)\equiv 0\,,$$

as we shall see shortly.

Non-Riemannian examples include

- $(1,0)$ Newton-Cartan gravity $\quad(\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}\mathbf{x}^2,\ \lim_{c\to\infty}g^{-1}$ is finite & degenerate$)$
- $(1,1)$ Gomis-Ooguri non-relativistic string — Melby-Thompson, Meyer, Ko, JHP 2015, Blair 2019
- $(D{-}1,0)$ ultra-relativistic Carroll gravity
- $(D,0)$ is uniquely given by $\mathcal{H}=\mathcal{J}$: maximally non-Riemannian with trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.
  This is the completely $\mathbf{O}(D,D)$-symmetric vacuum of DFT with no moduli, *c.f.* Siegel's chiral string.

  "*Spacetime emerges after SSB of* $\mathbf{O}(D,D)$*, identifying* $\{g,B\}$ *as Nambu–Goldstone boson moduli.*"
  Berman, Blair, and Otsuki 2019

  Further, taken as an internal space, it gives a 'moduli-free' (Scherk-Schwarz twistable) Kaluza-Klein reduction of pure DFT to heterotic DFT : Heterotic string has higher dimensional non-Riemannian origin.
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Unifications are characteristic of DFT & ExFT

⇒ Knowing the duality covariant definitions of the generalized metric/vielbein

and taking them as the fundamental variables of the theory,

'unifications' occur through different parametrizations:

- Maximally supersymmetric $D=10$ DFT with fixed chirality unifies IIA and IIB,
- ExFT unifies IIB and M-theory, Hohm-Samtleben 2013, Blair-Malek-JHP 2013
- Further, DFT and ExFT unify Riemannian SUGRA and non-Riemannian gravities.

"*Whatever happens for DFT happens for ExFT*" David Berman

Section condition

- **Diffeomorphisms** are generated by "generalized Lie derivative": Siegel 1993

$$\hat{\mathcal{L}}_{\xi} T_{A_1 \cdots A_n} := \xi^B \partial_B T_{A_1 \cdots A_n} + \omega_T \, \partial_B \xi^B T_{A_1 \cdots A_n} + \sum_{i=1}^{n} (\partial_{A_i} \xi_B - \partial_B \xi_{A_i}) T_{A_1 \cdots A_{i-1}}{}^{B}{}_{A_{i+1} \cdots A_n} ,$$

where ω_T is the weight, *e.g.* $\delta e^{-2d} = \partial_B(\xi^B e^{-2d})$, $\delta V_{Ap} = \xi^B \partial_B V_{Ap} + (\partial_A \xi_B - \partial_B \xi_A) V^B{}_p$.

- For consistency of closure, the so-called 'section condition' should be imposed: $\partial_M \partial^M = 0$.

  From $\partial_M \partial^M = 2\partial_\mu \tilde{\partial}^\mu$, the section condition can be easily solved by letting $\tilde{\partial}^\mu = 0$.

  The general solutions are then generated by the $\mathbf{O}(D, D)$ rotation of it.

- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_s(x) = \Phi_s(x + \Delta) , \qquad \Delta^M = \Phi_t \partial^M \Phi_u ,$$

where $\Phi_s, \Phi_t, \Phi_u \in \{ d, \mathcal{H}_{MN}, \xi^M, \partial_N d, \partial_L \mathcal{H}_{MN}, \cdots \}$, arbitrary functions appearing in DFT,

and Δ^M is said to be derivative-index-valued. JHP 2013

▶ 'Physics' should be invariant under such shifts of the doubled coordinates.

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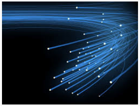

Doubled coordinates, $x^{M}=(\tilde{x}_{\mu}\,,x^{\nu})$, are gauged through an equivalence relation,

$$x^{M}\ \sim\ x^{M}+\Delta^{M}(x)\,,$$

where Δ^{M} is derivative-index-valued.

Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^{D}$.

- If we solve the section condition by letting $\tilde{\partial}^{\mu}\equiv 0$, and further choose $\Delta^{M}=c_{\mu}\partial^{M}x^{\mu}$, we note

$$(\tilde{x}_{\mu}\ ,\ x^{\nu})\ \ \sim\ \ (\tilde{x}_{\mu}+c_{\mu}\ ,\ x^{\nu})$$

- Then, $\mathbf{O}(D,D)$ rotates the gauged directions and hence the section.

Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$

JHP 1304.5946

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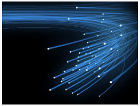

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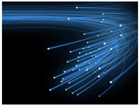
Doubled-yet-gauged coordinates, $(D+D)\times(1-1/2)$ JHP 1304.5946



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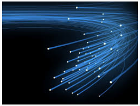

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Doubled-yet-gauged sigma models

The definition of the proper length readily leads to 'completely covariant' actions:

I. Particle action Ko-JHP-Suh 2016

$$\mathcal{S}_{\mathrm{particle}} = \int \mathrm{d}\tau \ \tfrac{1}{2} e^{-1} D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{2} m^2 e$$

II. String action Hull 2006, Lee-JHP 2013, Arvanitakis-Blair 2017

$$S_{\mathrm{string}} = \tfrac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h} h^{ij} D_i x^M D_j x^N \mathcal{H}_{MN}(x) - \epsilon^{ij} D_i x^M \mathcal{A}_{jM}$$

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III. κ-symmetric Green-Schwarz doubled-yet-gauged superstring, unifying IIA & IIB JHP 2016

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– The first part appears quadratically, which leads to Gaussian integral.

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i) Particle freezes over the $(n + \bar{n})$ dimensions

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Semi-covariant formalism

Imtak Jeon, Kanghoon Lee, & JHP 2010, 2011

- Semi-covariant derivative :

$$\nabla_C T_{A_1A_2\cdots A_n} := \partial_C T_{A_1A_2\cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

for which the 'DFT-Christoffel' connection can be uniquely fixed,

$$\Gamma_{CAB} = 2\big(P\partial_C P\bar{P}\big)_{[AB]} + 2\big(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\big)\partial_D P_{EC} - \tfrac{4}{D-1}\big(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\big)\big(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\big)$$

by demanding compatibility with $\{\mathcal{J}_{AB}, \mathcal{H}_{AB}, d\}$, torsionless condition, and projection property,

$$\nabla_A P_{BC} = \nabla_A \bar{P}_{BC} = \nabla_A d = 0\,, \qquad \hat{\mathcal{L}}_\xi^{\partial} = \hat{\mathcal{L}}_\xi^{\nabla} \;\Leftrightarrow\; \Gamma_{[ABC]} = 0\,, \qquad (\mathcal{P} + \bar{\mathcal{P}})_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,,$$

where multi-indexed projectors are

$$\mathcal{P}_{ABC}{}^{DEF} := P_A{}^D P_{[B}{}^{[E} P_{C]}{}^{F]} + \frac{2}{P_M{}^M - 1} P_{A[B} P_{C]}{}^{[E} P^{F]D}\,, \qquad \text{same for } \bar{\mathcal{P}}_{ABC}{}^{DEF} \text{ with } P_{AB} \leftrightarrow \bar{P}_{AB}\,.$$

– In particular, DFT-Killing equations can be defined from

$$\hat{\mathcal{L}}_\xi^{\nabla}\mathcal{H}_{AB} = 8\bar{P}_{(A}{}^{[C} P_{B)}{}^{D]}\nabla_C \xi_D\,, \qquad \hat{\mathcal{L}}_\xi^{\nabla} d = -\tfrac{1}{2}\nabla_A \xi^A\,.$$

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- Semi-covariant Riemann curvature :

$$S_{ABCD}=S_{[AB][CD]}=S_{CDAB}:=\tfrac{1}{2}\left(R_{ABCD}+R_{CDAB}-\Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,,\qquad S_{[ABC]D}=0\,,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}=\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as 'total derivative',

$$\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}\,,$$

which is useful for Lagrangian variation, *i.e.* action principle.

- Semi-covariant 'Master' derivative :

$$\mathcal{D}_A:=\partial_A+\Gamma_A+\Phi_A+\bar{\Phi}_A=\nabla_A+\Phi_A+\bar{\Phi}_A\,.$$

The two spin connections are determined in terms of the DFT-Christoffel connection,

$$\Phi_{Apq}=V^{B}{}_{p}\nabla_A V_{Bq}\,,\qquad\qquad \bar{\Phi}_{A\bar{p}\bar{q}}=\bar{V}^{B}{}_{\bar{p}}\nabla_A\bar{V}_{B\bar{q}}\,,$$

by requiring the compatibility with the vielbeins,

$$\mathcal{D}_A V_{Bp}=\nabla_A V_{Bp}+\Phi_{Ap}{}^{q}V_{Bq}=0\,,\qquad \mathcal{D}_A\bar{V}_{B\bar{p}}=\nabla_A\bar{V}_{B\bar{p}}+\bar{\Phi}_{A\bar{p}}{}^{\bar{q}}\bar{V}_{B\bar{q}}=0\,.$$

Semi-covariant formalism

Imtak Jeon, Kanghoon Lee, & JHP 2010, 2011

- Semi-covariant Riemann curvature :

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,, \qquad S_{[ABC]D} = 0\,,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}{=}\partial_A\Gamma_{BCD}{-}\partial_B\Gamma_{ACD}{+}\Gamma_{AC}{}^{E}\Gamma_{BED}{-}\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as 'total derivative',

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,,$$

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by requiring the compatibility with the vielbeins,

$$\mathcal{D}_A V_{Bp} = \nabla_A V_{Bp} + \Phi_{Ap}{}^{q}V_{Bq} = 0\,, \qquad \mathcal{D}_A \bar{V}_{B\bar{p}} = \nabla_A \bar{V}_{B\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}}\bar{V}_{B\bar{q}} = 0\,.$$

Anomaly is under control through the six-indexed projectors

- Semi-covariance:

$$\delta_\xi\left(\nabla_C T_{A_1\cdots A_n}\right) = \hat{\mathcal{L}}_\xi\left(\nabla_C T_{A_1\cdots A_n}\right) + \sum_{i=1}^{n} 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E\xi_F\, T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$\delta_\xi S_{ABCD} = \hat{\mathcal{L}}_\xi S_{ABCD} + 2\nabla_{[A}\left((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F\xi_G\right) + 2\nabla_{[C}\left((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F\xi_G\right).$$

- This is due to

$$\delta_\xi\Gamma_{CAB} = \hat{\mathcal{L}}_\xi\Gamma_{CAB} + 2\left[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^F\delta_A^D\delta_B^E\right]\partial_F\partial_{[D}\xi_{E]}\,.$$

Ideally one might desire to cancel these red-colored anomalies by adding extra terms to Γ_{CAB}.

But, since

$$\delta\mathcal{H}_{AB} = (P\delta\mathcal{H}\bar{P})_{AB} + (\bar{P}\delta\mathcal{H}P)_{AB}\,, \qquad \delta_\xi(\partial_C\mathcal{H}_{AB}) = \hat{\mathcal{L}}_\xi(\partial_C\mathcal{H}_{AB}) + 8\bar{P}_{(A}{}^D P_{B)}{}^E\partial_C\partial_{[D}\xi_{E]}\,,$$

it is impossible to construct such compensating terms out of the derivatives of $\mathcal{H}_{AB}$.

- However, we can easily project out the anomalies.

Complete covariantization: fixing the O(D, D) coupling to matter

– Tensors:

$$P_C{}^D \bar{P}_{A_1}{}^{B_1} \cdots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,,$$

$$\bar{P}_C{}^D P_{A_1}{}^{B_1} \cdots P_{A_n}{}^{B_n} \nabla_D T_{B_1 \cdots B_n} \quad \Longrightarrow \quad \mathcal{D}_{\bar{p}} T_{q_1 q_2 \cdots q_n} \,,$$

$$\mathcal{D}^p T_{p \bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p} q_1 q_2 \cdots q_n} \,; \qquad \mathcal{D}_p \mathcal{D}^p T_{\bar{q}_1 \bar{q}_2 \cdots \bar{q}_n} \,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_{q_1 q_2 \cdots q_n} \,.$$

– Yang-Mills:

$$\mathcal{F}_{p\bar{q}} := \mathcal{F}_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \qquad \text{where} \qquad \mathcal{F}_{AB} := \nabla_A W_B - \nabla_B W_A - i\,[W_A, W_B] \,.$$

– Spinors, ρ^α, $\psi^\alpha_{\bar{p}}$:

$$\gamma^p \mathcal{D}_p \rho \,, \qquad \mathcal{D}_{\bar{p}} \rho \,, \qquad \gamma^p \mathcal{D}_p \psi_{\bar{q}} \,, \qquad \mathcal{D}_{\bar{p}} \psi^{\bar{p}} \,,$$

– RR sector, $\mathcal{C}^\alpha{}_{\bar{\alpha}}$:

$$\mathcal{D}_\pm \mathcal{C} := \gamma^p \mathcal{D}_p \mathcal{C} \pm \gamma^{(D+1)} \mathcal{D}_{\bar{p}} \mathcal{C} \bar{\gamma}^{\bar{p}} \,, \quad (\mathcal{D}_\pm)^2 = 0 \qquad \Longrightarrow \qquad \mathcal{F} := \mathcal{D}_+ \mathcal{C} \qquad (\text{RR flux}) \,.$$

– Curvatures:

$$S_{p\bar{q}} := S_{AB} V^A{}_p \bar{V}^B{}_{\bar{q}} \quad (\text{Ricci}) \,, \qquad S_{(0)} := (P^{AC} P^{BD} - \bar{P}^{AC} \bar{P}^{BD}) S_{ABCD} \quad (\text{scalar} \Rightarrow \text{'pure' DFT}) \,.$$

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$\mathbf{O}(D,D)$ coupling to other superstring sectors or SM matter

- $D=10$ Maximally Supersymmetric DFT — Jeon-Lee-JHP-Suh 2012 [**Full order construction**]

$$\mathcal{L}_{\rm type\,II}=e^{-2d}\Big[\tfrac{1}{8}S_{(0)}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})+i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'$$
$$-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}}+i\bar{\psi}'^{p}\mathcal{D}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\Big]$$

which unifies IIA & IIB SUGRAs, and Gomis-Ooguri gravity as different solution/parametrization sectors.

- $\mathbf{O}(4,4)$ coupling to the $D=4$ Standard Model, — Kangsin Choi & JHP 2015

$$\mathcal{L}_{\rm SM}=e^{-2d}\left[\begin{array}{l}\frac{1}{16\pi G_N}S_{(0)}\\[4pt] +\sum_{\mathcal{V}}{\rm Tr}(\mathcal{F}_{p\bar{q}}\mathcal{F}^{p\bar{q}})+\sum_{\psi}\bar{\psi}\gamma^{a}\mathcal{D}_{a}\psi+\sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{a}}\mathcal{D}_{\bar{a}}\psi'\\[4pt] -\mathcal{H}^{AB}(\mathcal{D}_{A}\phi)^{\dagger}\mathcal{D}_{B}\phi-V(\phi)+y_{d}\,\bar{q}\cdot\phi\,d+y_{u}\,\bar{q}\cdot\tilde{\phi}\,u+y_{e}\,\bar{l}'\cdot\phi\,e'\end{array}\right]$$

⋆ Every single term above is completely covariant, w.r.t. $\mathbf{O}(D,D)$, DFT-diffeomorphisms, and twofold local Lorentz symmetries.

Derivation of EDFE from General Covariance 1/2

- Henceforth, we consider a general DFT action coupled to matter fields, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\,\tfrac{1}{16\pi G}S_{(0)} + L_{\rm matter}\big(\Upsilon_a, \mathcal{D}_A\Upsilon_b\big)\Big]\,,$$

and seek the variation of the action induced by all the fields, d, V_{Ap}, $\bar{V}_{Ap}$, Υ_a.

Note $\delta V_{Ap} = (\bar{P}+P)_A{}^B\delta V_{Bp} = \bar{V}_{A\bar{q}}\bar{V}^{B\bar{q}}\delta V_{Bp} + (\delta V_{B[p}V^B{}_{q]})V_A{}^q$. The 2nd term is a local Lorentz rotation and can be absorbed into $\delta\Upsilon_a$. Thus, only the projected variation, $\bar{V}^B{}_{\bar{q}}\delta V_{Bp} = -V^B{}_p\delta\bar{V}_{B\bar{q}}$, appears.

- Firstly, the 'pure' DFT part transforms, up to total derivatives ($\simeq$), as

$$\delta\Big(e^{-2d}S_{(0)}\Big) \simeq 4e^{-2d}\Big(\bar{V}^{B\bar{q}}\delta V_B{}^p S_{p\bar{q}} - \tfrac{1}{2}\delta d\, S_{(0)}\Big)\,.$$

- Secondly, the variation of the matter part,

$$\delta\Big(e^{-2d}L_{\rm matter}\Big) \simeq -2e^{-2d}\Big(\bar{V}^{A\bar{q}}\delta V_A{}^p K_{p\bar{q}} - \tfrac{1}{2}\delta d\, T_{(0)} - \tfrac{1}{2}\delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\Big)$$

naturally defines

$$K_{p\bar{q}} := \frac{1}{2}\left(V_{Ap}\frac{\delta L_{\rm matter}}{\delta\bar{V}_A{}^{\bar{q}}} - \bar{V}_{A\bar{q}}\frac{\delta L_{\rm matter}}{\delta V_A{}^p}\right)\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\rm matter}\right)}{\delta d}\,.$$

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Derivation of EDFE from General Covariance 2/2

Angus-Cho-JHP 1804.00964

- Combining the two results, the variation of the action reads

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right].$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a = \hat{\mathcal{L}}_\xi\Upsilon_a$ and

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- The Diffeomorphic General Covariance of the Action then implies

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\Big[4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)} - 8\pi G T_{(0)})\Big] + \delta_\xi\Upsilon_a\frac{\delta L_{\rm matter}}{\delta\Upsilon_a}\right].$$

This gives the **O**(D,D) completion of Einstein curvature, JHP-Rey-Rim-Sakatani 2015

$$G_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)}\,,$$

the **O**(D,D) completion of Energy-Momentum tensor, Angus-Cho-JHP 2018

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and hence, the **O**(D,D) completion of Einstein Field Equations, $G_{AB} = 8\pi G T_{AB}$.

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Angus-Cho-JHP 1804.00964

- Combining the two results, the variation of the action reads

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{A\bar{q}}\delta V_A{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

- Specifically when the variation is generated by diffeomorphisms, we have $\delta_\xi\Upsilon_a = \hat{\mathcal{L}}_\xi\Upsilon_a$ and

$$\bar{V}^{A\bar{q}}\delta_\xi V_A{}^p = \bar{V}^{A\bar{q}}\hat{\mathcal{L}}_\xi V_A{}^p = 2\mathcal{D}_{[A}\xi_{B]}\bar{V}^{A\bar{q}}V^{Bp}\,, \qquad \delta_\xi d = -\tfrac{1}{2}e^{2d}\hat{\mathcal{L}}_\xi\left(e^{-2d}\right) = -\tfrac{1}{2}\mathcal{D}_A\xi^A\,.$$

- The Diffeomorphic General Covariance of the Action then implies

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^B\mathcal{D}^A\left[4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{AB}(S_{(0)} - 8\pi G T_{(0)})\right] + \delta_\xi\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right].$$

This gives the **O**(D,D) completion of Einstein curvature, JHP-Rey-Rim-Sakatani 2015

$$G_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}S_{(0)}\,, \qquad \mathcal{D}_A G^{AB} = 0 \qquad \text{(off-shell)}\,,$$

the **O**(D,D) completion of Energy-Momentum tensor, Angus-Cho-JHP 2018

$$T_{AB} := 4V_{[A}{}^p\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{AB}T_{(0)}\,, \qquad \mathcal{D}_A T^{AB} = 0 \qquad \text{(on-shell)}\,,$$

and hence, the **O**(D,D) completion of Einstein Field Equations, $G_{AB} = 8\pi G T_{AB}$.

DFT = O(D, D) completion of GR

- One single master formula unifies all the EOMs of the whole massless NS-NS sector,

$$G_{AB} = 8\pi G T_{AB} \quad : \quad \text{Einstein Double Field Equations}$$

which is naturally consistent with our central idea that DFT treats the closed-string massless sector as the geometrical graviton multiplet.

- The $(0, 0)$ Riemannian parametrization reduces EDFE to

$$R_{\mu\nu} + 2\nabla_\mu(\partial_\nu \phi) - \tfrac{1}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} = 8\pi G K_{(\mu\nu)} \,,$$

$$e^{2\phi} \nabla^\rho \left(e^{-2\phi} H_{\rho\mu\nu} \right) = 16\pi G K_{[\mu\nu]} \,,$$

$$R + 4\Box\phi - 4\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} = 8\pi G T_{(0)} \,.$$

- EDFE also govern the generic $(n, \bar{n})$ non-Riemannian geometries, *e.g.* Newton–Cartan, Carroll, Gomis–Ooguri, generalizing the above three equations.

  We shall come back to this later.

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Examples: $T_{AB} := 4\,V_{[A}{}^{p}\bar{V}_{B]}{}^{\bar{q}}K_{p\bar{q}} - \frac{1}{2}\mathcal{J}_{AB}T_{(0)}$

- Scalar field,

$$L_{\varphi} = -\tfrac{1}{2}\mathcal{H}^{MN}\partial_M\varphi\partial_N\varphi - V(\varphi)\,, \qquad K_{p\bar{q}} = \partial_p\varphi\partial_{\bar{q}}\varphi\,, \qquad T_{(0)} = -2L_{\varphi}\,.$$

- Spinor field,

$$L_{\psi} = \bar{\psi}\gamma^p\mathcal{D}_p\psi + m_{\psi}\bar{\psi}\psi\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}(\bar{\psi}\gamma_p\mathcal{D}_{\bar{q}}\psi - \mathcal{D}_{\bar{q}}\bar{\psi}\gamma_p\psi)\,, \qquad T_{(0)} \equiv 0\,.$$

- RR sector,

$$L_{\mathrm{RR}} = \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})\,, \qquad K_{p\bar{q}} = -\tfrac{1}{4}\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})\,, \qquad T_{(0)} = 0\,.$$

- Fundamental string: with $D_i y^M = \partial_i y^M - \mathcal{A}_i^M$ (doubled-yet-gauged),

$$e^{-2d}L_{\mathrm{string}} = \tfrac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}D_iy^MD_jy^N\mathcal{H}_{MN}(y) - \epsilon^{ij}D_iy^M\mathcal{A}_{jM}\right]\delta^D(x-y(\sigma))\,,$$

$$K_{p\bar{q}} = \tfrac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\sqrt{-h}h^{ij}D_iy^MD_jy^N\,V_{Mp}\bar{V}_{N\bar{q}}\,e^{2d(x)}\delta^D(x-y(\sigma))\,, \qquad T_{(0)} = 0\,.$$

– More examples include Yang-Mills, point particle, Green-Schwarz superstring, *etc.* 1804.00964

Cosmology

Stephen Angus, Kyoungho Cho, G. Franzmann, S. Mukohyama & JHP 1905.03620

▶ **O(D, D) completion of the Friendamann equations:**

$$\frac{8\pi G}{3}\rho e^{2\phi}+\frac{h^2}{12a^6} = H^2-2\left(\frac{\phi'}{N}\right)H+\frac{2}{3}\left(\frac{\phi'}{N}\right)^2+\frac{k}{a^2}$$

$$\frac{4\pi G}{3}(\rho+3p)e^{2\phi}+\frac{h^2}{6a^6} = -H^2-\frac{H'}{N}+\left(\frac{\phi'}{N}\right)H-\frac{2}{3}\left(\frac{\phi'}{N}\right)^2+\frac{1}{N}\left(\frac{\phi'}{N}\right)'$$

$$\frac{8\pi G}{3}\left(\rho e^{2\phi}-\frac{1}{2}T_{(0)}\right) = -H^2-\frac{H'}{N}+\frac{2}{3N}\left(\frac{\phi'}{N}\right)'$$

which imply the conservation equation,

$$\rho'+3NH(\rho+p)+\phi' T_{(0)}e^{-2\phi}=0\,.$$

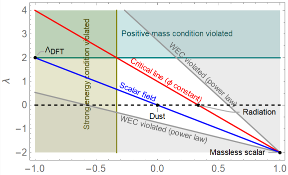
Here most general cosmological (homogeneous, isotropic, & Riemannian) ansatzes have been adopted:

$$\rho := \left(-K^t{}_t+\tfrac{1}{2}T_{(0)}\right)e^{-2\phi}\,,\qquad p := \left(K^r{}_r-\tfrac{1}{2}T_{(0)}\right)e^{-2\phi}\,,\qquad H_{(3)}=\frac{hr^2}{\sqrt{1-kr^2}}\sin\vartheta\,\mathrm{d}r\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi\,.$$

* This gives an enriched and novel framework beyond typical string cosmology, enjoying two equation-of-state parameters, $w=p/\rho$ (conventional) and $\lambda=T_{(0)}e^{-2\phi}/\rho$ (new).

In particular, de Sitter is unnatural as incompatible with DFT C.C. term, $e^{-2d}\Lambda_{\mathrm{DFT}}$. It might be an artifact of GR.

c.f. Swampland *a la* Vafa



Non-Riemannian $(n,\bar{n})$ sectors in DFT

Kyoungho Cho & JHP 1909.10711

- After $\tilde{\partial}^{\mu}\equiv 0$, the semi-covariant formalism naturally induces a 'upper-indexed' covariant derivative for the undoubled ordinary diffeomorphisms and $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ local rotations,

$$\mathbb{D}^{\mu}=H^{\mu\rho}\partial_{\rho}+\Omega^{\mu}+\Upsilon^{\mu}+\bar{\Upsilon}^{\mu}\,,$$

$$\begin{aligned}\Omega^{\mu\nu}{}_{\lambda}=&-\tfrac{1}{2}\partial_{\lambda}H^{\mu\nu}-H^{\rho[\mu}\partial_{\rho}H^{\nu]\sigma}K_{\sigma\lambda}-H^{\rho[\mu}\partial_{\rho}Y_{i}^{\nu]}X_{\lambda}^{i}-H^{\rho[\mu}\partial_{\rho}\bar{Y}_{\bar{\imath}}^{\nu]}\bar{X}_{\lambda}^{\bar{\imath}}\\&+\Big(2H^{\rho[\mu}Y_{i}^{\nu]}\partial_{[\tau}X_{\rho]}^{i}-2H^{\rho[\mu}\bar{Y}_{\bar{\imath}}^{\nu]}\partial_{[\tau}\bar{X}_{\rho]}^{\bar{\imath}}\Big)\Big(Y_{j}^{\tau}X_{\lambda}^{j}-\bar{Y}_{\bar{\jmath}}^{\tau}\bar{X}_{\lambda}^{\bar{\jmath}}\Big),\\\Upsilon^{\mu\,i}{}_{j}=&-2H^{\mu\rho}Y_{j}^{\sigma}\partial_{[\rho}X_{\sigma]}^{i}\,,\qquad\qquad\bar{\Upsilon}^{\mu\,\bar{\imath}}{}_{\bar{\jmath}}=-2H^{\mu\rho}\bar{Y}_{\bar{\jmath}}^{\sigma}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{\imath}}\,.\end{aligned}$$

- For $(0,0)$, it reduces to $\mathbb{D}^{\mu}=g^{\mu\nu}\bigtriangledown_{\nu}$,while for generic $(n,\bar{n})$ it enables us to expand

$$\int e^{-2d}S_{(0)}=\int e^{-2d}\Big[R_{\Omega}+4K_{\mu\nu}\mathbb{D}^{\mu}d\mathbb{D}^{\nu}d-\tfrac{1}{12}H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}\mathbb{H}_{\lambda\mu\nu}\mathbb{H}_{\rho\sigma\tau}-\mathbb{H}_{\lambda\mu\nu}H^{\lambda\rho}\Big(Y_{i}^{\mu}\widehat{\mathbb{D}}^{\nu}X_{\rho}^{i}-\bar{Y}_{\bar{\imath}}^{\mu}\widehat{\mathbb{D}}^{\nu}\bar{X}_{\rho}^{\bar{\imath}}\Big)\Big],$$

where $\widehat{\mathbb{H}}^{\lambda\mu\nu}$ is a diffeomorphism covariant, $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ invariant, and Milne-shift invariant $\mathbb{H}$-flux,

$$\widehat{\mathbb{H}}^{\lambda\mu\nu}=\widehat{\mathbb{H}}^{[\lambda\mu\nu]}:=H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}H_{\rho\sigma\tau}+6H^{\rho[\lambda}Y_{i}^{\mu}\mathbb{D}^{\nu]}X_{\rho}^{i}-6H^{\rho[\lambda}\bar{Y}_{\bar{\imath}}^{\mu}\mathbb{D}^{\nu]}\bar{X}_{\rho}^{\bar{\imath}}\,.$$

- This might have provided the action principle for each non-Riemannian gravity with fixed $(n,\bar{n})$, but it cannot produce the full EDFE, since $\delta\mathcal{H}_{AB}$ may involve $(n,\bar{n})\rightarrow(n-1,\bar{n}-1)$.

$\Rightarrow$ We conclude that the various non-Riemannian gravities should better be identified as different solution sectors of DFT rather than viewed as independent theories.

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Connection to Graded Poisson Geometry

Basile-Joung-JHP 1910.13120

- In 2016, **Deser and Sämann** formulates the generalized Lie derivative using a graded Poisson bracket:

$$\Big[T(x,\theta)\,,\ \big[p_A\theta^A,\xi_B\theta^B\big\}\Big\} = \hat{\mathcal{L}}_\xi T(x,\theta)\,,\qquad [F,G\} := \frac{\partial F}{\partial x^A}\frac{\partial G}{\partial p_A} - \frac{\partial F}{\partial p_A}\frac{\partial G}{\partial x^A} - (-1)^{\deg(F)}\frac{\partial F}{\partial \theta^A}\frac{\partial G}{\partial \theta_A}$$

where $T(x,\theta) = \frac{1}{p!}T_{C_1C_2\cdots A_p}(x)\theta^{C_1}\theta^{C_2}\cdots\theta^{C_p}$.

- Recently, we have identified this graded Poisson bracket as the Dirac bracket in the Hamiltonian formulation of the Faddeev–Popov doubled-yet-gagued particle action,

$$S_{\rm F.P.} = \int {\rm d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^A D_\tau x^B\mathcal{H}_{AB}(x) - \tfrac{1}{2}m^2 e + k_A\mathcal{A}^A + k(e-1) + \tfrac{1}{2}\theta_A\dot{\theta}^A + \sum_{\alpha=1}^{2}\tfrac{1}{2}\vartheta_\alpha\dot{\vartheta}^\alpha\,,$$

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Further, intringuingly, the *bc* ghost system for the worldline diffeomorphisms has also $\mathbf{O}(1,1)$ symmetry,

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EDFE as the master formula for massless NS-NS & non-Riemannian geometry

– **DFT as a modified gravity ?** *c.f.* **1804.00964**

- **The $D = 4$ spherical 'star' solution has three free parameters due to the enriched T_{AB}, and modifies Schwarzschild geometry for small R/MG.**
- **H-flux as dark matter? It does not couple to particle geodesics but contributes to the mass formula along with (visible) matter, ${K_t}^t$:**

$$M = \int_0^\infty \mathrm{d}r \int_0^\pi \mathrm{d}\vartheta \int_0^{2\pi} \mathrm{d}\varphi \; e^{-2d} \left(-2{K_t}^t + \frac{1}{8\pi G} \left| H_{t\vartheta\varphi} H^{t\vartheta\varphi} \right| \right) .$$

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Danke schön