Exploring the moduli space of toroidal string compactifications

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In collaboration with

B. Fraiman, C. Nuñez arXiv:1805.11128

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Motivation

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Simplest: bosonic string on S^1:

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momentum # winding #

Hamiltonian $$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2}$$ $$\tilde{R} = \frac{\alpha'}{R}$$

Level-matching $$\bar{N} - N = p\tilde{p}$$

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Massless states: $N = \bar{N} = 1 \quad : \quad g_{\mu\nu}, B_{\mu\nu}, \phi$

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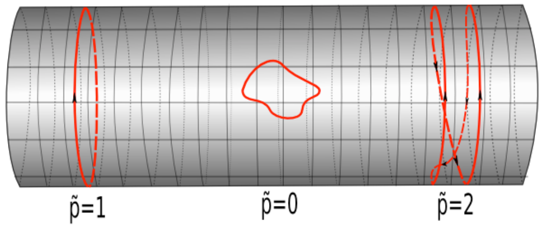

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At $R = \tilde{R} = \sqrt{\alpha'}$ Extra massless states for ex: $\bar{N} = 1, N = 0 \quad p = \tilde{p} = \pm 1$

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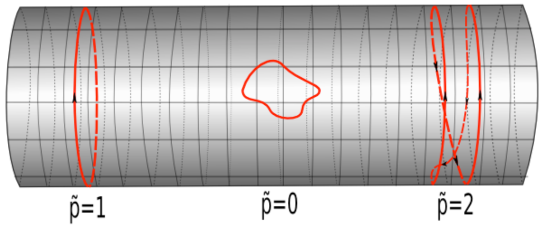

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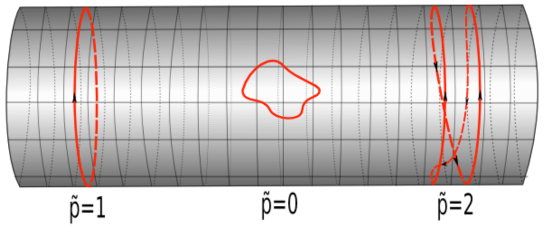

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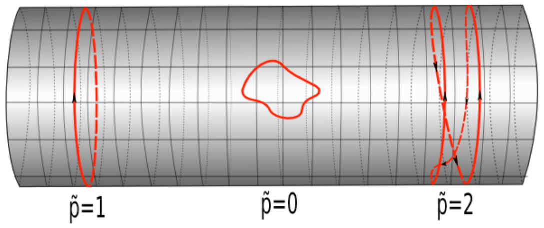

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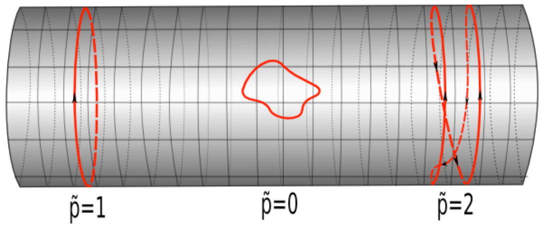

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Extra massless states
have momentum
and/or winding on circle

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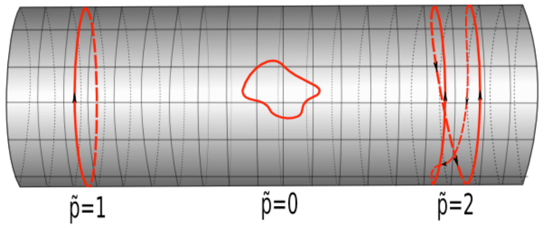

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From string scattering amplitudes ⟷ Double Field Theory

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have momentum
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Bosonic string on S^1

Mass $M^2 = 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2}$

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$$k_{L,R} = \frac{1}{\sqrt{2}}\left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}}\right)$$

Massless states at $R = \tilde{R} = 1$

Vectors $\bar{N}_x = 1$

- $N_y = 1 \quad (g_{\mu y} + B_{\mu y})$
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$$V \sim J^3(z) \cdot (\bar{\partial} X^\mu e^{ikX})$$

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$$J^3(z) = \partial Y^L(z)$$

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- SU(2)$_R$ Vectors $\quad N_x = 1 \qquad A^i \to \bar{A}^i$

$$J^i(z) \to \bar{J}^i(\bar{z}) \qquad Y^L(z) \to Y^R(\bar{z})$$

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$$J^3(z) = \partial Y^L(z)$$

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- Scalars $N_x = \bar{N}_x = 0$

$N_y = 1, \bar{N}_y = 1 \ (g_{yy})$

$N_y = 1, p = -\tilde{p} = \pm 1 \ (k_R = \pm\sqrt{2})$

$\bar{N}_y = 1, p = \tilde{p} = \pm 1 \ (k_L = \pm\sqrt{2})$

$p = \pm 2, \tilde{p} = 0 \ (k_L = k_R = \pm\sqrt{2})$

$p = 0, \tilde{p} = \pm 2 \ (k_L = -k_R = \pm\sqrt{2})$

$$\boxed{\begin{aligned} J^3(z) &= \partial Y^L(z) \\ J^\pm(z) &= e^{\pm\sqrt{2} i Y^L(z)} \end{aligned}}$$

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Bosonic string on S^1

Mass $M^2 = 2(N+\bar{N}-2)+\dfrac{p^2}{R^2}+\dfrac{\tilde{p}^2}{\tilde{R}^2}$

Level-matching $\bar{N}-N = p\tilde{p}$

Massless states at $R=\tilde{R}=1$

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- SU(2)L Vectors $\bar{N}_x = 1$
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- SU(2)R Vectors $N_x = 1 \qquad A^i\rightarrow\bar{A}^i \qquad J^i(z)\rightarrow\bar{J}^i(\bar{z}) \qquad Y^L(z)\rightarrow Y^R(\bar{z})$
- Scalars (3,3) $N_x=\bar{N}_x=0$

$N_y=1, \bar{N}_y=1 \ (g_{yy}) \quad : \quad M^{33}$

$N_y=1, p=-\tilde{p}=\pm1 \ (k_R=\pm\sqrt{2}) \quad : \quad M^{3\pm}$

$\bar{N}_y=1, p=\tilde{p}=\pm1 \ (k_L=\pm\sqrt{2}) \quad : \quad M^{\pm 3}$

$p=\pm2, \tilde{p}=0 \ (k_L=k_R=\pm\sqrt{2}) \quad : \quad M^{\pm\pm}$

$p=0, \tilde{p}=\pm2 \ (k_L=-k_R=\pm\sqrt{2}) \quad : \quad M^{\pm\mp}$

$$V^{ij}\sim J^i J^j e^{ikX}$$

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Symmetry enhancement (recap)

$$R \neq 1 (\neq \tilde{R})$$

$$U(1) \times U(1)$$

A $\bar{A}$

2 vectors

$(g_{\mu y} \pm B_{\mu y})$

M

1 scalar

(g_{yy})

Symmetry enhancement (recap)

$R \neq 1 (\neq \tilde{R})$

$R = \tilde{R} = 1$

$$U(1) \times U(1) \longrightarrow SU(2) \times SU(2)$$

A^3 $\bar{A}^3$

2 vectors

$(g_{\mu y} \pm B_{\mu y})$

A^i $\bar{A}^i$

6 vectors

$i = \pm, 3$

$i = \pm, 3$

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2 vectors $\qquad$ 6 vectors

$(g_{\mu y} \pm B_{\mu y})$

M^{33} $\qquad$ M^{ij}

1 scalar $\qquad$ 9 scalars

(g_{yy})

Bosonic string on T^d

Narain 86

Massless states:

$g_{\mu m}, B_{\mu m}$ 2d vectors: U(1)^d x U(1)^d

g_{mn}, B_{mn} d^2 scalars

Bosonic string on T^d

Narain 86

Massless states:

$g_{\mu m}, B_{\mu m}$ 2d vectors: $U(1)^d \times U(1)^d$

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+

lots of extra vectors & scalars
with mom & winding at points of
enhancement

Bosonic string on T^d

Narain 86

S^1 $$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

$$0 = N - \bar{N} + p\tilde{p}$$

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Mass $$M^2 = 2(N + \bar{N} - 2) + Z^t \mathcal{H} Z \qquad Z = \begin{pmatrix} p_m \\ \tilde{p}^m \end{pmatrix}$$

Level-matching $$0 = (N - \bar{N}) + \tfrac{1}{2}\, Z^t \eta Z$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \in \frac{O(d,d)}{O(d)\times O(d)}$$

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\eta^{LR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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S^1

$$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

$$0 = N - \bar{N} + p\tilde{p}$$

$$p_{L,R} = \frac{1}{\sqrt{2}}\left(\frac{p}{R} \pm \tilde{p}R\right)$$

Mass

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Level-matching

$$0 = (N - \bar{N}) + \frac{1}{2} Z^t \eta Z$$

$$p = EZ$$

$$\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e_a^m\left[p_m + (g_{mn} + B_{mn})\tilde{p}^n\right] \\ e_a^m\left[p_m - (g_{mn} - B_{mn})\tilde{p}^n\right] \end{pmatrix}$$

Bosonic string on T^d

Narain 86

Massless states:

$g_{\mu m}, B_{\mu m}$ 2d vectors: U(1)^d x U(1)^d

g_{mn}, B_{mn} d² scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \in \frac{O(d,d)}{O(d)\times O(d)}$$

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\eta^{LR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

S¹ $$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

$$0 = N - \bar{N} + p\tilde{p}$$

$$p_{L,R} = \frac{1}{\sqrt{2}}\left(\frac{p}{R} \pm \tilde{p}R\right)$$

Mass $$M^2 = 2(N + \bar{N} - 2) + Z^t \underbrace{\mathcal{H}}_{E^T E} Z \qquad Z = \begin{pmatrix} p_m \\ \tilde{p}^m \end{pmatrix}$$

$$p_L^2 + p_R^2$$

Level-matching $$0 = (N - \bar{N}) + \frac{1}{2} Z^t \underbrace{\eta}_{E^T \eta E} Z$$

$$p_L^2 - p_R^2$$

$$p = EZ \qquad \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m\left[p_m + (g_{mn} + B_{mn})\tilde{p}^n\right] \\ e_a^m\left[p_m - (g_{mn} - B_{mn})\tilde{p}^n\right] \end{pmatrix}$$

form a lattice: Lorentzian (d,d), even, self-dual

$\Gamma_{d,d}$

Massless vectors

$\mathcal{M}_D \times T^d$

$U(1)^d \times U(1)^d$

$A^m \quad \bar{A}^m$

2d vectors

$g_{\mu m} \pm B_{\mu m}$

Massless vectors

$$\mathcal{M}_D \times T^d$$

0= $M^2 = 2(N + \bar{N} - 2) + \left(p_L^2 + p_R^2\right)$

$$U(1)^d \times U(1)^d$$

LMC $0 = 2(N - \bar{N}) + \left(p_L^2 - p_R^2\right)$

$A^m \quad \bar{A}^m$

$$p = EZ$$

Extra vectors

2d vectors

$N = 0,\ \bar{N} = 1$ or $N = 1, \bar{N} = 0$

$$g_{\mu m} \pm B_{\mu m}$$

Massless vectors

$\mathcal{M}_D \times T^d$

$U(1)^d \times U(1)^d$

0= $M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$

LMC $0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$

$A^m \quad \bar{A}^m$

$p = EZ$

Extra vectors

2d vectors

$g_{\mu m} \pm B_{\mu m}$

$N = 0, \ \bar{N} = 1$ or $N = 1, \bar{N} = 0$

$p_L^2 - p_R^2 = \pm 2$ LMC

$p_L^2 + p_R^2 = 2$ M2=0

Massless vectors

$$\mathcal{M}_D \times T^d$$

0= $M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$

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$p_L^2 = 2, p_R = 0$

Massless vectors

$$\mathcal{M}_D \times T^d$$

$$0= \; M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

$$\text{LMC} \quad 0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

$$p = EZ$$

$$U(1)^d \times U(1)^d$$

$$A^m \qquad \bar{A}^m$$

Extra vectors

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

$$N = 0,\ \bar{N} = 1 \quad \text{or} \quad N = 1, \bar{N} = 0$$

$$p_L^2 - p_R^2 = \pm 2 \qquad \text{LMC}$$

$$p_L^2 + p_R^2 = 2 \qquad M^2{=}0$$

$$p_L^2 = 2, p_R = 0 \qquad p_L = 0, p_R^2 = 2$$

Massless vectors

$$\mathcal{M}_D \times T^d$$

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$p_L^2 = 2, p_R = 0$ $p_L = 0, p_R^2 = 2$

$$p = EZ = \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m\left[p_m + (g_{mn} + B_{mn})\tilde{p}^n\right] \\ e_a^m\left[p_m - (g_{mn} - B_{mn})\tilde{p}^n\right] \end{pmatrix}$$

Massless vectors

$$\mathcal{M}_D \times T^d$$

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2d vectors

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0= $M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$

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Extra vectors

$N = 0,\ \bar{N} = 1$ or $N = 1, \bar{N} = 0$

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at generic point in moduli space, no solution

Massless vectors

$$\mathcal{M}_D \times T^d$$

$$U(1)^d \times U(1)^d$$

$$A^m \qquad \bar{A}^m$$

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

0= $M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$

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$$p = EZ$$

Extra vectors

$N = 0,\ \bar{N} = 1$ or $N = 1, \bar{N} = 0$

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$$p_L^2 = 2, p_R = 0 \qquad p_L = 0, p_R^2 = 2$$

$p = EZ$ **=** $\begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n\right] \\ e_a^m \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n\right] \end{pmatrix}$

at generic point in moduli space, no solution

at special points:
solutions are roots of simply-laced gauge group

$$G \times G$$

Massless vectors

$$\mathcal{M}_D \times T^d$$

rank d rank d
dim n dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

$$A^m \qquad \bar{A}^m$$

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

$$0 = M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

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at generic point in moduli space, no solution

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Massless vectors

$\mathcal{M}_D \times T^d$

$$U(1)^d \times U(1)^d \longrightarrow \overset{\substack{\text{rank d}\\ \text{dim n}}}{G} \times \overset{\substack{\text{rank d}\\ \text{dim n}}}{G}$$

positive root

negative root

$$A^m \qquad \bar{A}^m \qquad\qquad A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$$

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

0= $M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$

LMC $\quad 0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$

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rank d dim n rank d dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

positive root

negative root

$$A^m \qquad \bar{A}^m \qquad A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$$

2d vectors

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at generic point in moduli space, no solution

at special points:
solutions are roots of simply-laced gauge group

$$G \times G$$

Symmetry enhancement, bosonic string on T^d

G simply-laced Lie algebra rank d, dim n G

maximal enhancement

$$U(1)^d \times U(1)^d \longrightarrow G \times G \qquad E = g + B$$

d=1 ● ● $SU(2) \times SU(2)$ $A_1 \times A_1$ 1

Symmetry enhancement, bosonic string on T^d

G simply-laced Lie algebra rank d, dim n G

maximal enhancement

$$U(1)^d \times U(1)^d \longrightarrow G \times G \qquad E = g + B$$

d=1	● ●	$SU(2) \times SU(2)$	$A_1 \times A_1$	1
d=2	● ● ● ●	$SU(2)^2 \times SU(2)^2$		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Symmetry enhancement, bosonic string on T^d

G simply-laced Lie algebra rank d, dim n G

maximal enhancement

$$U(1)^d \times U(1)^d \longrightarrow G \times G \qquad E = g + B$$

d=1

$$SU(2) \times SU(2) \qquad A_1 \times A_1 \qquad 1$$

d=2

$$SU(2)^2 \times SU(2)^2 \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

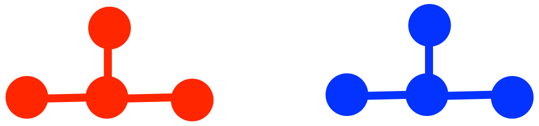
$$SU(3) \times SU(3) \qquad A_2 \times A_2 \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Symmetry enhancement, bosonic string on T^d

G simply-laced Lie algebra rank d, dim n G

maximal enhancement

$$U(1)^d \times U(1)^d \longrightarrow G \times G \qquad E = g + B$$

d=1		$SU(2) \times SU(2)$	$A_1 \times A_1$	1
d=2		$SU(2)^2 \times SU(2)^2$		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
		$SU(3) \times SU(3)$	$A_2 \times A_2$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
d=3		$SU(2)^3 \times SU(2)^3$		
		$SU(2) \times SU(3) \times SU(3) \times SU(2)$		
		$SU(4) \times SU(4)$	$A_3 \times A_3$	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$
d=4		$SU(2)^4 \times SU(2)^4$		
	⋮	⋮		
		$SU(5) \times SU(5)$	$A_4 \times A_4$	
		$SO(8) \times SO(8)$	$D_4 \times D_4$	

ADE series

Heterotic string on T^d

bosonic x superstring

Massless states:

$I = 1, ..., 16$ “chiral heterotic directions”

$g_{\mu m}, B_{\mu m}, A^I_\mu$ 2d+16 vectors: U(1)$^{d+16}$ x U(1)^d

g_{mn}, B_{mn}, A^I_m (d+16) x d scalars

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+

lots of extra vectors & scalars
with mom & winding at points of
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$$M^2 = 2\left(N + \bar{N} - \tfrac{3}{2}\right) + Z^t \mathcal{H} Z$$

Heterotic string on T^d

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$$M^2 = 2\left(N + \bar{N} - \tfrac{3}{2}\right) + Z^t \mathcal{H} Z$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^t g^{-1} & g + C^t g^{-1} C + AA^t & (1 + C^t g^{-1})A \\ -A^t g^{-1} & A^t(1 + g^{-1}C) & 1 + A^t g^{-1} A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16)\times O(d)}$$

1

$$C = B + \tfrac{1}{2} AA^t$$

Heterotic string on T^d

bosonic x superstring

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+

lots of extra vectors & scalars
with mom & winding at points of
enhancement

$$Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix}$$

quantized momenta

$\in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

SO(32) E$_8$ x E$_8$

$$M^2 = 2\left(N + \bar{N} - \tfrac{3}{2}\right) + Z^t \mathcal{H} Z$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^t g^{-1} & g + C^t g^{-1} C + AA^t & (1 + C^t g^{-1})A \\ -A^t g^{-1} & A^t(1 + g^{-1}C) & 1 + A^t g^{-1} A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)}$$

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Level-matching

$$0 = 2\left(N - \bar{N} - \tfrac{1}{2}\right) + Z^t \eta Z$$

1

$$C = B - \tfrac{1}{2} A A^t$$

$$\eta = \begin{pmatrix} 0 & 1_d & 0 \\ 1_d & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

$$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

Heterotic string on T^d

bosonic x superstring

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$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^t g^{-1} & g + C^t g^{-1} C + AA^t & (1 + C^t g^{-1})A \\ -A^t g^{-1} & A^t(1 + g^{-1}C) & 1 + A^t g^{-1} A \end{pmatrix} \in \frac{O(d{+}16,d)}{O(d{+}16)\times O(d)}$$

1

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$$\eta = \begin{pmatrix} 0 & 1_d & 0 \\ 1_d & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

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$$Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix}$$

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$\in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

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$$p_L^2 + p_R^2$$

Level-matching

$$0 = 2\left(N - \bar{N} - \tfrac{1}{2}\right) + Z^t \underbrace{\eta}_{E^T \eta E} Z$$

$$p_L^2 - p_R^2$$

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1

$$C = B - \tfrac{1}{2}AA^t$$

Level-matching

$$0 = 2\left(N - \bar{N} - \tfrac{1}{2}\right) + Z^t \underbrace{\eta}_{E^T \eta E} Z$$

$$p_L^2 - p_R^2$$

$$\eta = \begin{pmatrix} 0 & 1_d & 0 \\ 1_d & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

$$p = EZ$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m\left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2}A^I_m A^I_n \tilde{p}^n\right] \\ e_a^m\left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2}A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix}$$

$$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

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$\in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

SO(32) E$_8$ x E$_8$

$$M^2 = 2\left(N + \bar{N} - \tfrac{3}{2}\right) + Z^t \underbrace{\mathcal{H}}_{E^T E} Z$$

$$p_L^2 + p_R^2$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^t g^{-1} & g + C^t g^{-1} C + AA^t & (1 + C^t g^{-1})A \\ -A^t g^{-1} & A^t(1 + g^{-1}C) & 1 + A^t g^{-1} A \end{pmatrix} \in \frac{O(d{+}16,d)}{O(d{+}16)\times O(d)}$$

$$C = B + \tfrac{1}{2}AA^t$$

Level-matching

$$0 = 2\left(N - \bar{N} - \tfrac{1}{2}\right) + Z^t \underbrace{\eta}_{E^T \eta E} Z$$

$$p_L^2 - p_R^2$$

$$\eta = \begin{pmatrix} 0 & 1_d & 0 \\ 1_d & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

$$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

$$p = EZ$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^m_a \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ e^m_a \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix}$$

form a lattice Lorentzian (d+16,d), even, self-dual

$\Gamma^{16+d,d}$

Heterotic string on T^d

bosonic x superstring

Massless states:

$I = 1, \ldots, 16$ “chiral heterotic directions”

$g_{\mu m}, B_{\mu m}, A^I_\mu$ 2d+16 vectors: U(1)$^{d+16}$ x U(1)^d

g_{mn}, B_{mn}, A^I_m (d+16) x d scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

$$Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix}$$

quantized momenta

$\in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

SO(32) E$_8$ x E$_8$

$$M^2 = 2\left(N + \bar{N} - \tfrac{3}{2}\right) + Z^t \underbrace{\mathcal{H}}_{E^T E} Z$$

$$p_L^2 + p_R^2$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^t g^{-1} & g + C^t g^{-1} C + AA^t & (1 + C^t g^{-1})A \\ -A^t g^{-1} & A^t(1 + g^{-1}C) & 1 + A^t g^{-1} A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)}$$

$$C = B + \tfrac{1}{2} AA^t$$

Level-matching

$$0 = 2\left(N - \bar{N} - \tfrac{1}{2}\right) + Z^t \underbrace{\eta}_{E^T \eta E} Z$$

$$p_L^2 - p_R^2$$

$$\eta = \begin{pmatrix} 0 & 1_d & 0 \\ 1_d & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

$$p = EZ$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^m_a \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ e^m_a \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix}$$

$$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

form a lattice Lorentzian (d+16,d), even, self-dual

Extra massless vectors $N = 0, \bar{N} = \frac{1}{2}$ $p_L^2 = 2, p_R = 0$ $\Gamma^{16+d,d}$

Symmetry breaking, symmetry enhancement

quantized momenta

$$\pi^I \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8$$

SO(32) E8 x E8

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}^m_a \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \hat{e}^m_a \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix}$$

$$p_{aR} = 0$$

$$p^2_{aL} = 2$$

Symmetry breaking, symmetry enhancement

quantized momenta

$$\pi^I \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8$$

SO(32) E₈ x E₈

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_a^m \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A_m^I - \frac{1}{2} A_m^I A_n^I \tilde{p}^n\right] \\ \hat{e}_a^m \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A_m^I - \frac{1}{2} A_m^I A_n^I \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A_m^I \, \tilde{p}^m\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + \left(g + B - \frac{1}{2} A A^t\right)\, \tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p + \left(-g + B - \frac{1}{2} A A^t\right)\, \tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$p_{aR} = 0$$

$$p_{aL}^2 = 2$$

Symmetry breaking, symmetry enhancement

quantized momenta

$$\pi^I \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8$$

SO(32) E$_8$ x E$_8$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}^m_a \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \hat{e}^m_a \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + \left(g + B - \frac{1}{2} A A^t\right)\,\tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p + \left(-g + B - \frac{1}{2} A A^t\right)\,\tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

for S^1

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}R} \begin{pmatrix} p + \left(R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ \sqrt{2}R\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$p_{aR} = 0$$

$$p^2_{aL} = 2$$

Symmetry breaking, symmetry enhancement

quantized momenta

$$\pi^I \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8$$

SO(32)　　E8 x E8

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}^m_a \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \hat{e}^m_a \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \, \tilde{p}^m\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + \left(g + B - \frac{1}{2} A A^t\right)\, \tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p + \left(-g + B - \frac{1}{2} A A^t\right)\, \tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\, \tilde{p}\right] \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + E\, \tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p - E^t\, \tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\, \tilde{p}\right] \end{pmatrix}$$

for S^1

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}R} \begin{pmatrix} p + \left(R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ \sqrt{2}R\left[\pi + A\, \tilde{p}\right] \end{pmatrix}$$

$$p_{aR} = 0$$

$$p^2_{aL} = 2$$

Symmetry breaking, symmetry enhancement

quantized momenta

$\pi^I \in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

SO(32) E8 x E8

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_a^m \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A_m^I - \frac{1}{2} A_m^I A_n^I \tilde{p}^n\right] \\ \hat{e}_a^m \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A_m^I - \frac{1}{2} A_m^I A_n^I \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A_m^I \tilde{p}^m\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + \left(g + B - \frac{1}{2} A A^t\right) \tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p + \left(-g + B - \frac{1}{2} A A^t\right) \tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + E\,\tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p - E^t\,\tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

for S¹

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}R} \begin{pmatrix} p + \left(R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ \sqrt{2}R\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$p_{aR} = 0 \;\Rightarrow\; \underset{\tilde{p} = 0}{p} = \underbrace{\pi \cdot A}_{\in \mathbb{Z}} - \overset{\overset{R^2 - \frac{1}{2}|A|^2}{\shortparallel}}{E}\,\tilde{p} \;\in \mathbb{Z}$$

symmetry breaking

$$p_{aL}^2 = 2$$

Symmetry breaking, symmetry enhancement

quantized momenta

$\pi^I \in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$

SO(32) E$_8$ x E$_8$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_a^m \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \hat{e}_a^m \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n - \pi^I A^I_m - \frac{1}{2} A^I_m A^I_n \tilde{p}^n\right] \\ \sqrt{2}\left[\pi^I + A^I_m \tilde{p}^m\right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + \left(g + B - \frac{1}{2} A A^t\right)\tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p + \left(-g + B - \frac{1}{2} A A^t\right)\tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}\left[p + E\,\tilde{p} - \pi \cdot A\right] \\ \hat{e}\left[p - E^t\,\tilde{p} - \pi \cdot A\right] \\ \sqrt{2}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

for S^1

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p^I_L \end{pmatrix} = \frac{1}{\sqrt{2}R} \begin{pmatrix} p + \left(R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ \sqrt{2}R\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$p_{aR} = 0 \;\Rightarrow\; p = \underbrace{\pi \cdot A}_{\in \mathbb{Z}} - \overset{R^2 - \frac{1}{2}|A|^2}{\overset{\parallel}{E}}\,\tilde{p} \;\in \mathbb{Z}$$

$\tilde{p} = 0$

symmetry breaking

$$p^2_{aL} = 2 \;\Rightarrow\; |\pi + A\tilde{p}|^2 = 2\left(1 - R^2 \tilde{p}^2\right)$$

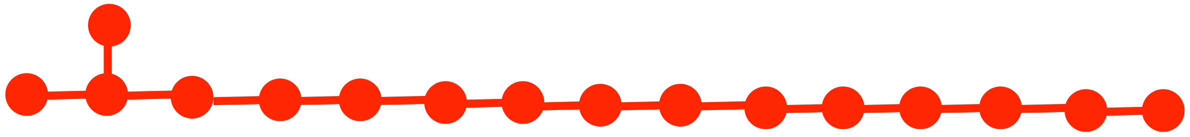
symmetry enhancement

Symmetry enhancement, heterotic string on S1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$

$SO(32)$

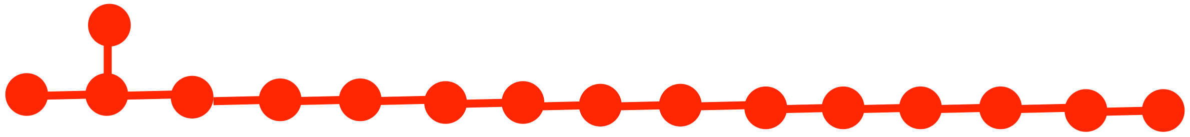
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$

$SO(32) \times U(1)$

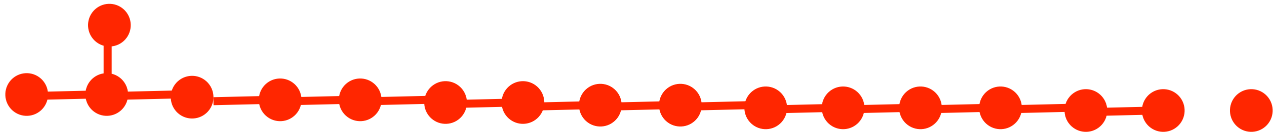
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S¹

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$

$R^2 = 1$

$SO(32) \times U(1)$

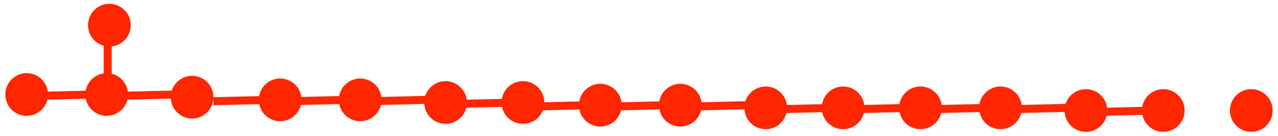
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$

$R^2 = 1$

$SO(32)$

$\times\ SU(2)$

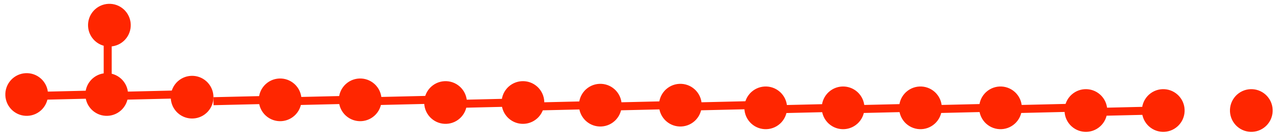
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$

$SO(32)$

$R^2 = 1$

$\times\ SU(2)$

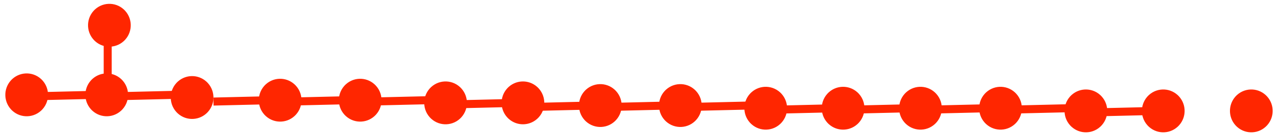
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



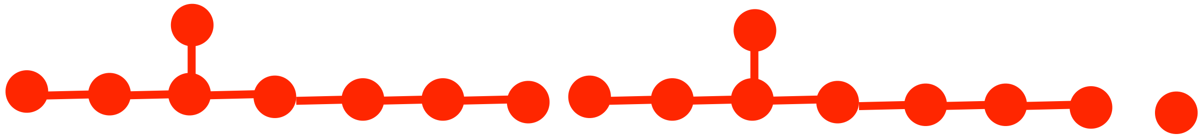
$A^I = 0$ or $A \in \Gamma_{16}$

$SO(32)$

$R^2 = 1$

$\times\ SU(2)$

E$_8$ x E$_8$



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$

$E_8 \times E_8$

$R^2 = 1$

$\times\ SU(2)$

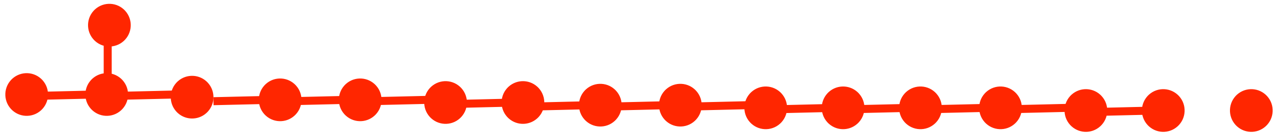
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$

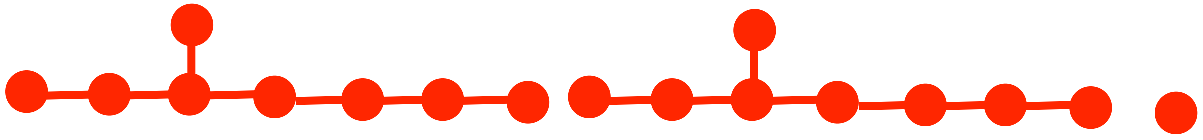
$SO(32)$

$R^2 = 1$

$\times\, SU(2)$

$A = (1, 0_{15})$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$

$E_8 \times E_8$

$R^2 = 1$

$\times\, SU(2)$

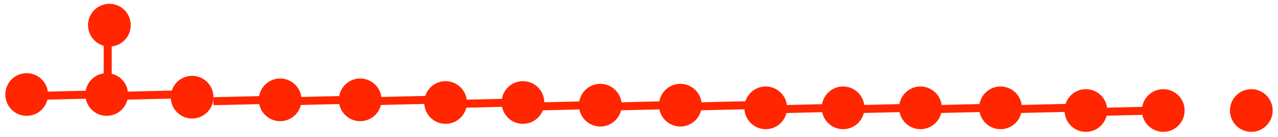
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S¹

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

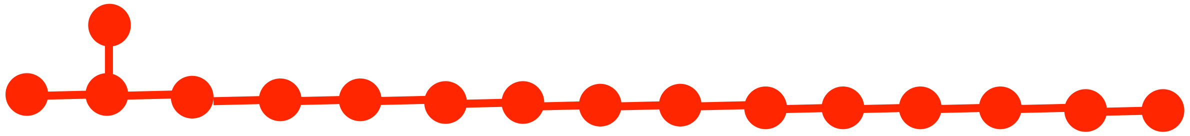
SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$

$R^2 = 1$

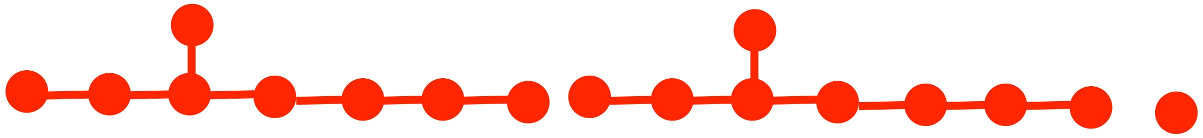
$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$

$SO(32)$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$

$R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$

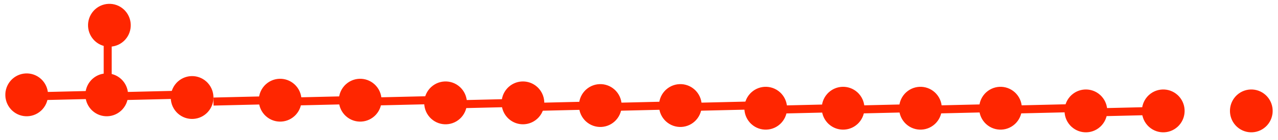
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

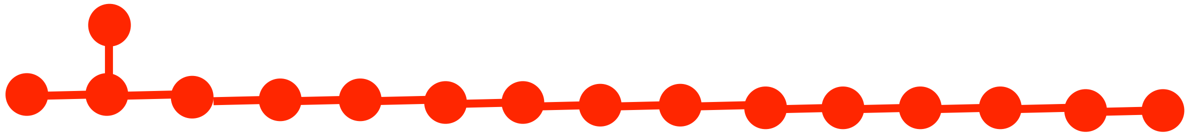
simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

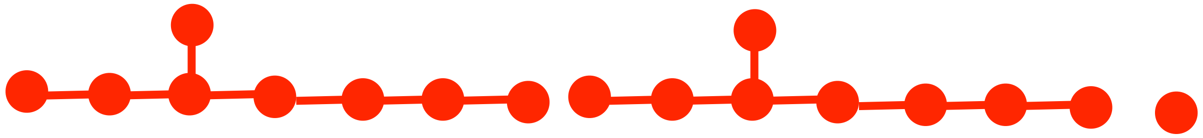
$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$

$SO(32) \times U(1)$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$

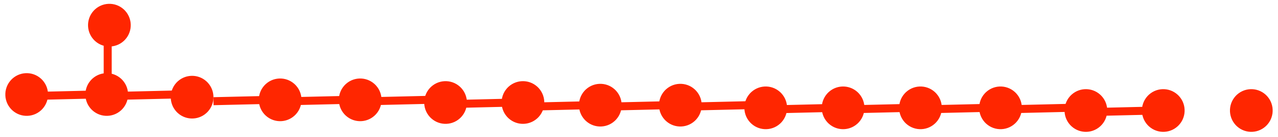
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I \tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

simply-laced
Lie algebra
rank 17

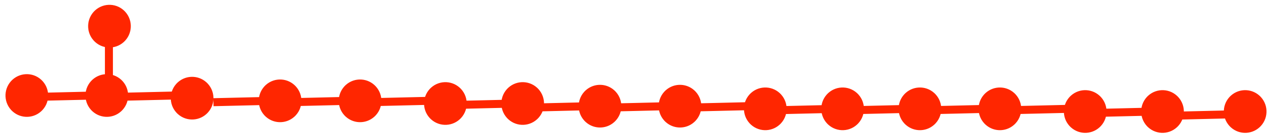
SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$

$R^2 = 1$

$SO(32)$ $\times SU(2)$

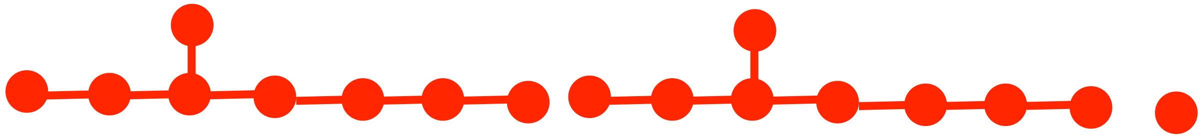


$A = (1, 0_{15}) \in \Gamma_v$

$R^2 = 1/2$

$SO(34)$

E$_8$ x E$_8$



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$

$R^2 = 1$

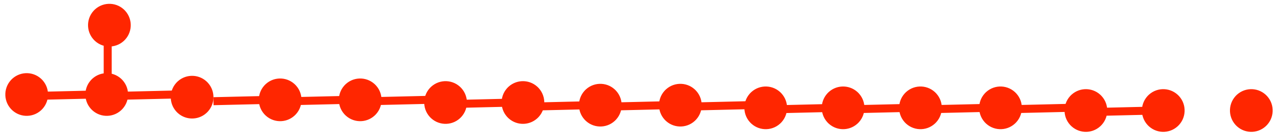
$E_8 \times E_8$ $\times SU(2)$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\, \tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S1 $U(1)^{17} \longrightarrow$ G

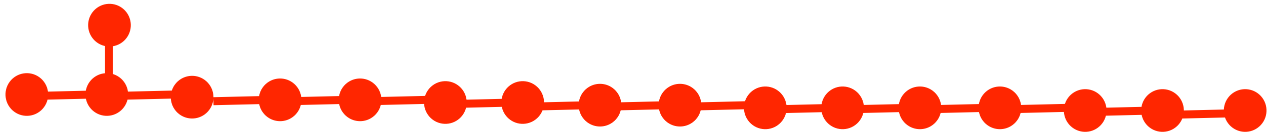
simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

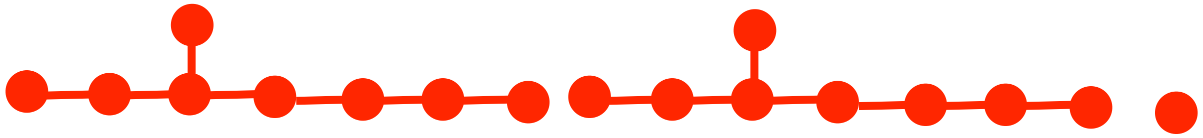
$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

$SO(34)$

E8 x E8



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

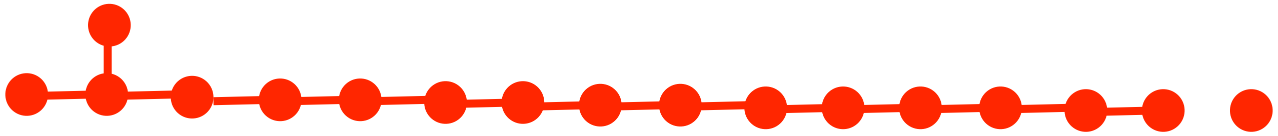
$E_8 \times E_8$ $\times SU(2)$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\,[\pi^I + A^I\,\tilde{p}] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S¹ $U(1)^{17} \longrightarrow$ G

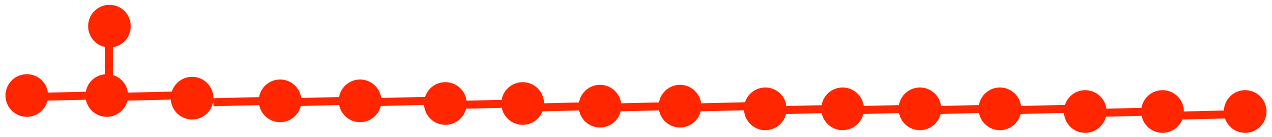
simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

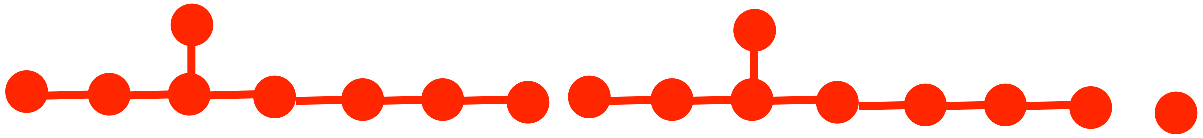
$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

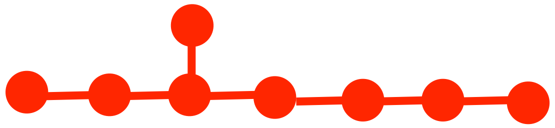
$SO(34)$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$

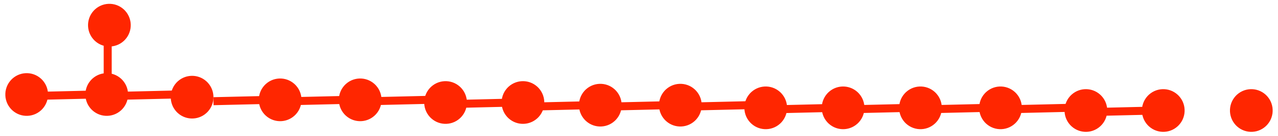
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I \tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

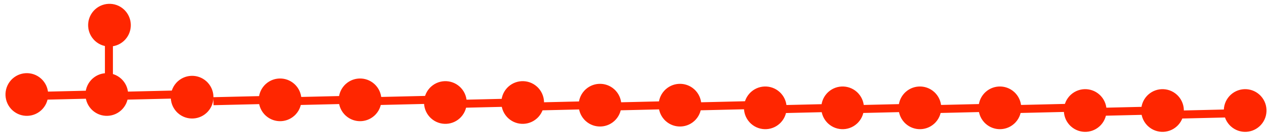
simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$ $\quad R^2 = 1$

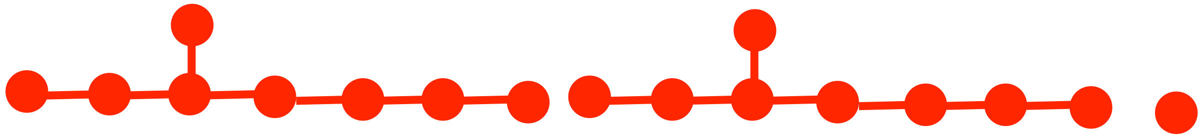
$SO(32)$ $\quad \times\, SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $\quad R^2 = 1/2$

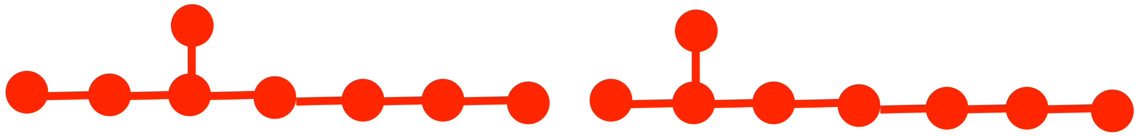
$SO(34)$

E8 x E8



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $\quad R^2 = 1$

$E_8 \times E_8$ $\quad \times\, SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$

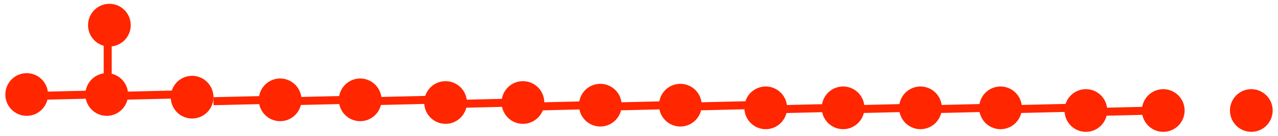
$E_8 \times SO(16)$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1 $U(1)^{17} \longrightarrow$ G

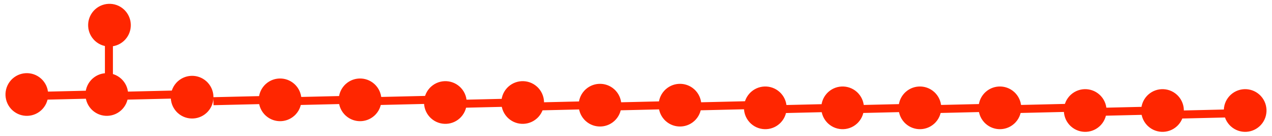
simply-laced
Lie algebra
rank 17

SO(32)



$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

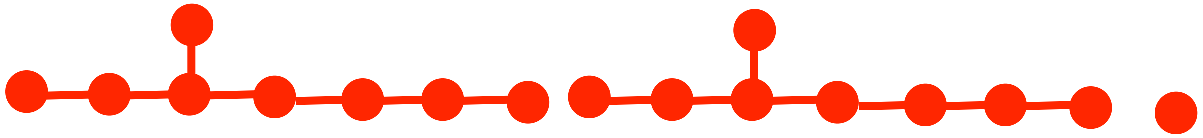
$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

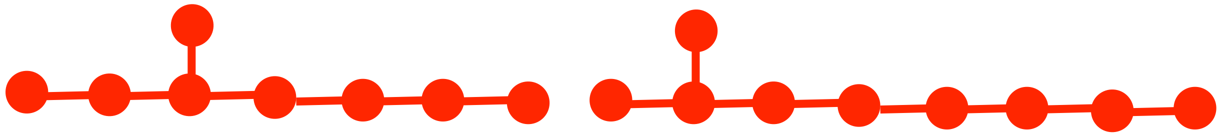
$SO(34)$

E$_8$ x E$_8$



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$ $R^2 = 1/2$

$E_8 \times SO(18)$

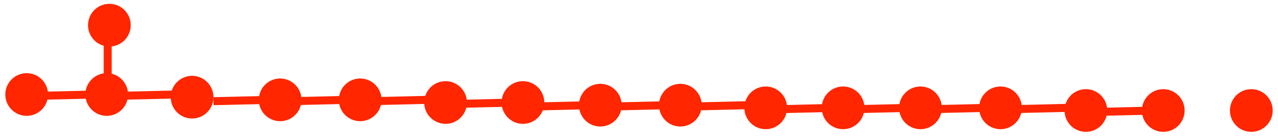
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p} R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \frac{p}{R} - \tilde{p} R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \sqrt{2}\,[\pi^I + A^I\,\tilde{p}] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

$U(1)^{17} \longrightarrow$ G

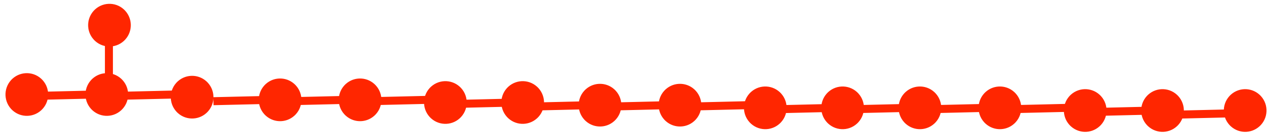
simply-laced
Lie algebra
rank 17

SO(32)



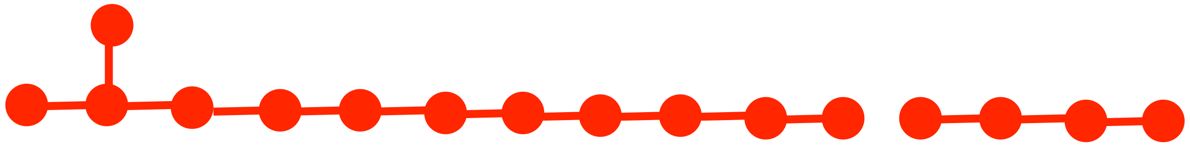
$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

$SO(32)$ $\times\, SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

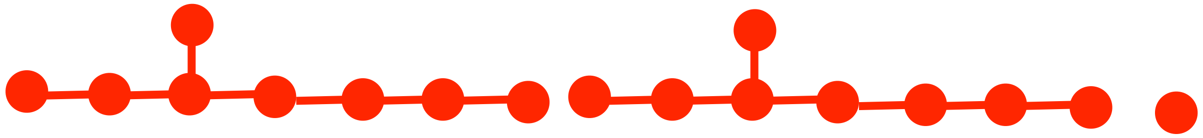
$SO(34)$



$SO(32-2p) \times SU(p)$

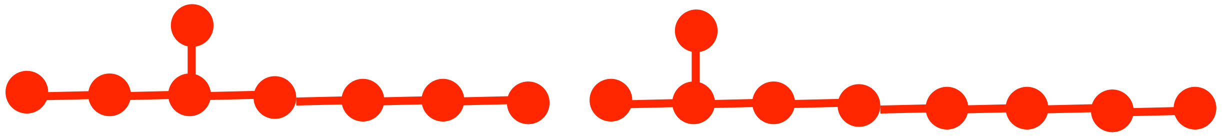
$$A = ((\tfrac{1}{4})_p, 0_{16-p})$$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times\, SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$ $R^2 = 1/2$

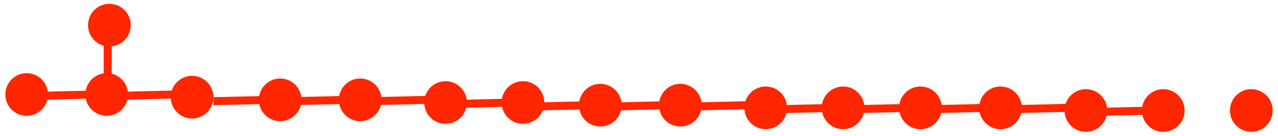
$E_8 \times SO(18)$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\,[\pi^I + A^I\,\tilde{p}] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S¹ $U(1)^{17} \longrightarrow$ G

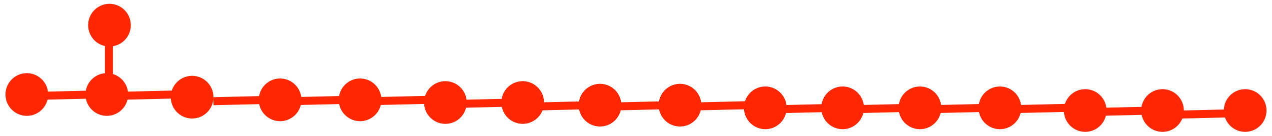
simply-laced
Lie algebra
rank 17

SO(32)



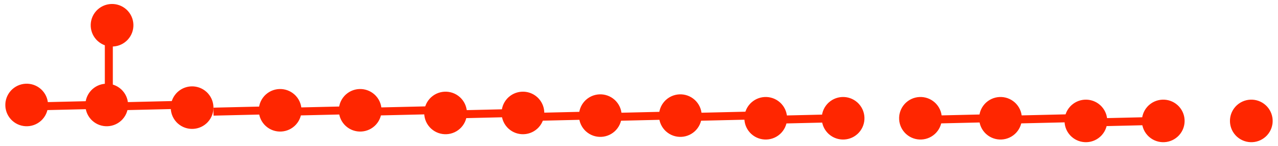
$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

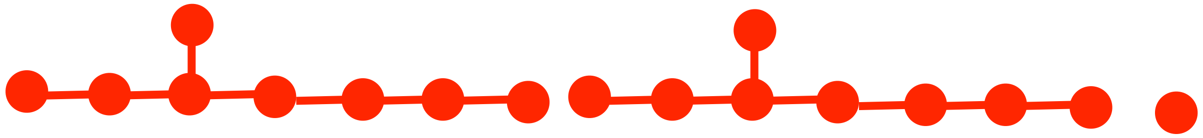
$SO(34)$



$SO(32-2p) \times SU(p)$ $\times SU(2)$

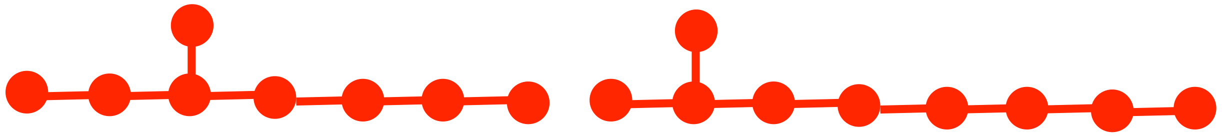
$A = ((\frac{1}{4})_p, 0_{16-p})$ $R^2 = 1 - \frac{p}{32}$

E₈ x E₈



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$ $R^2 = 1/2$

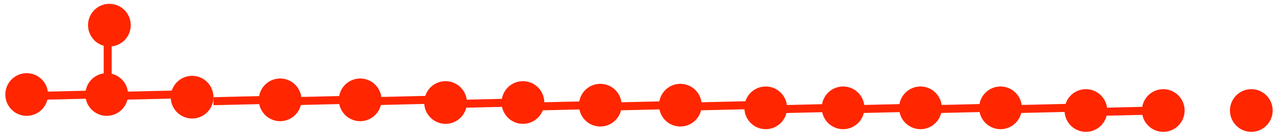
$E_8 \times SO(18)$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2 \tilde{p}) \\ \sqrt{2}\,[\pi^I + A^I \tilde{p}] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S1 $U(1)^{17} \longrightarrow$ G

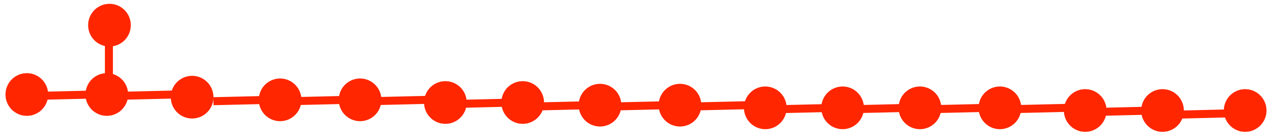
simply-laced
Lie algebra
rank 17

SO(32)



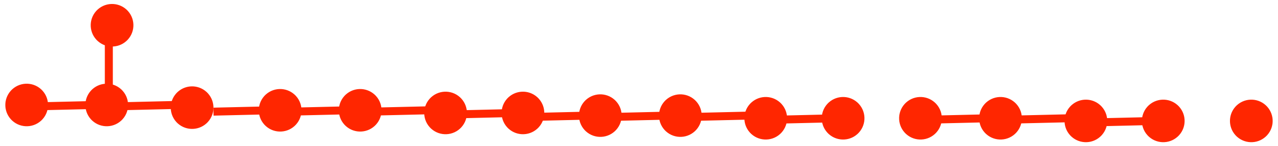
$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

$SO(34)$

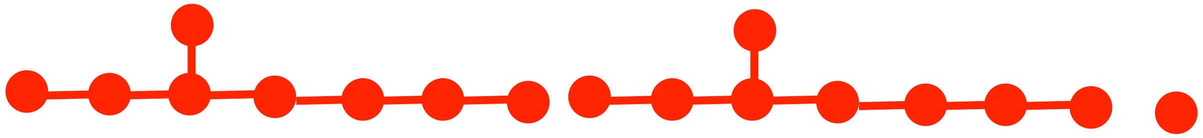


$SO(32-2p) \times SU(p)$ $\times SU(2)$

$A = ((\frac{1}{4})_p, 0_{16-p})$ $R^2 = 1 - \frac{p}{32}$

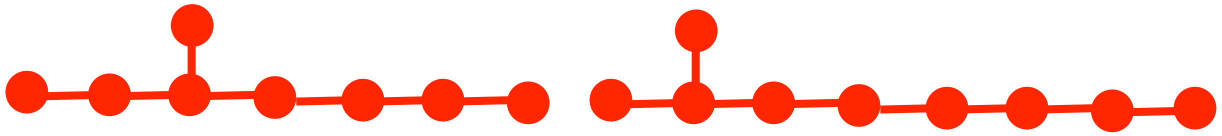
⋮

E8 x E8



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$ $R^2 = 1/2$

$E_8 \times SO(18)$

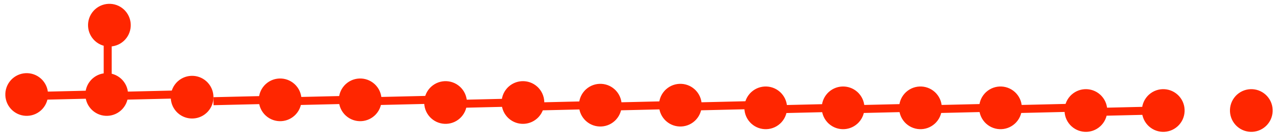
⋮

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^1

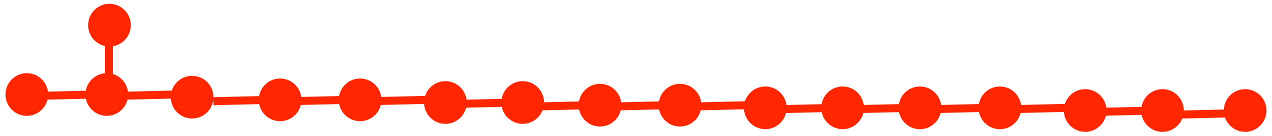
$U(1)^{17} \longrightarrow G$ simply-laced Lie algebra rank 17

SO(32)



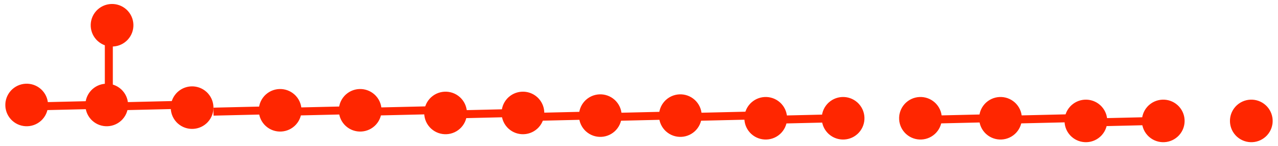
$A^I = 0$ or $A \in \Gamma_{16}$ $R^2 = 1$

$SO(32)$ $\times SU(2)$



$A = (1, 0_{15}) \in \Gamma_v$ or $A \in \Gamma_c$ $R^2 = 1/2$

$SO(34)$

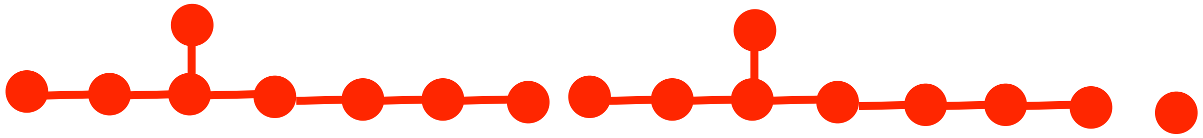


$SO(32-2p) \times SU(p)$ $\times SU(2)$

$A = ((\frac{1}{4})_p, 0_{16-p})$ $R^2 = 1 - \frac{p}{32}$

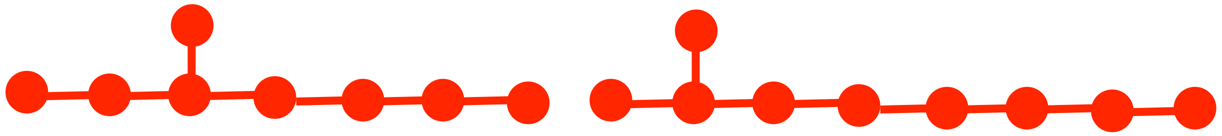
⋮

E8 x E8



$A^I = 0$ or $A \in \Gamma_8 \times \Gamma_8$ $R^2 = 1$

$E_8 \times E_8$ $\times SU(2)$



$A = (0_8, -1, 0_7) \in \Gamma_0 \times \Gamma_v$ $R^2 = 1/2$

$E_8 \times SO(18)$

⋮

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_L^I \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^I A^I - \frac{1}{2}|A|^2\tilde{p}) \\ \sqrt{2}\left[\pi^I + A^I\,\tilde{p}\right] \end{pmatrix}$$

Same groups? What are all the groups one can get?

All enhancement groups can be obtained from the Generalized Dynkin Diagram of $\Gamma^{17,1}$

Goddard, Olive 85
Cachazo, Vafa 00

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$E_8 \times E_8$

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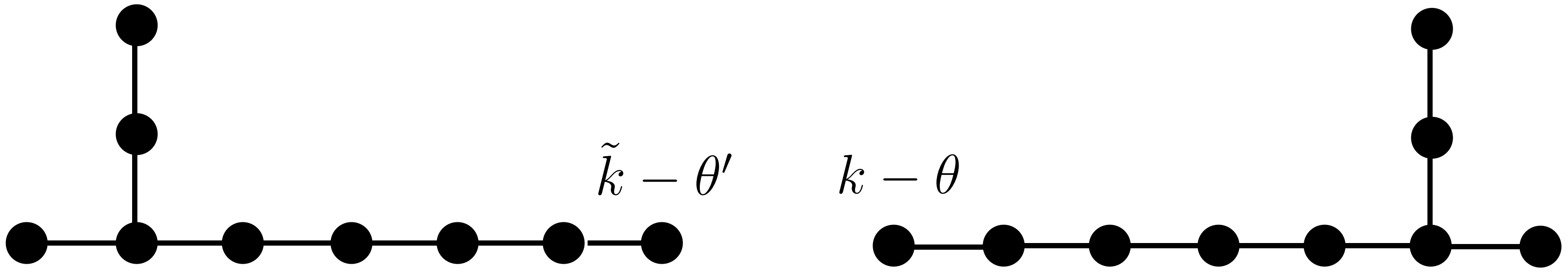

$\hat{E}_8 \times \hat{E}_8$

$$k = \frac{1}{\sqrt{2}}(1,1)$$

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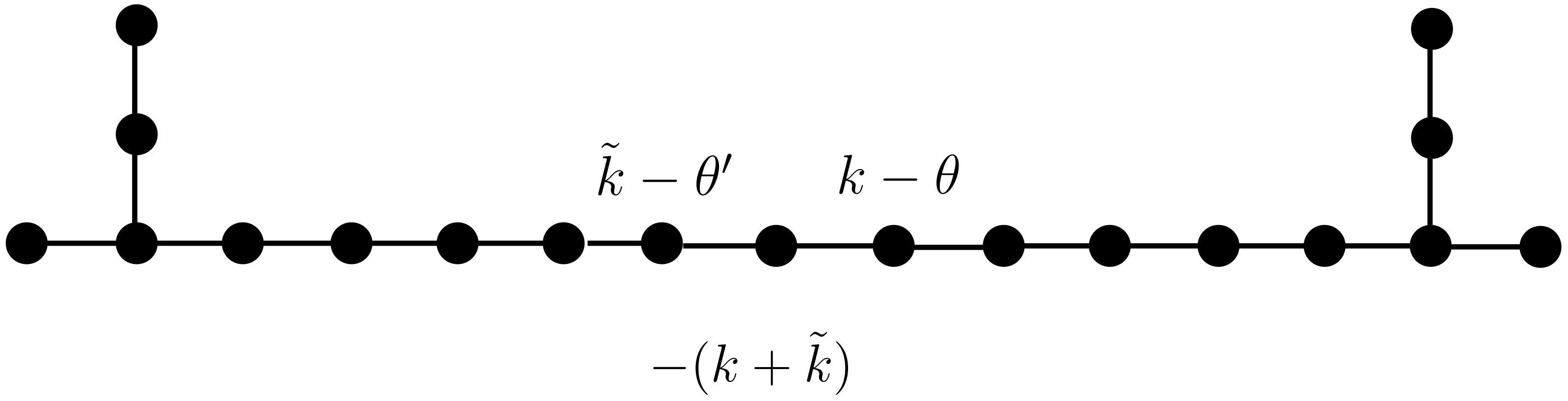

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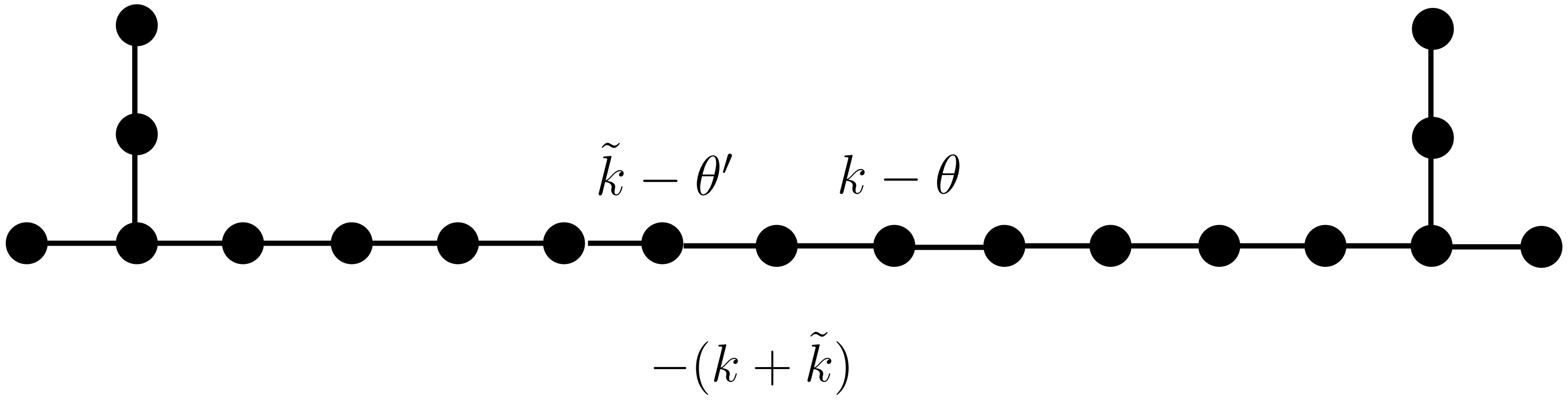

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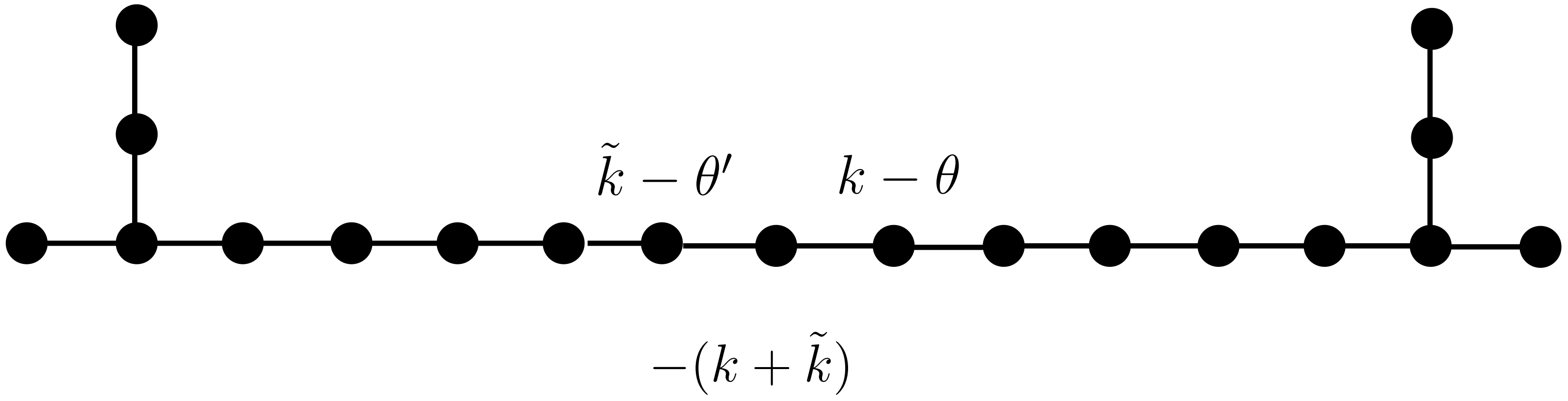

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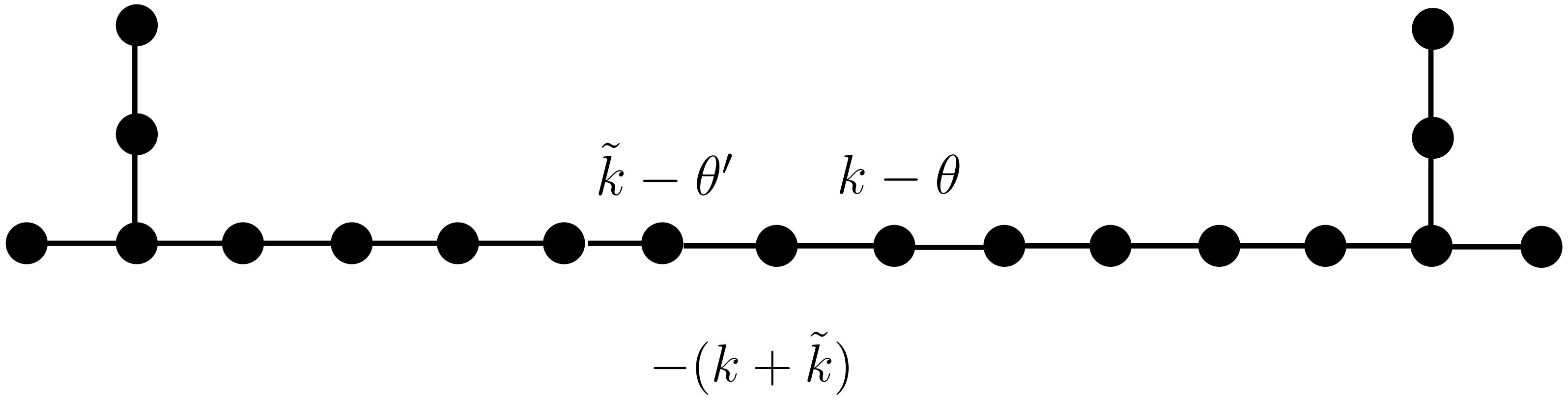

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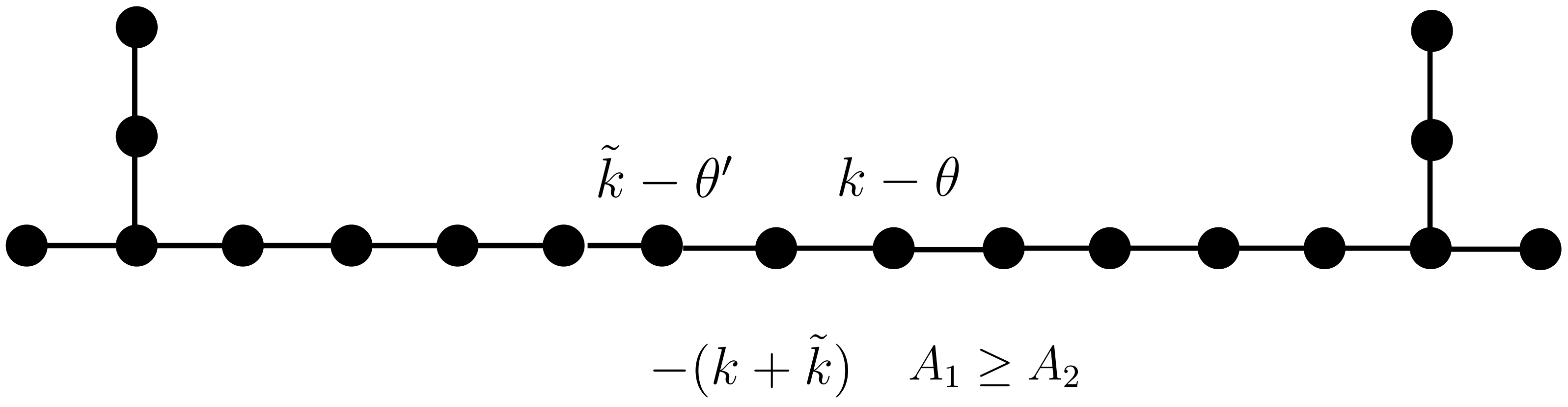

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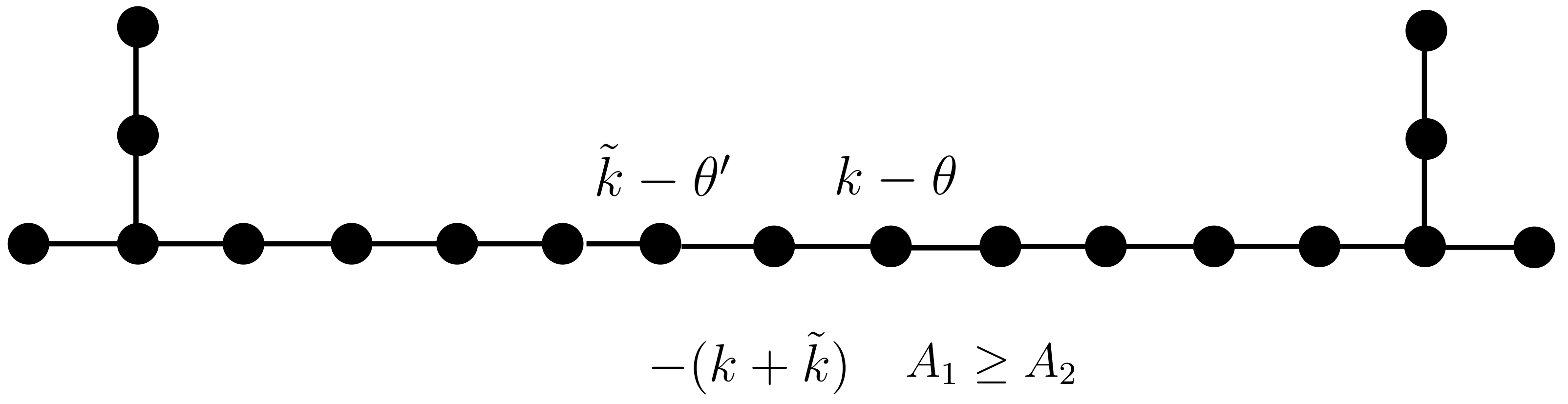

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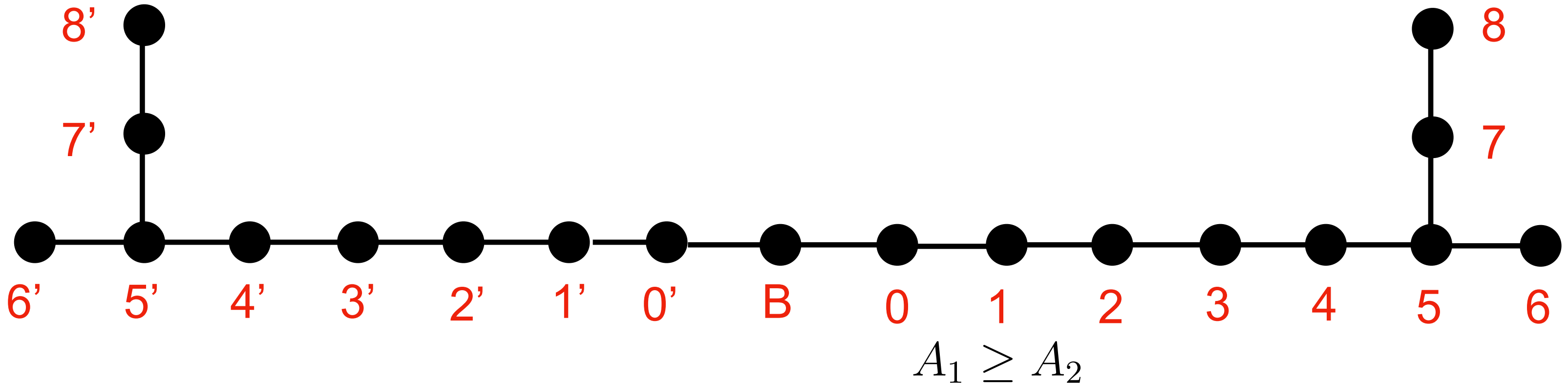

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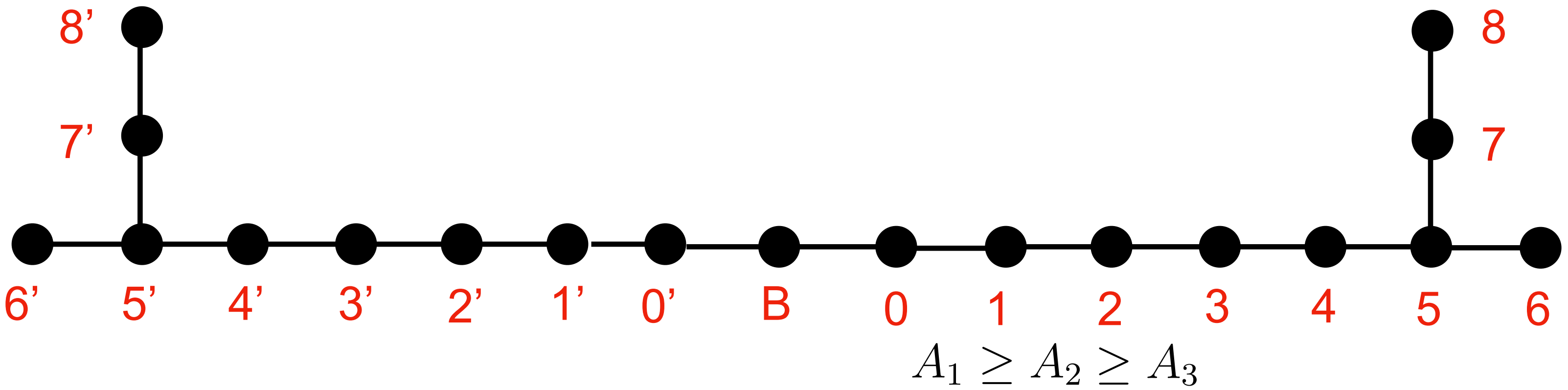

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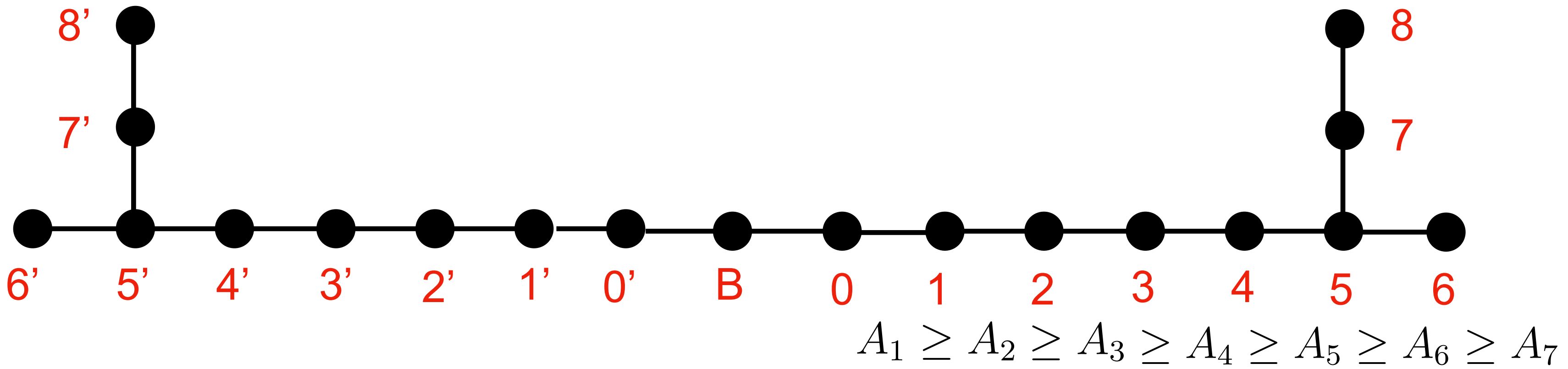

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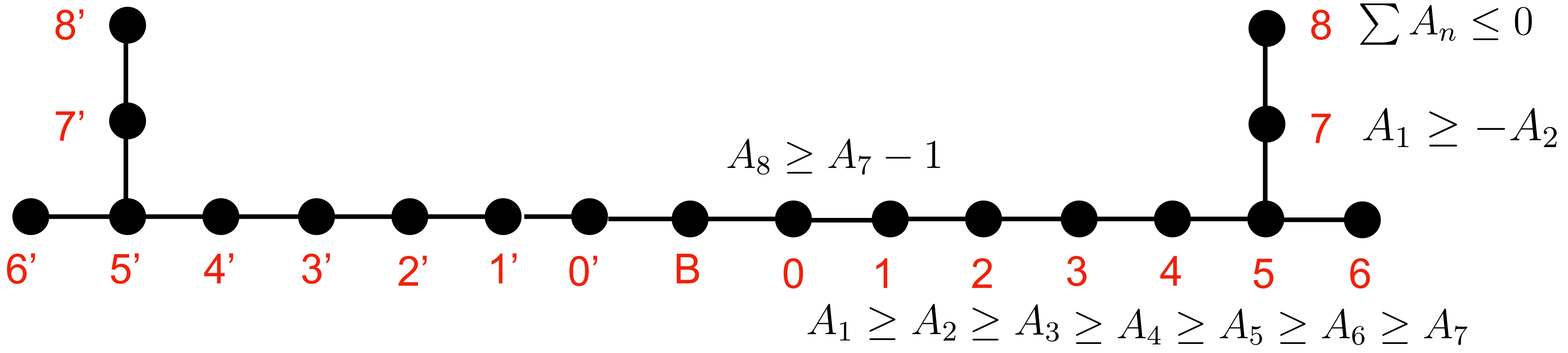

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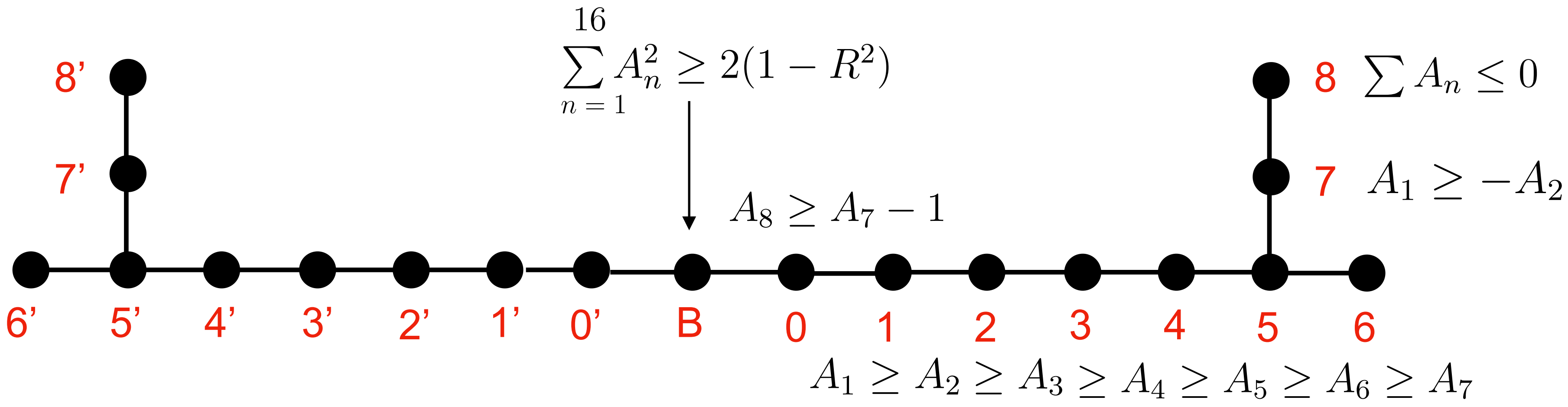

$$\sum_{n=1}^{16} A_n^2 \geq 2(1-R^2)$$

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$$\sum A_n \leq 0$$

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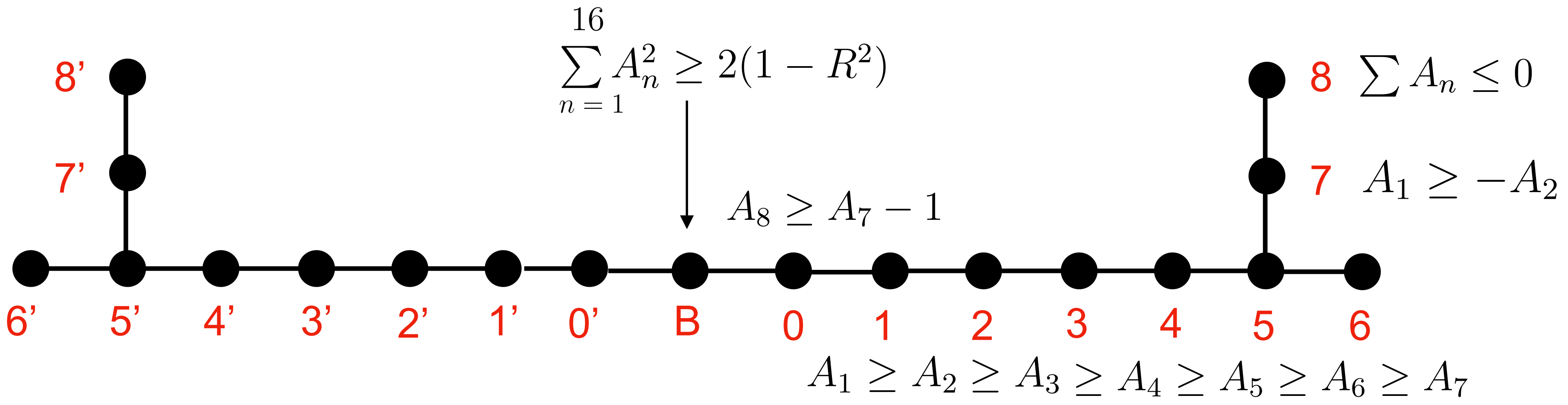

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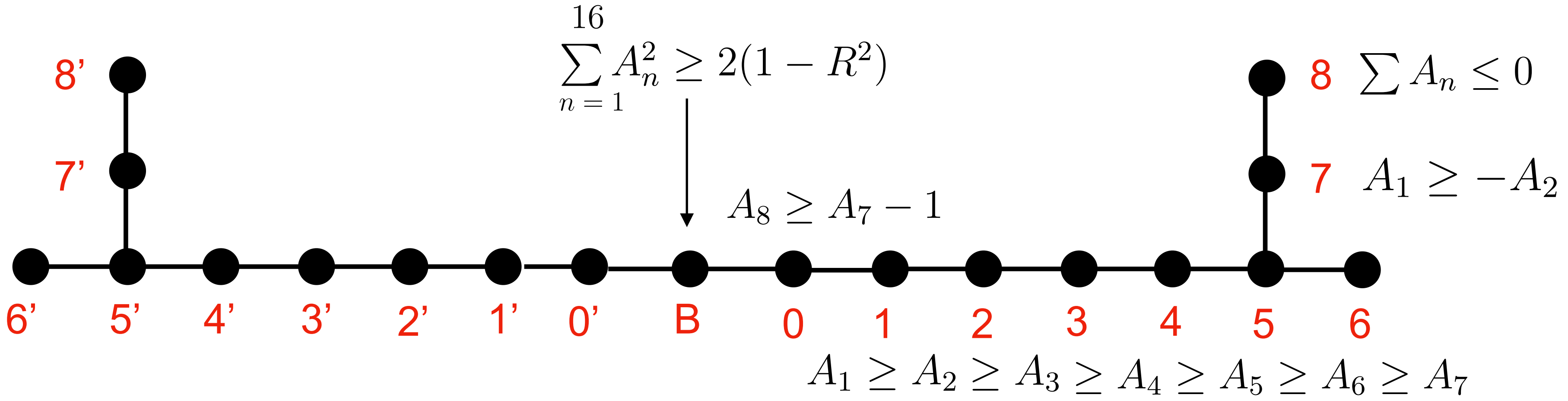

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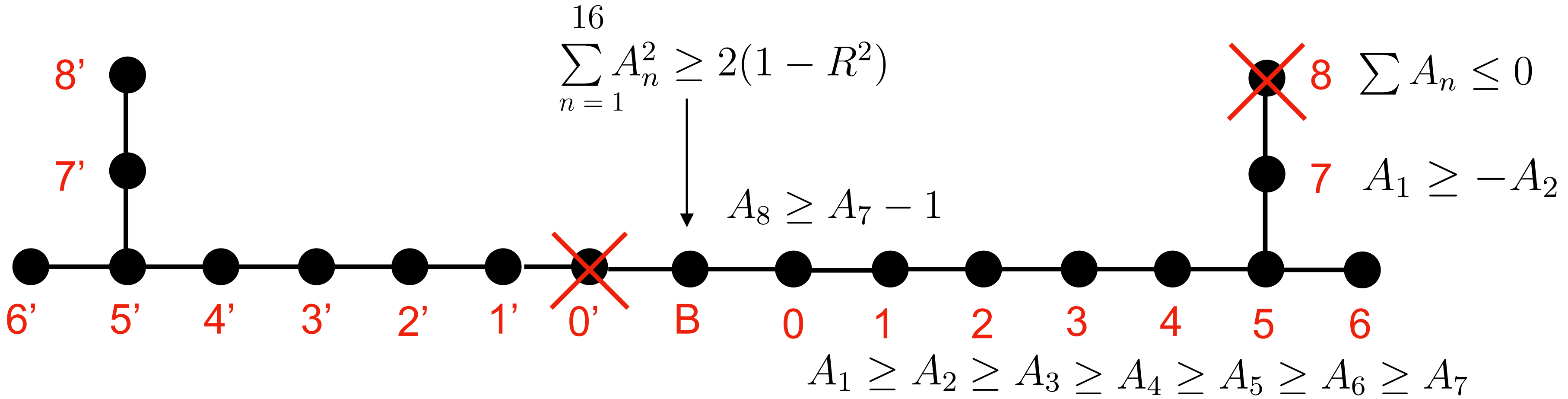

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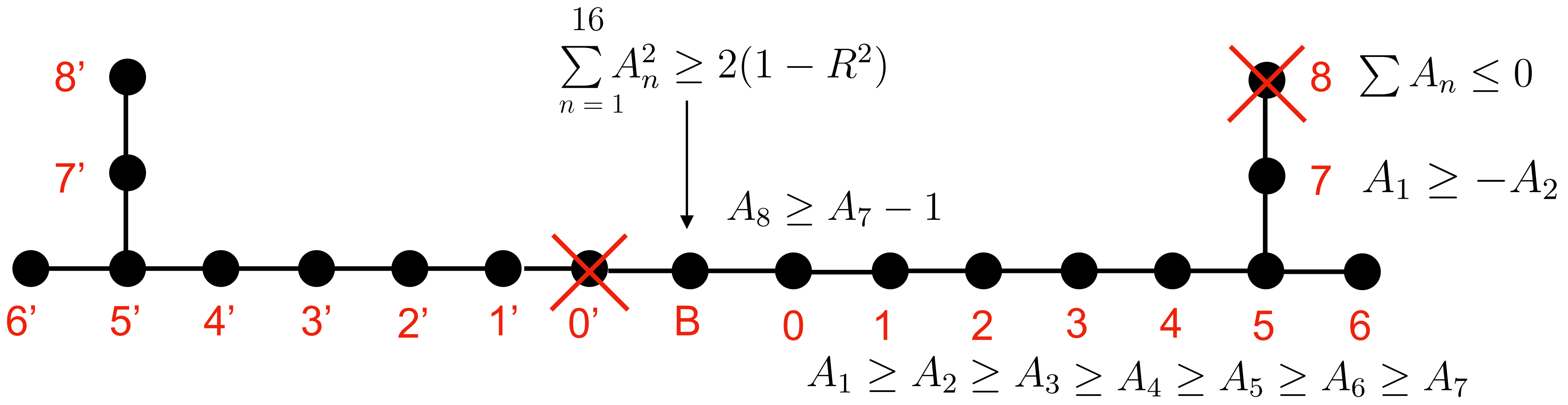

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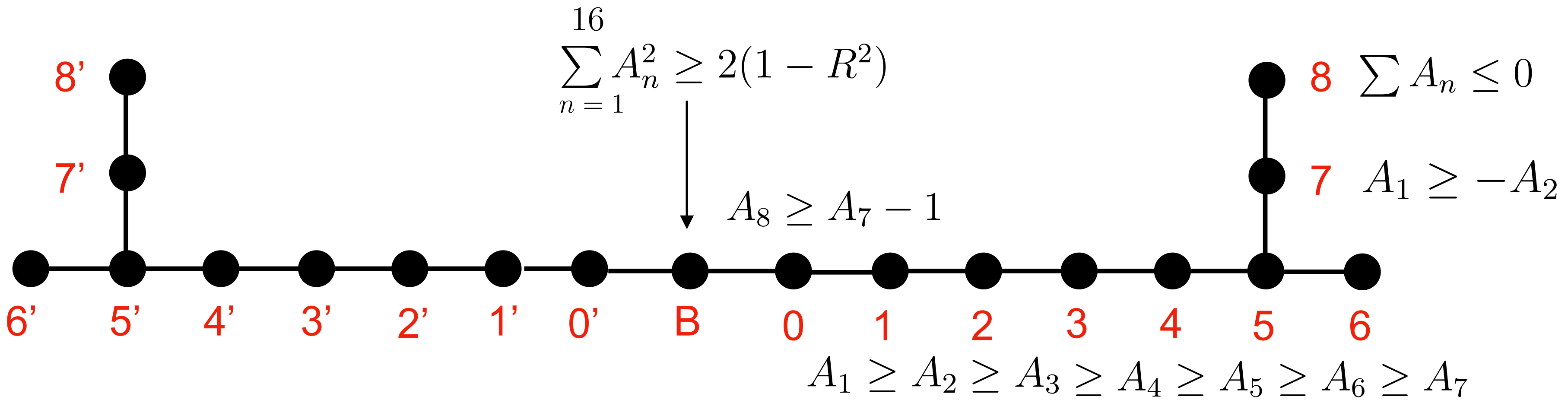

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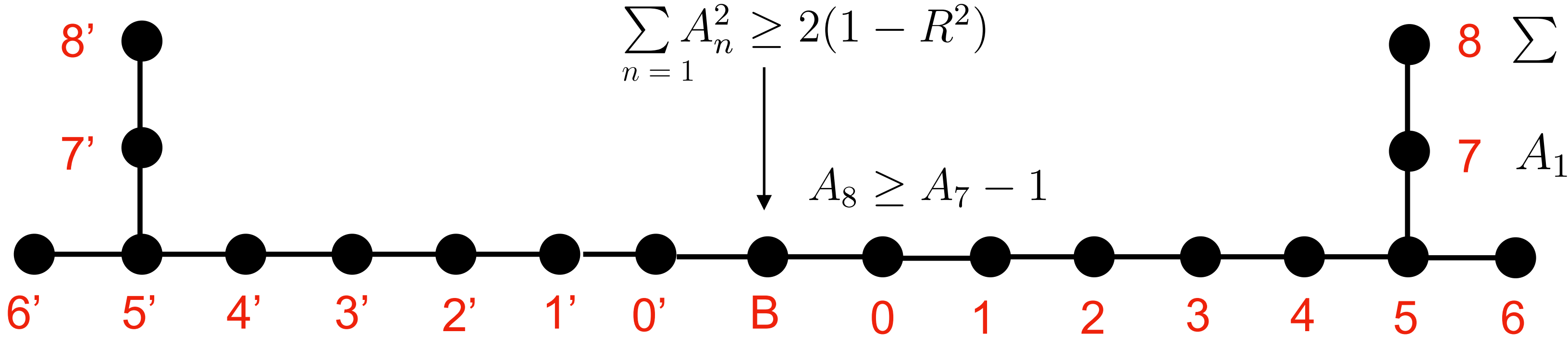

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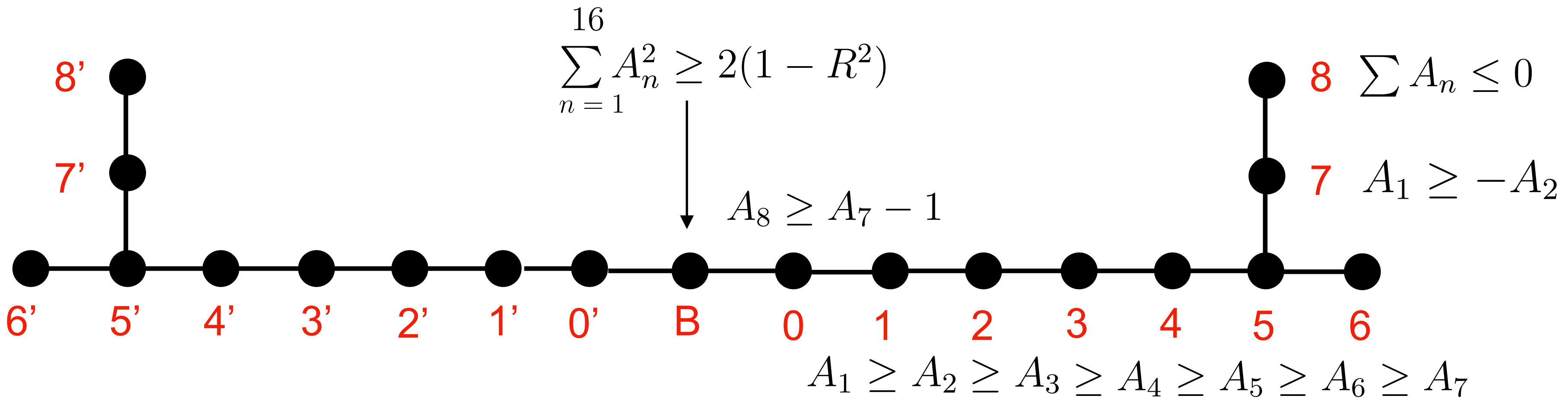

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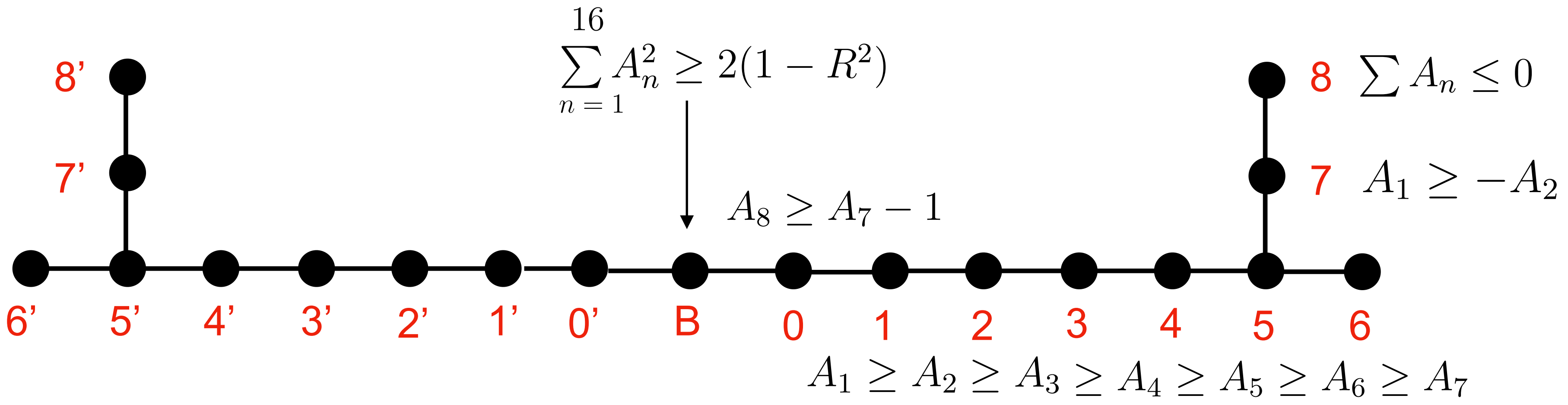

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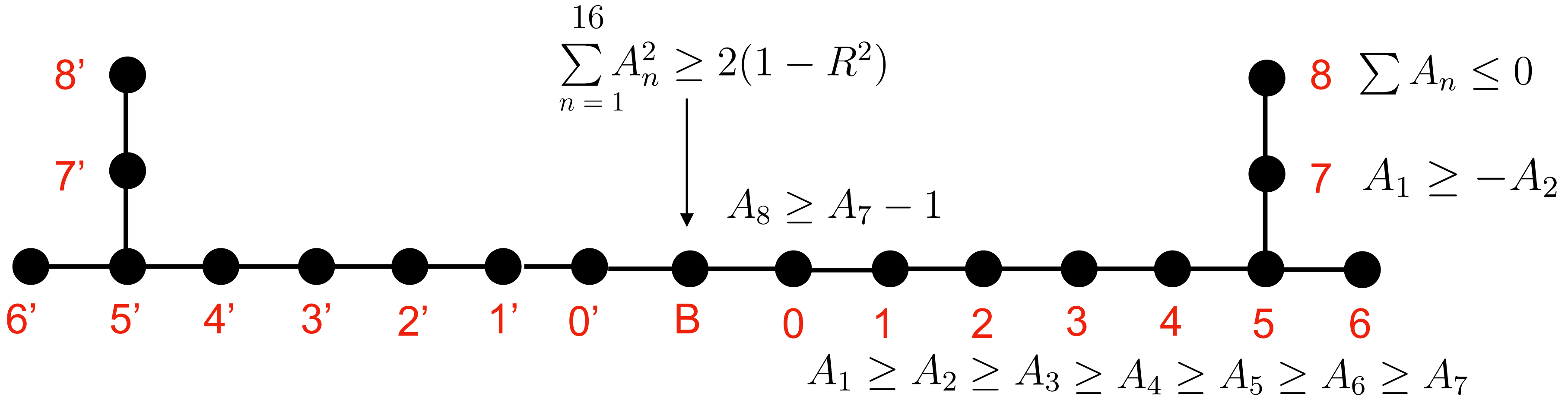

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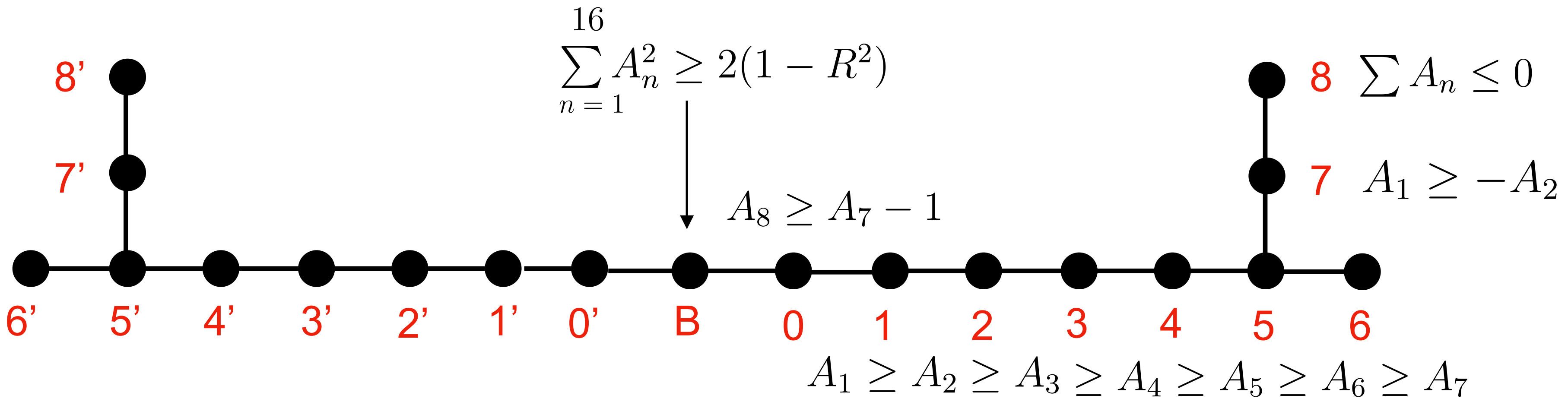
Goddard, Olive 85
Cachazo, Vafa 00



$\hat{E}_8 \times \hat{E}_8$

$$k = \frac{1}{\sqrt{2}}(1,1)$$

$$\in \Gamma^{1,1}$$

$$\tilde{k} = \frac{1}{\sqrt{2}}(1,-1)$$

$$\alpha_0 = k - \theta$$

$$\alpha_0' = \tilde{k} - \theta'$$

$$\alpha_B = -(k+\tilde{k})$$

- 19 roots: $|\alpha|^2 = 2$
- Associated to Weyl reflections in $\Gamma^{17,1}$

  $SO(17,1,\mathbb{Z})$ is generated by these 19 Weyl reflections
- Weyl group divides the moduli space into chambers
  Fundamental chamber: fundamental region in mod space
- Each root corresponds to a boundary in mod space
- Deleting 2 roots we get rank 17 maximal enhancement groups

  E.g.: $E_8 \times SO(18)$ $A = (0_7, -1, 0_8)$ $R^2 = 1/2$
- Possible groups: $E_{9-p} \times E_{9-q} \times SU(p+q)$ $SU(18)$

  $E_{9-p} \times SU(9+p)$ $SO(34)$

  $E_{9-p} \times SO(16+2p)$

Wilson line	R^{-2}	Gauge group
$\left(\left(\frac{q}{2(p+q)}\right)_p, 0_{8-p}, \left(\frac{q}{2(p+q)}\right)_q, \left(\frac{1}{2}\right)_{8-q}\right)$	$8\left(\frac{1}{p}+\frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(\left(\frac{q}{2(6+q)}\right)_7, -\frac{q}{2(6+q)}, \left(\frac{q}{2(6+q)}\right)_q, \left(\frac{1}{2}\right)_{8-q}\right)$	$2-\frac{2}{q+9}+\frac{8}{q}$	$SU(9+q) \times E_{9-q}$
$\left(\left(\frac{1}{4}\right)_7, -\frac{1}{4}, \left(\frac{1}{4}\right)_7, -\frac{1}{4}\right) + (0_{15}, 1)$	4	$SU(18)$
$\left(0_{8+q}, \left(\frac{1}{2}\right)_{8-q}\right)$	$\frac{8}{q}$	$SO(16+2q) \times E_{9-q}$
$(0_{15}, 1)$	2	$SO(34)$

Table 1: Maximal enhancements for the $SO(32)$ theory.

Wilson line	R^{2}	Gauge group
$\left(\left(\frac{1}{p}\right)_p, 0_{8-p}, \left(\frac{1}{q}\right)_q, 0_{8-q}\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{2}\left(\frac{1}{p}+\frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(\left(\frac{1}{6}\right)_7, -\frac{1}{6}, \left(\frac{1}{q}\right)_q, 0_{8-q}\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{2}\left(\frac{1}{9}+\frac{1}{q}\right)$	$SU(9+q) \times E_{9-q}$
$\left(\left(\frac{1}{6}\right)_7, -\frac{1}{6}, \left(-\frac{1}{6}\right)_7\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{9}$	$SU(18)$
$\left(0_8, \left(\frac{1}{q}\right)_q, 0_{8-q}\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{2q}$	$SO(16+2q) \times E_{9-q}$
$\left(0_8, \left(\frac{1}{6}\right)_7, -\frac{1}{6}\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{18}$	$SO(34)$

Table 2: Maximal enhancements for the $E_8 \times E_8$ theory.

Heterotic string on T^2

$$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}$$

$$(p_L, p_R) \in \Gamma^{18,2}$$

Heterotic string on T^2

$$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}$$

$$(p_L, p_R) \in \Gamma^{18,2}$$

- What are all the possible enhancement groups?

Heterotic string on T2

$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}$

$(p_L, p_R) \in \Gamma^{18,2}$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$

Heterotic string on T2

$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}$

$(p_L, p_R) \in \Gamma^{18,2}$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

Heterotic string on T²

$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}$ $(p_L, p_R) \in \Gamma^{18,2}$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

- Just play and find them by hand…

Heterotic string on T2

$$\mathcal{H} \in \underbrace{\frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}}_{\text{mod space of F-theory on K3}} \qquad (p_L, p_R) \in \Gamma^{18,2}$$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

- Just play and find them by hand…

or

- Heterotic on T2 ↔ F-theory on K3

Heterotic string on T^2

$$\mathcal{H} \in \underbrace{\frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}}_{\text{mod space of F-theory on K3}} \qquad\qquad (p_L, p_R) \in \Gamma^{18,2}$$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

- Just play and find them by hand…

or

- Heterotic on T2 ↔ F-theory on K3
- Search for all groups on K3 Chabrol 19

Heterotic string on T^2

$$\mathcal{H} \in \underbrace{\frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R})\times O(2,\mathbb{R})\times O(18,2,\mathbb{Z})}}_{\text{mod space of F-theory on K3}} \qquad (p_L, p_R) \in \Gamma^{18,2}$$

- What are all the possible enhancement groups?

  All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

- Just play and find them by hand…

or

- Heterotic on T2 ↔ F-theory on K3
- Search for all groups on K3 Chabrol 19
- Actually, have been all listed Shimada Zhang 00

No	Σ
1	$6\,A_3$
2	$2\,A_1 + 4\,A_4$
3	$2\,A_2 + 2\,A_3 + 2\,A_4$
4	$3\,A_1 + 3\,A_5$
5	$4\,A_2 + 2\,A_5$
6	$A_3 + 3\,A_5$
7	$2\,A_1 + 2\,A_3 + 2\,A_5$
8	$A_1 + 2\,A_2 + A_3 + 2\,A_5$
9	$2\,A_4 + 2\,A_5$
10	$2\,A_2 + A_4 + 2\,A_5$
11	$A_1 + A_3 + A_4 + 2\,A_5$
12	$A_1 + A_2 + 2\,A_3 + A_4 + A_5$
13	$3\,A_6$
14	$2\,A_1 + 2\,A_2 + 2\,A_6$
15	$2\,A_3 + 2\,A_6$
16	$A_2 + A_4 + 2\,A_6$
17	$2\,A_1 + A_2 + 2\,A_4 + A_6$
18	$A_1 + A_3 + 2\,A_4 + A_6$
19	$A_2 + 2\,A_3 + A_4 + A_6$
20	$A_1 + 2\,A_2 + A_3 + A_4 + A_6$
21	$2\,A_1 + 2\,A_5 + A_6$
22	$A_1 + 2\,A_3 + A_5 + A_6$
23	$A_1 + A_2 + A_4 + A_5 + A_6$
24	$A_3 + A_4 + A_5 + A_6$
25	$4\,A_1 + 2\,A_7$
26	$2\,A_2 + 2\,A_7$
27	$A_1 + A_3 + 2\,A_7$
28	$2\,A_1 + 3\,A_3 + A_7$
29	$A_2 + 3\,A_3 + A_7$
30	$2\,A_2 + A_3 + A_4 + A_7$
31	$2\,A_1 + A_2 + A_3 + A_4 + A_7$
32	$A_1 + 2\,A_5 + A_7$
33	$3\,A_1 + A_3 + A_5 + A_7$
34	$A_1 + A_2 + A_3 + A_5 + A_7$
35	$2\,A_1 + A_4 + A_5 + A_7$
36	$A_2 + A_4 + A_5 + A_7$
37	$A_1 + 2\,A_2 + A_6 + A_7$
38	$2\,A_1 + A_3 + A_6 + A_7$
39	$A_2 + A_3 + A_6 + A_7$
40	$A_1 + A_4 + A_6 + A_7$
41	$A_5 + A_6 + A_7$
42	$2\,A_1 + 2\,A_8$
43	$A_1 + 3\,A_2 + A_3 + A_8$
44	$2\,A_1 + 2\,A_4 + A_8$
45	$3\,A_2 + A_4 + A_8$
46	$A_1 + A_2 + A_3 + A_4 + A_8$
47	$A_1 + 2\,A_2 + A_5 + A_8$
48	$A_2 + A_3 + A_5 + A_8$
49	$A_1 + A_4 + A_5 + A_8$
50	$2\,A_1 + A_2 + A_6 + A_8$
51	$A_1 + A_3 + A_6 + A_8$
52	$A_4 + A_6 + A_8$
53	$A_1 + A_2 + A_7 + A_8$
54	$2\,A_9$
55	$A_1 + A_2 + 2\,A_3 + A_9$
56	$2\,A_1 + 2\,A_2 + A_3 + A_9$
57	$A_1 + 2\,A_4 + A_9$
58	$3\,A_1 + A_2 + A_4 + A_9$
59	$2\,A_1 + A_3 + A_4 + A_9$
60	$2\,A_1 + A_2 + A_5 + A_9$
61	$A_1 + A_3 + A_5 + A_9$
62	$A_4 + A_5 + A_9$
63	$3\,A_1 + A_6 + A_9$
64	$A_1 + A_2 + A_6 + A_9$
65	$A_3 + A_6 + A_9$
66	$A_2 + A_7 + A_9$
67	$A_1 + A_8 + A_9$
68	$A_2 + 2\,A_3 + A_{10}$
69	$A_1 + 2\,A_2 + A_3 + A_{10}$
70	$2\,A_4 + A_{10}$
71	$2\,A_2 + A_4 + A_{10}$
72	$2\,A_1 + A_2 + A_4 + A_{10}$
73	$A_1 + A_3 + A_4 + A_{10}$
74	$A_1 + A_2 + A_5 + A_{10}$
75	$A_3 + A_5 + A_{10}$
76	$2\,A_1 + A_6 + A_{10}$
77	$A_2 + A_6 + A_{10}$
78	$A_1 + A_7 + A_{10}$
79	$A_8 + A_{10}$
80	$A_1 + 3\,A_2 + A_{11}$
81	$3\,A_1 + 2\,A_2 + A_{11}$
82	$A_1 + 2\,A_3 + A_{11}$
83	$2\,A_2 + A_3 + A_{11}$
84	$2\,A_1 + A_2 + A_3 + A_{11}$
85	$3\,A_1 + A_4 + A_{11}$
86	$A_1 + A_2 + A_4 + A_{11}$
87	$2\,A_1 + A_5 + A_{11}$
88	$A_2 + A_5 + A_{11}$
89	$A_1 + A_6 + A_{11}$
90	$2\,A_1 + 2\,A_2 + A_{12}$
91	$A_1 + A_2 + A_3 + A_{12}$
92	$2\,A_1 + A_4 + A_{12}$
93	$A_2 + A_4 + A_{12}$
94	$A_1 + A_5 + A_{12}$
95	$A_6 + A_{12}$
96	$A_1 + 2\,A_2 + A_{13}$
97	$3\,A_1 + A_2 + A_{13}$
98	$2\,A_1 + A_3 + A_{13}$
99	$A_2 + A_3 + A_{13}$
100	$A_1 + A_4 + A_{13}$
101	$A_5 + A_{13}$
102	$2\,A_2 + A_{14}$
103	$2\,A_1 + A_2 + A_{14}$
104	$A_1 + A_3 + A_{14}$
105	$A_4 + A_{14}$
106	$3\,A_1 + A_{15}$
107	$A_1 + A_2 + A_{15}$
108	$A_3 + A_{15}$
109	$2\,A_1 + A_{16}$
110	$A_2 + A_{16}$
111	$A_1 + A_{17}$
112	A_{18}
113	$2\,A_4 + 2\,D_5$
114	$A_3 + 2\,A_5 + D_5$
115	$2\,A_4 + A_5 + D_5$
116	$A_1 + A_3 + A_4 + A_5 + D_5$
117	$A_1 + 2\,A_6 + D_5$
118	$2\,A_2 + A_3 + A_6 + D_5$
119	$A_1 + A_2 + A_4 + A_6 + D_5$
120	$A_2 + A_5 + A_6 + D_5$
121	$A_1 + A_7 + 2\,D_5$
122	$A_1 + A_2 + A_3 + A_7 + D_5$
123	$2\,A_1 + A_4 + A_7 + D_5$
124	$A_8 + 2\,D_5$
125	$A_1 + A_4 + A_8 + D_5$
126	$A_5 + A_8 + D_5$
127	$2\,A_2 + A_9 + D_5$
128	$2\,A_1 + A_2 + A_9 + D_5$
129	$A_1 + A_3 + A_9 + D_5$
130	$A_4 + A_9 + D_5$
131	$A_1 + A_2 + A_{10} + D_5$
132	$2\,A_1 + A_{11} + D_5$
133	$A_2 + A_{11} + D_5$
134	$A_1 + A_{12} + D_5$
135	$A_{13} + D_5$
136	$3\,D_6$
137	$2\,A_3 + 2\,D_6$
138	$2\,A_2 + 2\,A_4 + D_6$
139	$2\,A_1 + 2\,A_5 + D_6$
140	$A_1 + 2\,A_3 + A_5 + D_6$
141	$A_3 + A_4 + A_5 + D_6$
142	$2\,A_6 + D_6$
143	$A_2 + A_4 + A_6 + D_6$
144	$A_1 + 2\,A_2 + A_7 + D_6$
145	$A_2 + A_3 + A_7 + D_6$
146	$A_1 + A_4 + A_7 + D_6$
147	$A_4 + A_8 + D_6$
148	$A_1 + A_2 + A_9 + D_6$
149	$A_3 + A_9 + D_6$
150	$A_2 + A_{10} + D_6$
151	$A_1 + A_{11} + D_6$

No	Σ
152	$A_{12} + D_6$
153	$A_2 + A_5 + D_5 + D_6$
154	$A_7 + D_5 + D_6$
155	$2 A_2 + 2 D_7$
156	$A_2 + 3 A_3 + D_7$
157	$A_1 + A_2 + 2 A_4 + D_7$
158	$A_2 + A_3 + A_6 + D_7$
159	$A_1 + A_4 + A_6 + D_7$
160	$A_5 + A_6 + D_7$
161	$2 A_1 + A_2 + A_7 + D_7$
162	$A_1 + A_3 + A_7 + D_7$
163	$2 A_1 + A_9 + D_7$
164	$A_2 + A_9 + D_7$
165	$A_1 + A_{10} + D_7$
166	$A_{11} + D_7$
167	$A_1 + A_5 + D_5 + D_7$
168	$A_5 + D_6 + D_7$
169	$2 A_1 + 2 D_8$
170	$2 A_2 + 2 A_3 + D_8$
171	$2 A_5 + D_8$
172	$2 A_1 + A_3 + A_5 + D_8$
173	$A_1 + A_4 + A_5 + D_8$
174	$2 A_2 + A_6 + D_8$
175	$A_1 + A_2 + A_7 + D_8$
176	$A_1 + A_9 + D_8$
177	$2 D_5 + D_8$
178	$A_1 + A_3 + D_6 + D_8$
179	$2 D_9$
180	$A_1 + 2 A_2 + A_4 + D_9$
181	$A_1 + A_3 + A_5 + D_9$
182	$A_4 + A_5 + D_9$
183	$A_1 + A_2 + A_6 + D_9$
184	$2 A_1 + A_7 + D_9$
185	$A_1 + A_8 + D_9$
186	$A_9 + D_9$
187	$A_4 + D_5 + D_9$

No	Σ
188	$2 A_1 + 2 A_3 + D_{10}$
189	$2 A_4 + D_{10}$
190	$A_1 + A_3 + A_4 + D_{10}$
191	$3 A_1 + A_5 + D_{10}$
192	$A_3 + A_5 + D_{10}$
193	$A_2 + A_6 + D_{10}$
194	$A_8 + D_{10}$
195	$A_1 + A_2 + D_5 + D_{10}$
196	$A_2 + D_6 + D_{10}$
197	$A_1 + D_7 + D_{10}$
198	$2 A_2 + A_3 + D_{11}$
199	$A_1 + A_2 + A_4 + D_{11}$
200	$A_2 + A_5 + D_{11}$
201	$A_1 + A_6 + D_{11}$
202	$2 A_1 + 2 A_2 + D_{12}$
203	$A_1 + A_2 + A_3 + D_{12}$
204	$2 A_1 + A_4 + D_{12}$
205	$A_1 + D_5 + D_{12}$
206	$D_6 + D_{12}$
207	$A_1 + A_4 + D_{13}$
208	$A_5 + D_{13}$
209	$D_5 + D_{13}$
210	$2 A_2 + D_{14}$
211	$2 A_1 + A_2 + D_{14}$
212	$A_1 + A_3 + D_{14}$
213	$A_4 + D_{14}$
214	$A_1 + A_2 + D_{15}$
215	$2 A_1 + D_{16}$
216	$A_2 + D_{16}$
217	$A_1 + D_{17}$
218	D_{18}
219	$3 E_6$
220	$2 A_3 + 2 E_6$
221	$A_1 + A_3 + 2 A_4 + E_6$
222	$A_1 + A_5 + 2 E_6$
223	$A_2 + 2 A_5 + E_6$

No	Σ
224	$2 A_2 + A_3 + A_5 + E_6$
225	$A_3 + A_4 + A_5 + E_6$
226	$A_6 + 2 E_6$
227	$A_1 + A_2 + A_3 + A_6 + E_6$
228	$2 A_1 + A_4 + A_6 + E_6$
229	$A_2 + A_4 + A_6 + E_6$
230	$A_1 + A_5 + A_6 + E_6$
231	$A_1 + A_4 + A_7 + E_6$
232	$A_5 + A_7 + E_6$
233	$2 A_2 + A_8 + E_6$
234	$2 A_1 + A_2 + A_8 + E_6$
235	$A_1 + A_3 + A_8 + E_6$
236	$A_4 + A_8 + E_6$
237	$A_1 + A_2 + A_9 + E_6$
238	$A_3 + A_9 + E_6$
239	$2 A_1 + A_{10} + E_6$
240	$A_2 + A_{10} + E_6$
241	$A_1 + A_{11} + E_6$
242	$A_{12} + E_6$
243	$A_3 + A_4 + D_5 + E_6$
244	$A_1 + A_6 + D_5 + E_6$
245	$A_7 + D_5 + E_6$
246	$D_6 + 2 E_6$
247	$A_2 + A_4 + D_6 + E_6$
248	$A_6 + D_6 + E_6$
249	$A_1 + A_4 + D_7 + E_6$
250	$D_5 + D_7 + E_6$
251	$A_4 + D_8 + E_6$
252	$A_1 + A_2 + D_9 + E_6$
253	$A_3 + D_9 + E_6$
254	$A_1 + D_{11} + E_6$
255	$D_{12} + E_6$
256	$2 A_2 + 2 E_7$
257	$A_1 + A_3 + 2 E_7$

No	Σ
258	$A_4 + 2 E_7$
259	$A_1 + 2 A_3 + A_4 + E_7$
260	$2 A_2 + A_3 + A_4 + E_7$
261	$2 A_3 + A_5 + E_7$
262	$A_1 + A_2 + A_3 + A_5 + E_7$
263	$2 A_1 + A_4 + A_5 + E_7$
264	$A_2 + A_4 + A_5 + E_7$
265	$A_1 + 2 A_2 + A_6 + E_7$
266	$A_2 + A_3 + A_6 + E_7$
267	$A_1 + A_4 + A_6 + E_7$
268	$A_5 + A_6 + E_7$
269	$2 A_2 + A_7 + E_7$
270	$2 A_1 + A_2 + A_7 + E_7$
271	$A_1 + A_3 + A_7 + E_7$
272	$A_4 + A_7 + E_7$
273	$A_1 + A_2 + A_8 + E_7$
274	$A_3 + A_8 + E_7$
275	$2 A_1 + A_9 + E_7$
276	$A_2 + A_9 + E_7$
277	$A_1 + A_{10} + E_7$
278	$A_{11} + E_7$
279	$D_4 + 2 E_7$
280	$A_2 + A_4 + D_5 + E_7$
281	$A_1 + A_5 + D_5 + E_7$
282	$A_6 + D_5 + E_7$
283	$A_2 + A_3 + D_6 + E_7$
284	$A_5 + D_6 + E_7$
285	$D_5 + D_6 + E_7$
286	$A_1 + A_3 + D_7 + E_7$
287	$A_4 + D_7 + E_7$
288	$A_1 + A_2 + D_8 + E_7$
289	$A_2 + D_9 + E_7$
290	$A_1 + D_{10} + E_7$

No	Σ
291	$D_{11} + E_7$
292	$A_2 + A_3 + E_6 + E_7$
293	$A_1 + A_4 + E_6 + E_7$
294	$A_5 + E_6 + E_7$
295	$D_5 + E_6 + E_7$
296	$2 A_1 + 2 E_8$
297	$A_2 + 2 E_8$
298	$2 A_2 + 2 A_3 + E_8$
299	$2 A_1 + 2 A_4 + E_8$
300	$A_1 + A_2 + A_3 + A_4 + E_8$
301	$2 A_5 + E_8$
302	$A_2 + A_3 + A_5 + E_8$
303	$A_1 + A_4 + A_5 + E_8$
304	$2 A_2 + A_6 + E_8$
305	$2 A_1 + A_2 + A_6 + E_8$
306	$A_1 + A_3 + A_6 + E_8$
307	$A_4 + A_6 + E_8$
308	$A_1 + A_2 + A_7 + E_8$
309	$2 A_1 + A_8 + E_8$
310	$A_2 + A_8 + E_8$
311	$A_1 + A_9 + E_8$
312	$A_{10} + E_8$
313	$2 D_5 + E_8$
314	$A_1 + A_4 + D_5 + E_8$
315	$A_5 + D_5 + E_8$
316	$2 A_2 + D_6 + E_8$
317	$A_4 + D_6 + E_8$
318	$A_1 + A_2 + D_7 + E_8$
319	$A_1 + D_9 + E_8$
320	$D_{10} + E_8$
321	$A_1 + A_3 + E_6 + E_8$
322	$A_4 + E_6 + E_8$
323	$D_4 + E_6 + E_8$
324	$A_1 + A_2 + E_7 + E_8$
325	$A_3 + E_7 + E_8$

No	Σ
152	$A_{12}+D_6$
153	$A_2+A_5+D_5+D_6$
154	$A_7+D_5+D_6$
155	$2\,A_2+2\,D_7$
156	$A_2+3\,A_3+D_7$
157	$A_1+A_2+2\,A_4+D_7$
158	$A_2+A_3+A_6+D_7$
159	$A_1+A_4+A_6+D_7$
160	$A_5+A_6+D_7$
161	$2\,A_1+A_2+A_7+D_7$
162	$A_1+A_3+A_7+D_7$
163	$2\,A_1+A_9+D_7$
164	$A_2+A_9+D_7$
165	$A_1+A_{10}+D_7$
166	$A_{11}+D_7$
167	$A_1+A_5+D_5+D_7$
168	$A_5+D_6+D_7$
169	$2\,A_1+2\,D_8$
170	$2\,A_2+2\,A_3+D_8$
171	$2\,A_5+D_8$
172	$2\,A_1+A_3+A_5+D_8$
173	$A_1+A_4+A_5+D_8$
174	$2\,A_2+A_6+D_8$
175	$A_1+A_2+A_7+D_8$
176	$A_1+A_9+D_8$
177	$2\,D_5+D_8$
178	$A_1+A_3+D_6+D_8$
179	$2\,D_9$
180	$A_1+2\,A_2+A_4+D_9$
181	$A_1+A_3+A_5+D_9$
182	$A_4+A_5+D_9$
183	$A_1+A_2+A_6+D_9$
184	$2\,A_1+A_7+D_9$
185	$A_1+A_8+D_9$
186	A_9+D_9
187	$A_4+D_5+D_9$
188	$2\,A_1+2\,A_3+D_{10}$
189	$2\,A_4+D_{10}$
190	$A_1+A_3+A_4+D_{10}$
191	$3\,A_1+A_5+D_{10}$
192	$A_3+A_5+D_{10}$
193	$A_2+A_6+D_{10}$
194	A_8+D_{10}
195	$A_1+A_2+D_5+D_{10}$
196	$A_2+D_6+D_{10}$
197	$A_1+D_7+D_{10}$
198	$2\,A_2+A_3+D_{11}$
199	$A_1+A_2+A_4+D_{11}$
200	$A_2+A_5+D_{11}$
201	$A_1+A_6+D_{11}$
202	$2\,A_1+2\,A_2+D_{12}$
203	$A_1+A_2+A_3+D_{12}$
204	$2\,A_1+A_4+D_{12}$
205	$A_1+D_5+D_{12}$
206	D_6+D_{12}
207	$A_1+A_4+D_{13}$
208	A_5+D_{13}
209	D_5+D_{13}
210	$2\,A_2+D_{14}$
211	$2\,A_1+A_2+D_{14}$
212	$A_1+A_3+D_{14}$
213	A_4+D_{14}
214	$A_1+A_2+D_{15}$
215	$2\,A_1+D_{16}$
216	A_2+D_{16}
217	A_1+D_{17}
218	D_{18}
219	$3\,E_6$
220	$2\,A_3+2\,E_6$
221	$A_1+A_3+2\,A_4+E_6$
222	$A_1+A_5+2\,E_6$
223	$A_2+2\,A_5+E_6$
224	$2\,A_2+A_3+A_5+E_6$
225	$A_3+A_4+A_5+E_6$
226	$A_6+2\,E_6$
227	$A_1+A_2+A_3+A_6+E_6$
228	$2\,A_1+A_4+A_6+E_6$
229	$A_2+A_4+A_6+E_6$
230	$A_1+A_5+A_6+E_6$
231	$A_1+A_4+A_7+E_6$
232	$A_5+A_7+E_6$
233	$2\,A_2+A_8+E_6$
234	$2\,A_1+A_2+A_8+E_6$
235	$A_1+A_3+A_8+E_6$
236	$A_4+A_8+E_6$
237	$A_1+A_2+A_9+E_6$
238	$A_3+A_9+E_6$
239	$2\,A_1+A_{10}+E_6$
240	$A_2+A_{10}+E_6$
241	$A_1+A_{11}+E_6$
242	$A_{12}+E_6$
243	$A_3+A_4+D_5+E_6$
244	$A_1+A_6+D_5+E_6$
245	$A_7+D_5+E_6$
246	$D_6+2\,E_6$
247	$A_2+A_4+D_6+E_6$
248	$A_6+D_6+E_6$
249	$A_1+A_4+D_7+E_6$
250	$D_5+D_7+E_6$
251	$A_4+D_8+E_6$
252	$A_1+A_2+D_9+E_6$
253	$A_3+D_9+E_6$
254	$A_1+D_{11}+E_6$
255	$D_{12}+E_6$
256	$2\,A_2+2\,E_7$
257	$A_1+A_3+2\,E_7$
258	$A_4+2\,E_7$
259	$A_1+2\,A_3+A_4+E_7$
260	$2\,A_2+A_3+A_4+E_7$
261	$2\,A_3+A_5+E_7$
262	$A_1+A_2+A_3+A_5+E_7$
263	$2\,A_1+A_4+A_5+E_7$
264	$A_2+A_4+A_5+E_7$
265	$A_1+2\,A_2+A_6+E_7$
266	$A_2+A_3+A_6+E_7$
267	$A_1+A_4+A_6+E_7$
268	$A_5+A_6+E_7$
269	$2\,A_2+A_7+E_7$
270	$2\,A_1+A_2+A_7+E_7$
271	$A_1+A_3+A_7+E_7$
272	$A_4+A_7+E_7$
273	$A_1+A_2+A_8+E_7$
274	$A_3+A_8+E_7$
275	$2\,A_1+A_9+E_7$
276	$A_2+A_9+E_7$
277	$A_1+A_{10}+E_7$
278	$A_{11}+E_7$
279	$D_4+2\,E_7$
280	$A_2+A_4+D_5+E_7$
281	$A_1+A_5+D_5+E_7$
282	$A_6+D_5+E_7$
283	$A_2+A_3+D_6+E_7$
284	$A_5+D_6+E_7$
285	$D_5+D_6+E_7$
286	$A_1+A_3+D_7+E_7$
287	$A_4+D_7+E_7$
288	$A_1+A_2+D_8+E_7$
289	$A_2+D_9+E_7$
290	$A_1+D_{10}+E_7$
291	$D_{11}+E_7$
292	$A_2+A_3+E_6+E_7$
293	$A_1+A_4+E_6+E_7$
294	$A_5+E_6+E_7$
295	$D_5+E_6+E_7$
296	$2\,A_1+2\,E_8$ A=0
297	$A_2+2\,E_8$
298	$2\,A_2+2\,A_3+E_8$
299	$2\,A_1+2\,A_4+E_8$
300	$A_1+A_2+A_3+A_4+E_8$
301	$2\,A_5+E_8$
302	$A_2+A_3+A_5+E_8$
303	$A_1+A_4+A_5+E_8$
304	$2\,A_2+A_6+E_8$
305	$2\,A_1+A_2+A_6+E_8$
306	$A_1+A_3+A_6+E_8$
307	$A_4+A_6+E_8$
308	$A_1+A_2+A_7+E_8$
309	$2\,A_1+A_8+E_8$
310	$A_2+A_8+E_8$
311	$A_1+A_9+E_8$
312	$A_{10}+E_8$
313	$2\,D_5+E_8$
314	$A_1+A_4+D_5+E_8$
315	$A_5+D_5+E_8$
316	$2\,A_2+D_6+E_8$
317	$A_4+D_6+E_8$
318	$A_1+A_2+D_7+E_8$
319	$A_1+D_9+E_8$
320	$D_{10}+E_8$
321	$A_1+A_3+E_6+E_8$
322	$A_4+E_6+E_8$
323	$D_4+E_6+E_8$
324	$A_1+A_2+E_7+E_8$
325	$A_3+E_7+E_8$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

No	Σ
152	$A_{12} + D_6$
153	$A_2 + A_5 + D_5 + D_6$
154	$A_7 + D_5 + D_6$
155	$2\,A_2 + 2\,D_7$
156	$A_2 + 3\,A_3 + D_7$
157	$A_1 + A_2 + 2\,A_4 + D_7$
158	$A_2 + A_3 + A_6 + D_7$
159	$A_1 + A_4 + A_6 + D_7$
160	$A_5 + A_6 + D_7$
161	$2\,A_1 + A_2 + A_7 + D_7$
162	$A_1 + A_3 + A_7 + D_7$
163	$2\,A_1 + A_9 + D_7$
164	$A_2 + A_9 + D_7$
165	$A_1 + A_{10} + D_7$
166	$A_{11} + D_7$
167	$A_1 + A_5 + D_5 + D_7$
168	$A_5 + D_6 + D_7$
169	$2\,A_1 + 2\,D_8$
170	$2\,A_2 + 2\,A_3 + D_8$
171	$2\,A_5 + D_8$
172	$2\,A_1 + A_3 + A_5 + D_8$
173	$A_1 + A_4 + A_5 + D_8$
174	$2\,A_2 + A_6 + D_8$
175	$A_1 + A_2 + A_7 + D_8$
176	$A_1 + A_9 + D_8$
177	$2\,D_5 + D_8$
178	$A_1 + A_3 + D_6 + D_8$
179	$2\,D_9$
180	$A_1 + 2\,A_2 + A_4 + D_9$
181	$A_1 + A_3 + A_5 + D_9$
182	$A_4 + A_5 + D_9$
183	$A_1 + A_2 + A_6 + D_9$
184	$2\,A_1 + A_7 + D_9$
185	$A_1 + A_8 + D_9$
186	$A_9 + D_9$
187	$A_4 + D_5 + D_9$

No	Σ
188	$2\,A_1 + 2\,A_3 + D_{10}$
189	$2\,A_4 + D_{10}$
190	$A_1 + A_3 + A_4 + D_{10}$
191	$3\,A_1 + A_5 + D_{10}$
192	$A_3 + A_5 + D_{10}$
193	$A_2 + A_6 + D_{10}$
194	$A_8 + D_{10}$
195	$A_1 + A_2 + D_5 + D_{10}$
196	$A_2 + D_6 + D_{10}$
197	$A_1 + D_7 + D_{10}$
198	$2\,A_2 + A_3 + D_{11}$
199	$A_1 + A_2 + A_4 + D_{11}$
200	$A_2 + A_5 + D_{11}$
201	$A_1 + A_6 + D_{11}$
202	$2\,A_1 + 2\,A_2 + D_{12}$
203	$A_1 + A_2 + A_3 + D_{12}$
204	$2\,A_1 + A_4 + D_{12}$
205	$A_1 + D_5 + D_{12}$
206	$D_6 + D_{12}$
207	$A_1 + A_4 + D_{13}$
208	$A_5 + D_{13}$
209	$D_5 + D_{13}$
210	$2\,A_2 + D_{14}$
211	$2\,A_1 + A_2 + D_{14}$
212	$A_1 + A_3 + D_{14}$
213	$A_4 + D_{14}$
214	$A_1 + A_2 + D_{15}$
215	$2\,A_1 + D_{16}$
216	$A_2 + D_{16}$
217	$A_1 + D_{17}$
218	D_{18}
219	$3\,E_6$
220	$2\,A_3 + 2\,E_6$
221	$A_1 + A_3 + 2\,A_4 + E_6$
222	$A_1 + A_5 + 2\,E_6$
223	$A_2 + 2\,A_5 + E_6$

No	Σ
224	$2\,A_2 + A_3 + A_5 + E_6$
225	$A_3 + A_4 + A_5 + E_6$
226	$A_6 + 2\,E_6$
227	$A_1 + A_2 + A_3 + A_6 + E_6$
228	$2\,A_1 + A_4 + A_6 + E_6$
229	$A_2 + A_4 + A_6 + E_6$
230	$A_1 + A_5 + A_6 + E_6$
231	$A_1 + A_4 + A_7 + E_6$
232	$A_5 + A_7 + E_6$
233	$2\,A_2 + A_8 + E_6$
234	$2\,A_1 + A_2 + A_8 + E_6$
235	$A_1 + A_3 + A_8 + E_6$
236	$A_4 + A_8 + E_6$
237	$A_1 + A_2 + A_9 + E_6$
238	$A_3 + A_9 + E_6$
239	$2\,A_1 + A_{10} + E_6$
240	$A_2 + A_{10} + E_6$
241	$A_1 + A_{11} + E_6$
242	$A_{12} + E_6$
243	$A_3 + A_4 + D_5 + E_6$
244	$A_1 + A_6 + D_5 + E_6$
245	$A_7 + D_5 + E_6$
246	$D_6 + 2\,E_6$
247	$A_2 + A_4 + D_6 + E_6$
248	$A_6 + D_6 + E_6$
249	$A_1 + A_4 + D_7 + E_6$
250	$D_5 + D_7 + E_6$
251	$A_4 + D_8 + E_6$
252	$A_1 + A_2 + D_9 + E_6$
253	$A_3 + D_9 + E_6$
254	$A_1 + D_{11} + E_6$
255	$D_{12} + E_6$
256	$2\,A_2 + 2\,E_7$
257	$A_1 + A_3 + 2\,E_7$

No	Σ
258	$A_4 + 2\,E_7$
259	$A_1 + 2\,A_3 + A_4 + E_7$
260	$2\,A_2 + A_3 + A_4 + E_7$
261	$2\,A_3 + A_5 + E_7$
262	$A_1 + A_2 + A_3 + A_5 + E_7$
263	$2\,A_1 + A_4 + A_5 + E_7$
264	$A_2 + A_4 + A_5 + E_7$
265	$A_1 + 2\,A_2 + A_6 + E_7$
266	$A_2 + A_3 + A_6 + E_7$
267	$A_1 + A_4 + A_6 + E_7$
268	$A_5 + A_6 + E_7$
269	$2\,A_2 + A_7 + E_7$
270	$2\,A_1 + A_2 + A_7 + E_7$
271	$A_1 + A_3 + A_7 + E_7$
272	$A_4 + A_7 + E_7$
273	$A_1 + A_2 + A_8 + E_7$
274	$A_3 + A_8 + E_7$
275	$2\,A_1 + A_9 + E_7$
276	$A_2 + A_9 + E_7$
277	$A_1 + A_{10} + E_7$
278	$A_{11} + E_7$
279	$D_4 + 2\,E_7$
280	$A_2 + A_4 + D_5 + E_7$
281	$A_1 + A_5 + D_5 + E_7$
282	$A_6 + D_5 + E_7$
283	$A_2 + A_3 + D_6 + E_7$
284	$A_5 + D_6 + E_7$
285	$D_5 + D_6 + E_7$
286	$A_1 + A_3 + D_7 + E_7$
287	$A_4 + D_7 + E_7$
288	$A_1 + A_2 + D_8 + E_7$
289	$A_2 + D_9 + E_7$
290	$A_1 + D_{10} + E_7$

No	Σ
291	$D_{11} + E_7$
292	$A_2 + A_3 + E_6 + E_7$
293	$A_1 + A_4 + E_6 + E_7$
294	$A_5 + E_6 + E_7$
295	$D_5 + E_6 + E_7$
296	$2\,A_1 + 2\,E_8$
297	$A_2 + 2\,E_8$
298	$2\,A_2 + 2\,A_3 + E_8$
299	$2\,A_1 + 2\,A_4 + E_8$
300	$A_1 + A_2 + A_3 + A_4 + E_8$
301	$2\,A_5 + E_8$
302	$A_2 + A_3 + A_5 + E_8$
303	$A_1 + A_4 + A_5 + E_8$
304	$2\,A_2 + A_6 + E_8$
305	$2\,A_1 + A_2 + A_6 + E_8$
306	$A_1 + A_3 + A_6 + E_8$
307	$A_4 + A_6 + E_8$
308	$A_1 + A_2 + A_7 + E_8$
309	$2\,A_1 + A_8 + E_8$
310	$A_2 + A_8 + E_8$
311	$A_1 + A_9 + E_8$
312	$A_{10} + E_8$
313	$2\,D_5 + E_8$
314	$A_1 + A_4 + D_5 + E_8$
315	$A_5 + D_5 + E_8$
316	$2\,A_2 + D_6 + E_8$
317	$A_4 + D_6 + E_8$
318	$A_1 + A_2 + D_7 + E_8$
319	$A_1 + D_9 + E_8$
320	$D_{10} + E_8$
321	$A_1 + A_3 + E_6 + E_8$
322	$A_4 + E_6 + E_8$
323	$D_4 + E_6 + E_8$
324	$A_1 + A_2 + E_7 + E_8$
325	$A_3 + E_7 + E_8$

A=0

$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

A=0

No	Σ
152	$A_{12} + D_6$
153	$A_2 + A_5 + D_5 + D_6$
154	$A_7 + D_5 + D_6$
155	$2\,A_2 + 2\,D_7$
156	$A_2 + 3\,A_3 + D_7$
157	$A_1 + A_2 + 2\,A_4 + D_7$
158	$A_2 + A_3 + A_6 + D_7$
159	$A_1 + A_4 + A_6 + D_7$
160	$A_5 + A_6 + D_7$
161	$2\,A_1 + A_2 + A_7 + D_7$
162	$A_1 + A_3 + A_7 + D_7$
163	$2\,A_1 + A_9 + D_7$
164	$A_2 + A_9 + D_7$
165	$A_1 + A_{10} + D_7$
166	$A_{11} + D_7$
167	$A_1 + A_5 + D_5 + D_7$
168	$A_5 + D_6 + D_7$
169	$2\,A_1 + 2\,D_8$
170	$2\,A_2 + 2\,A_3 + D_8$
171	$2\,A_5 + D_8$
172	$2\,A_1 + A_3 + A_5 + D_8$
173	$A_1 + A_4 + A_5 + D_8$
174	$2\,A_2 + A_6 + D_8$
175	$A_1 + A_2 + A_7 + D_8$
176	$A_1 + A_9 + D_8$
177	$2\,D_5 + D_8$
178	$A_1 + A_3 + D_6 + D_8$
179	$2\,D_9$
180	$A_1 + 2\,A_2 + A_4 + D_9$
181	$A_1 + A_3 + A_5 + D_9$
182	$A_4 + A_5 + D_9$
183	$A_1 + A_2 + A_6 + D_9$
184	$2\,A_1 + A_7 + D_9$
185	$A_1 + A_8 + D_9$
186	$A_9 + D_9$
187	$A_4 + D_5 + D_9$

No	Σ
188	$2\,A_1 + 2\,A_3 + D_{10}$
189	$2\,A_4 + D_{10}$
190	$A_1 + A_3 + A_4 + D_{10}$
191	$3\,A_1 + A_5 + D_{10}$
192	$A_3 + A_5 + D_{10}$
193	$A_2 + A_6 + D_{10}$
194	$A_8 + D_{10}$
195	$A_1 + A_2 + D_5 + D_{10}$
196	$A_2 + D_6 + D_{10}$
197	$A_1 + D_7 + D_{10}$
198	$2\,A_2 + A_3 + D_{11}$
199	$A_1 + A_2 + A_4 + D_{11}$
200	$A_2 + A_5 + D_{11}$
201	$A_1 + A_6 + D_{11}$
202	$2\,A_1 + 2\,A_2 + D_{12}$
203	$A_1 + A_2 + A_3 + D_{12}$
204	$2\,A_1 + A_4 + D_{12}$
205	$A_1 + D_5 + D_{12}$
206	$D_6 + D_{12}$
207	$A_1 + A_4 + D_{13}$
208	$A_5 + D_{13}$
209	$D_5 + D_{13}$
210	$2\,A_2 + D_{14}$
211	$2\,A_1 + A_2 + D_{14}$
212	$A_1 + A_3 + D_{14}$
213	$A_4 + D_{14}$
214	$A_1 + A_2 + D_{15}$
215	$2\,A_1 + D_{16}$
216	$A_2 + D_{16}$
217	$A_1 + D_{17}$
218	D_{18}
219	$3\,E_6$
220	$2\,A_3 + 2\,E_6$
221	$A_1 + A_3 + 2\,A_4 + E_6$
222	$A_1 + A_5 + 2\,E_6$
223	$A_2 + 2\,A_5 + E_6$

No	Σ
224	$2\,A_2 + A_3 + A_5 + E_6$
225	$A_3 + A_4 + A_5 + E_6$
226	$A_6 + 2\,E_6$
227	$A_1 + A_2 + A_3 + A_6 + E_6$
228	$2\,A_1 + A_4 + A_6 + E_6$
229	$A_2 + A_4 + A_6 + E_6$
230	$A_1 + A_5 + A_6 + E_6$
231	$A_1 + A_4 + A_7 + E_6$
232	$A_5 + A_7 + E_6$
233	$2\,A_2 + A_8 + E_6$
234	$2\,A_1 + A_2 + A_8 + E_6$
235	$A_1 + A_3 + A_8 + E_6$
236	$A_4 + A_8 + E_6$
237	$A_1 + A_2 + A_9 + E_6$
238	$A_3 + A_9 + E_6$
239	$2\,A_1 + A_{10} + E_6$
240	$A_2 + A_{10} + E_6$
241	$A_1 + A_{11} + E_6$
242	$A_{12} + E_6$
243	$A_3 + A_4 + D_5 + E_6$
244	$A_1 + A_6 + D_5 + E_6$
245	$A_7 + D_5 + E_6$
246	$D_6 + 2\,E_6$
247	$A_2 + A_4 + D_6 + E_6$
248	$A_6 + D_6 + E_6$
249	$A_1 + A_4 + D_7 + E_6$
250	$D_5 + D_7 + E_6$
251	$A_4 + D_8 + E_6$
252	$A_1 + A_2 + D_9 + E_6$
253	$A_3 + D_9 + E_6$
254	$A_1 + D_{11} + E_6$
255	$D_{12} + E_6$
256	$2\,A_2 + 2\,E_7$
257	$A_1 + A_3 + 2\,E_7$

No	Σ
258	$A_4 + 2\,E_7$
259	$A_1 + 2\,A_3 + A_4 + E_7$
260	$2\,A_2 + A_3 + A_4 + E_7$
261	$2\,A_3 + A_5 + E_7$
262	$A_1 + A_2 + A_3 + A_5 + E_7$
263	$2\,A_1 + A_4 + A_5 + E_7$
264	$A_2 + A_4 + A_5 + E_7$
265	$A_1 + 2\,A_2 + A_6 + E_7$
266	$A_2 + A_3 + A_6 + E_7$
267	$A_1 + A_4 + A_6 + E_7$
268	$A_5 + A_6 + E_7$
269	$2\,A_2 + A_7 + E_7$
270	$2\,A_1 + A_2 + A_7 + E_7$
271	$A_1 + A_3 + A_7 + E_7$
272	$A_4 + A_7 + E_7$
273	$A_1 + A_2 + A_8 + E_7$
274	$A_3 + A_8 + E_7$
275	$2\,A_1 + A_9 + E_7$
276	$A_2 + A_9 + E_7$
277	$A_1 + A_{10} + E_7$
278	$A_{11} + E_7$
279	$D_4 + 2\,E_7$
280	$A_2 + A_4 + D_5 + E_7$
281	$A_1 + A_5 + D_5 + E_7$
282	$A_6 + D_5 + E_7$
283	$A_2 + A_3 + D_6 + E_7$
284	$A_5 + D_6 + E_7$
285	$D_5 + D_6 + E_7$
286	$A_1 + A_3 + D_7 + E_7$
287	$A_4 + D_7 + E_7$
288	$A_1 + A_2 + D_8 + E_7$
289	$A_2 + D_9 + E_7$
290	$A_1 + D_{10} + E_7$

No	Σ
291	$D_{11} + E_7$
292	$A_2 + A_3 + E_6 + E_7$
293	$A_1 + A_4 + E_6 + E_7$
294	$A_5 + E_6 + E_7$
295	$D_5 + E_6 + E_7$
296	$2\,A_1 + 2\,E_8$
297	$A_2 + 2\,E_8$
298	$2\,A_2 + 2\,A_3 + E_8$
299	$2\,A_1 + 2\,A_4 + E_8$
300	$A_1 + A_2 + A_3 + A_4 + E_8$
301	$2\,A_5 + E_8$
302	$A_2 + A_3 + A_5 + E_8$
303	$A_1 + A_4 + A_5 + E_8$
304	$2\,A_2 + A_6 + E_8$
305	$2\,A_1 + A_2 + A_6 + E_8$
306	$A_1 + A_3 + A_6 + E_8$
307	$A_4 + A_6 + E_8$
308	$A_1 + A_2 + A_7 + E_8$
309	$2\,A_1 + A_8 + E_8$
310	$A_2 + A_8 + E_8$
311	$A_1 + A_9 + E_8$
312	$A_{10} + E_8$
313	$2\,D_5 + E_8$
314	$A_1 + A_4 + D_5 + E_8$
315	$A_5 + D_5 + E_8$
316	$2\,A_2 + D_6 + E_8$
317	$A_4 + D_6 + E_8$
318	$A_1 + A_2 + D_7 + E_8$
319	$A_1 + D_9 + E_8$
320	$D_{10} + E_8$
321	$A_1 + A_3 + E_6 + E_8$
322	$A_4 + E_6 + E_8$
323	$D_4 + E_6 + E_8$
324	$A_1 + A_2 + E_7 + E_8$
325	$A_3 + E_7 + E_8$

$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

No	Σ
152	$A_{12} + D_6$
153	$A_2 + A_5 + D_5 + D_6$
154	$A_7 + D_5 + D_6$
155	$2\,A_2 + 2\,D_7$
156	$A_2 + 3\,A_3 + D_7$
157	$A_1 + A_2 + 2\,A_4 + D_7$
158	$A_2 + A_3 + A_6 + D_7$
159	$A_1 + A_4 + A_6 + D_7$
160	$A_5 + A_6 + D_7$
161	$2\,A_1 + A_2 + A_7 + D_7$
162	$A_1 + A_3 + A_7 + D_7$
163	$2\,A_1 + A_9 + D_7$
164	$A_2 + A_9 + D_7$
165	$A_1 + A_{10} + D_7$
166	$A_{11} + D_7$
167	$A_1 + A_5 + D_5 + D_7$
168	$A_5 + D_6 + D_7$
169	$2\,A_1 + 2\,D_8$
170	$2\,A_2 + 2\,A_3 + D_8$
171	$2\,A_5 + D_8$
172	$2\,A_1 + A_3 + A_5 + D_8$
173	$A_1 + A_4 + A_5 + D_8$
174	$2\,A_2 + A_6 + D_8$
175	$A_1 + A_2 + A_7 + D_8$
176	$A_1 + A_9 + D_8$
177	$2\,D_5 + D_8$
178	$A_1 + A_3 + D_6 + D_8$
179	$2\,D_9$
180	$A_1 + 2\,A_2 + A_4 + D_9$
181	$A_1 + A_3 + A_5 + D_9$
182	$A_4 + A_5 + D_9$
183	$A_1 + A_2 + A_6 + D_9$
184	$2\,A_1 + A_7 + D_9$
185	$A_1 + A_8 + D_9$
186	$A_9 + D_9$
187	$A_4 + D_5 + D_9$

No	Σ
188	$2\,A_1 + 2\,A_3 + D_{10}$
189	$2\,A_4 + D_{10}$
190	$A_1 + A_3 + A_4 + D_{10}$
191	$3\,A_1 + A_5 + D_{10}$
192	$A_3 + A_5 + D_{10}$
193	$A_2 + A_6 + D_{10}$
194	$A_8 + D_{10}$
195	$A_1 + A_2 + D_5 + D_{10}$
196	$A_2 + D_6 + D_{10}$
197	$A_1 + D_7 + D_{10}$
198	$2\,A_2 + A_3 + D_{11}$
199	$A_1 + A_2 + A_4 + D_{11}$
200	$A_2 + A_5 + D_{11}$
201	$A_1 + A_6 + D_{11}$
202	$2\,A_1 + 2\,A_2 + D_{12}$
203	$A_1 + A_2 + A_3 + D_{12}$
204	$2\,A_1 + A_4 + D_{12}$
205	$A_1 + D_5 + D_{12}$
206	$D_6 + D_{12}$
207	$A_1 + A_4 + D_{13}$
208	$A_5 + D_{13}$
209	$D_5 + D_{13}$
210	$2\,A_2 + D_{14}$
211	$2\,A_1 + A_2 + D_{14}$
212	$A_1 + A_3 + D_{14}$
213	$A_4 + D_{14}$
214	$A_1 + A_2 + D_{15}$
215	$2\,A_1 + D_{16}$
216	$A_2 + D_{16}$
217	$A_1 + D_{17}$
218	D_{18}
219	$3\,E_6$
220	$2\,A_3 + 2\,E_6$
221	$A_1 + A_3 + 2\,A_4 + E_6$
222	$A_1 + A_5 + 2\,E_6$
223	$A_2 + 2\,A_5 + E_6$

No	Σ
224	$2\,A_2 + A_3 + A_5 + E_6$
225	$A_3 + A_4 + A_5 + E_6$
226	$A_6 + 2\,E_6$
227	$A_1 + A_2 + A_3 + A_6 + E_6$
228	$2\,A_1 + A_4 + A_6 + E_6$
229	$A_2 + A_4 + A_6 + E_6$
230	$A_1 + A_5 + A_6 + E_6$
231	$A_1 + A_4 + A_7 + E_6$
232	$A_5 + A_7 + E_6$
233	$2\,A_2 + A_8 + E_6$
234	$2\,A_1 + A_2 + A_8 + E_6$
235	$A_1 + A_3 + A_8 + E_6$
236	$A_4 + A_8 + E_6$
237	$A_1 + A_2 + A_9 + E_6$
238	$A_3 + A_9 + E_6$
239	$2\,A_1 + A_{10} + E_6$
240	$A_2 + A_{10} + E_6$
241	$A_1 + A_{11} + E_6$
242	$A_{12} + E_6$
243	$A_3 + A_4 + D_5 + E_6$
244	$A_1 + A_6 + D_5 + E_6$
245	$A_7 + D_5 + E_6$
246	$D_6 + 2\,E_6$
247	$A_2 + A_4 + D_6 + E_6$
248	$A_6 + D_6 + E_6$
249	$A_1 + A_4 + D_7 + E_6$
250	$D_5 + D_7 + E_6$
251	$A_4 + D_8 + E_6$
252	$A_1 + A_2 + D_9 + E_6$
253	$A_3 + D_9 + E_6$
254	$A_1 + D_{11} + E_6$
255	$D_{12} + E_6$
256	$2\,A_2 + 2\,E_7$
257	$A_1 + A_3 + 2\,E_7$

No	Σ
258	$A_4 + 2\,E_7$
259	$A_1 + 2\,A_3 + A_4 + E_7$
260	$2\,A_2 + A_3 + A_4 + E_7$
261	$2\,A_3 + A_5 + E_7$
262	$A_1 + A_2 + A_3 + A_5 + E_7$
263	$2\,A_1 + A_4 + A_5 + E_7$
264	$A_2 + A_4 + A_5 + E_7$
265	$A_1 + 2\,A_2 + A_6 + E_7$
266	$A_2 + A_3 + A_6 + E_7$
267	$A_1 + A_4 + A_6 + E_7$
268	$A_5 + A_6 + E_7$
269	$2\,A_2 + A_7 + E_7$
270	$2\,A_1 + A_2 + A_7 + E_7$
271	$A_1 + A_3 + A_7 + E_7$
272	$A_4 + A_7 + E_7$
273	$A_1 + A_2 + A_8 + E_7$
274	$A_3 + A_8 + E_7$
275	$2\,A_1 + A_9 + E_7$
276	$A_2 + A_9 + E_7$
277	$A_1 + A_{10} + E_7$
278	$A_{11} + E_7$
279	$D_4 + 2\,E_7$
280	$A_2 + A_4 + D_5 + E_7$
281	$A_1 + A_5 + D_5 + E_7$
282	$A_6 + D_5 + E_7$
283	$A_2 + A_3 + D_6 + E_7$
284	$A_5 + D_6 + E_7$
285	$D_5 + D_6 + E_7$
286	$A_1 + A_3 + D_7 + E_7$
287	$A_4 + D_7 + E_7$
288	$A_1 + A_2 + D_8 + E_7$
289	$A_2 + D_9 + E_7$
290	$A_1 + D_{10} + E_7$

No	Σ
291	$D_{11} + E_7$
292	$A_2 + A_3 + E_6 + E_7$
293	$A_1 + A_4 + E_6 + E_7$
294	$A_5 + E_6 + E_7$
295	$D_5 + E_6 + E_7$
296	$\boxed{2\,A_1 + 2\,E_8}$
297	$\boxed{A_2 + 2\,E_8}$
298	$2\,A_2 + 2\,A_3 + E_8$
299	$2\,A_1 + 2\,A_4 + E_8$
300	$A_1 + A_2 + A_3 + A_4 + E_8$
301	$2\,A_5 + E_8$
302	$A_2 + A_3 + A_5 + E_8$
303	$A_1 + A_4 + A_5 + E_8$
304	$2\,A_2 + A_6 + E_8$
305	$2\,A_1 + A_2 + A_6 + E_8$
306	$A_1 + A_3 + A_6 + E_8$
307	$A_4 + A_6 + E_8$
308	$A_1 + A_2 + A_7 + E_8$
309	$2\,A_1 + A_8 + E_8$
310	$A_2 + A_8 + E_8$
311	$A_1 + A_9 + E_8$
312	$A_{10} + E_8$
313	$2\,D_5 + E_8$
314	$A_1 + A_4 + D_5 + E_8$
315	$A_5 + D_5 + E_8$
316	$2\,A_2 + D_6 + E_8$
317	$A_4 + D_6 + E_8$
318	$A_1 + A_2 + D_7 + E_8$
319	$A_1 + D_9 + E_8$
320	$\boxed{D_{10} + E_8}$
321	$A_1 + A_3 + E_6 + E_8$
322	$A_4 + E_6 + E_8$
323	$D_4 + E_6 + E_8$
324	$A_1 + A_2 + E_7 + E_8$
325	$A_3 + E_7 + E_8$

$A = 0$

$A = (1, 0_{15})$

$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

No	Σ
152	$A_{12} + D_6$
153	$A_2 + A_5 + D_5 + D_6$
154	$A_7 + D_5 + D_6$
155	$2\,A_2 + 2\,D_7$
156	$A_2 + 3\,A_3 + D_7$
157	$A_1 + A_2 + 2\,A_4 + D_7$
158	$A_2 + A_3 + A_6 + D_7$
159	$A_1 + A_4 + A_6 + D_7$
160	$A_5 + A_6 + D_7$
161	$2\,A_1 + A_2 + A_7 + D_7$
162	$A_1 + A_3 + A_7 + D_7$
163	$2\,A_1 + A_9 + D_7$
164	$A_2 + A_9 + D_7$
165	$A_1 + A_{10} + D_7$
166	$A_{11} + D_7$
167	$A_1 + A_5 + D_5 + D_7$
168	$A_5 + D_6 + D_7$
169	$2\,A_1 + 2\,D_8$
170	$2\,A_2 + 2\,A_3 + D_8$
171	$2\,A_5 + D_8$
172	$2\,A_1 + A_3 + A_5 + D_8$
173	$A_1 + A_4 + A_5 + D_8$
174	$2\,A_2 + A_6 + D_8$
175	$A_1 + A_2 + A_7 + D_8$
176	$A_1 + A_9 + D_8$
177	$2\,D_5 + D_8$
178	$A_1 + A_3 + D_6 + D_8$
179	$2\,D_9$
180	$A_1 + 2\,A_2 + A_4 + D_9$
181	$A_1 + A_3 + A_5 + D_9$
182	$A_4 + A_5 + D_9$
183	$A_1 + A_2 + A_6 + D_9$
184	$2\,A_1 + A_7 + D_9$
185	$A_1 + A_8 + D_9$
186	$A_9 + D_9$
187	$A_4 + D_5 + D_9$
188	$2\,A_1 + 2\,A_3 + D_{10}$
189	$2\,A_4 + D_{10}$
190	$A_1 + A_3 + A_4 + D_{10}$
191	$3\,A_1 + A_5 + D_{10}$
192	$A_3 + A_5 + D_{10}$
193	$A_2 + A_6 + D_{10}$
194	$A_8 + D_{10}$
195	$A_1 + A_2 + D_5 + D_{10}$
196	$A_2 + D_6 + D_{10}$
197	$A_1 + D_7 + D_{10}$
198	$2\,A_2 + A_3 + D_{11}$
199	$A_1 + A_2 + A_4 + D_{11}$
200	$A_2 + A_5 + D_{11}$
201	$A_1 + A_6 + D_{11}$
202	$2\,A_1 + 2\,A_2 + D_{12}$
203	$A_1 + A_2 + A_3 + D_{12}$
204	$2\,A_1 + A_4 + D_{12}$
205	$A_1 + D_5 + D_{12}$
206	$D_6 + D_{12}$
207	$A_1 + A_4 + D_{13}$
208	$A_5 + D_{13}$
209	$D_5 + D_{13}$
210	$2\,A_2 + D_{14}$
211	$2\,A_1 + A_2 + D_{14}$
212	$A_1 + A_3 + D_{14}$
213	$A_4 + D_{14}$
214	$A_1 + A_2 + D_{15}$
215	$2\,A_1 + D_{16}$
216	$A_2 + D_{16}$
217	$A_1 + D_{17}$
218	D_{18}
219	$3\,E_6$
220	$2\,A_3 + 2\,E_6$
221	$A_1 + A_3 + 2\,A_4 + E_6$
222	$A_1 + A_5 + 2\,E_6$
223	$A_2 + 2\,A_5 + E_6$
224	$2\,A_2 + A_3 + A_5 + E_6$
225	$A_3 + A_4 + A_5 + E_6$
226	$A_6 + 2\,E_6$
227	$A_1 + A_2 + A_3 + A_6 + E_6$
228	$2\,A_1 + A_4 + A_6 + E_6$
229	$A_2 + A_4 + A_6 + E_6$
230	$A_1 + A_5 + A_6 + E_6$
231	$A_1 + A_4 + A_7 + E_6$
232	$A_5 + A_7 + E_6$
233	$2\,A_2 + A_8 + E_6$
234	$2\,A_1 + A_2 + A_8 + E_6$
235	$A_1 + A_3 + A_8 + E_6$
236	$A_4 + A_8 + E_6$
237	$A_1 + A_2 + A_9 + E_6$
238	$A_3 + A_9 + E_6$
239	$2\,A_1 + A_{10} + E_6$
240	$A_2 + A_{10} + E_6$
241	$A_1 + A_{11} + E_6$
242	$A_{12} + E_6$
243	$A_3 + A_4 + D_5 + E_6$
244	$A_1 + A_6 + D_5 + E_6$
245	$A_7 + D_5 + E_6$
246	$D_6 + 2\,E_6$
247	$A_2 + A_4 + D_6 + E_6$
248	$A_6 + D_6 + E_6$
249	$A_1 + A_4 + D_7 + E_6$
250	$D_5 + D_7 + E_6$
251	$A_4 + D_8 + E_6$
252	$A_1 + A_2 + D_9 + E_6$
253	$A_3 + D_9 + E_6$
254	$A_1 + D_{11} + E_6$
255	$D_{12} + E_6$
256	$2\,A_2 + 2\,E_7$
257	$A_1 + A_3 + 2\,E_7$
258	$A_4 + 2\,E_7$
259	$A_1 + 2\,A_3 + A_4 + E_7$
260	$2\,A_2 + A_3 + A_4 + E_7$
261	$2\,A_3 + A_5 + E_7$
262	$A_1 + A_2 + A_3 + A_5 + E_7$
263	$2\,A_1 + A_4 + A_5 + E_7$
264	$A_2 + A_4 + A_5 + E_7$
265	$A_1 + 2\,A_2 + A_6 + E_7$
266	$A_2 + A_3 + A_6 + E_7$
267	$A_1 + A_4 + A_6 + E_7$
268	$A_5 + A_6 + E_7$
269	$2\,A_2 + A_7 + E_7$
270	$2\,A_1 + A_2 + A_7 + E_7$
271	$A_1 + A_3 + A_7 + E_7$
272	$A_4 + A_7 + E_7$
273	$A_1 + A_2 + A_8 + E_7$
274	$A_3 + A_8 + E_7$
275	$2\,A_1 + A_9 + E_7$
276	$A_2 + A_9 + E_7$
277	$A_1 + A_{10} + E_7$
278	$A_{11} + E_7$
279	$D_4 + 2\,E_7$
280	$A_2 + A_4 + D_5 + E_7$
281	$A_1 + A_5 + D_5 + E_7$
282	$A_6 + D_5 + E_7$
283	$A_2 + A_3 + D_6 + E_7$
284	$A_5 + D_6 + E_7$
285	$D_5 + D_6 + E_7$
286	$A_1 + A_3 + D_7 + E_7$
287	$A_4 + D_7 + E_7$
288	$A_1 + A_2 + D_8 + E_7$
289	$A_2 + D_9 + E_7$
290	$A_1 + D_{10} + E_7$
291	$D_{11} + E_7$
292	$A_2 + A_3 + E_6 + E_7$
293	$A_1 + A_4 + E_6 + E_7$
294	$A_5 + E_6 + E_7$
295	$D_5 + E_6 + E_7$
296	$2\,A_1 + 2\,E_8$
297	$A_2 + 2\,E_8$
298	$2\,A_2 + 2\,A_3 + E_8$
299	$2\,A_1 + 2\,A_4 + E_8$
300	$A_1 + A_2 + A_3 + A_4 + E_8$
301	$2\,A_5 + E_8$
302	$A_2 + A_3 + A_5 + E_8$
303	$A_1 + A_4 + A_5 + E_8$
304	$2\,A_2 + A_6 + E_8$
305	$2\,A_1 + A_2 + A_6 + E_8$
306	$A_1 + A_3 + A_6 + E_8$
307	$A_4 + A_6 + E_8$
308	$A_1 + A_2 + A_7 + E_8$
309	$2\,A_1 + A_8 + E_8$
310	$A_2 + A_8 + E_8$
311	$A_1 + A_9 + E_8$
312	$A_{10} + E_8$
313	$2\,D_5 + E_8$
314	$A_1 + A_4 + D_5 + E_8$
315	$A_5 + D_5 + E_8$
316	$2\,A_2 + D_6 + E_8$
317	$A_4 + D_6 + E_8$
318	$A_1 + A_2 + D_7 + E_8$
319	$A_1 + D_9 + E_8$
320	$D_{10} + E_8$
321	$A_1 + A_3 + E_6 + E_8$
322	$A_4 + E_6 + E_8$
323	$D_4 + E_6 + E_8$
324	$A_1 + A_2 + E_7 + E_8$
325	$A_3 + E_7 + E_8$

$A=0$

$A=(1,0_{15})$

No	Σ
1	$\boxed{6\,A_3}$!!
2	$2\,A_1 + 4\,A_4$
3	$2\,A_2 + 2\,A_3 + 2\,A_4$
4	$3\,A_1 + 3\,A_5$
5	$4\,A_2 + 2\,A_5$
6	$A_3 + 3\,A_5$
7	$2\,A_1 + 2\,A_3 + 2\,A_5$
8	$A_1 + 2\,A_2 + A_3 + 2\,A_5$
9	$2\,A_4 + 2\,A_5$
10	$2\,A_2 + A_4 + 2\,A_5$
11	$A_1 + A_3 + A_4 + 2\,A_5$
12	$A_1 + A_2 + 2\,A_3 + A_4 + A_5$
13	$\boxed{3\,A_6}$!!
14	$2\,A_1 + 2\,A_2 + 2\,A_6$
15	$2\,A_3 + 2\,A_6$
16	$A_2 + A_4 + 2\,A_6$
17	$2\,A_1 + A_2 + 2\,A_4 + A_6$
18	$A_1 + A_3 + 2\,A_4 + A_6$
19	$A_2 + 2\,A_3 + A_4 + A_6$
20	$A_1 + 2\,A_2 + A_3 + A_4 + A_6$
21	$2\,A_1 + 2\,A_5 + A_6$
22	$A_1 + 2\,A_3 + A_5 + A_6$
23	$A_1 + A_2 + A_4 + A_5 + A_6$
24	$A_3 + A_4 + A_5 + A_6$
25	$4\,A_1 + 2\,A_7$
26	$2\,A_2 + 2\,A_7$
27	$A_1 + A_3 + 2\,A_7$
28	$2\,A_1 + 3\,A_3 + A_7$
29	$A_2 + 3\,A_3 + A_7$
30	$2\,A_2 + A_3 + A_4 + A_7$
31	$2\,A_1 + A_2 + A_3 + A_4 + A_7$
32	$A_1 + 2\,A_5 + A_7$
33	$3\,A_1 + A_3 + A_5 + A_7$
34	$A_1 + A_2 + A_3 + A_5 + A_7$
35	$2\,A_1 + A_4 + A_5 + A_7$
36	$A_2 + A_4 + A_5 + A_7$
37	$A_1 + 2\,A_2 + A_6 + A_7$
38	$2\,A_1 + A_3 + A_6 + A_7$
39	$A_2 + A_3 + A_6 + A_7$
40	$A_1 + A_4 + A_6 + A_7$
41	$A_5 + A_6 + A_7$
42	$2\,A_1 + 2\,A_8$
43	$A_1 + 3\,A_2 + A_3 + A_8$
44	$2\,A_1 + 2\,A_4 + A_8$
45	$3\,A_2 + A_4 + A_8$
46	$A_1 + A_2 + A_3 + A_4 + A_8$
47	$A_1 + 2\,A_2 + A_5 + A_8$
48	$A_2 + A_3 + A_5 + A_8$
49	$A_1 + A_4 + A_5 + A_8$
50	$2\,A_1 + A_2 + A_6 + A_8$
51	$A_1 + A_3 + A_6 + A_8$
52	$A_4 + A_6 + A_8$
53	$A_1 + A_2 + A_7 + A_8$
54	$2\,A_9$
55	$A_1 + A_2 + 2\,A_3 + A_9$
56	$2\,A_1 + 2\,A_2 + A_3 + A_9$
57	$A_1 + 2\,A_4 + A_9$
58	$3\,A_1 + A_2 + A_4 + A_9$
59	$2\,A_1 + A_3 + A_4 + A_9$
60	$2\,A_1 + A_2 + A_5 + A_9$
61	$A_1 + A_3 + A_5 + A_9$
62	$A_4 + A_5 + A_9$
63	$3\,A_1 + A_6 + A_9$
64	$A_1 + A_2 + A_6 + A_9$
65	$A_3 + A_6 + A_9$
66	$A_2 + A_7 + A_9$
67	$A_1 + A_8 + A_9$
68	$A_2 + 2\,A_3 + A_{10}$
69	$A_1 + 2\,A_2 + A_3 + A_{10}$
70	$2\,A_4 + A_{10}$
71	$2\,A_2 + A_4 + A_{10}$
72	$2\,A_1 + A_2 + A_4 + A_{10}$
73	$A_1 + A_3 + A_4 + A_{10}$
74	$A_1 + A_2 + A_5 + A_{10}$
75	$A_3 + A_5 + A_{10}$
76	$2\,A_1 + A_6 + A_{10}$
77	$A_2 + A_6 + A_{10}$
78	$A_1 + A_7 + A_{10}$
79	$A_8 + A_{10}$
80	$A_1 + 3\,A_2 + A_{11}$
81	$3\,A_1 + 2\,A_2 + A_{11}$
82	$A_1 + 2\,A_3 + A_{11}$
83	$2\,A_2 + A_3 + A_{11}$
84	$2\,A_1 + A_2 + A_3 + A_{11}$
85	$3\,A_1 + A_4 + A_{11}$
86	$A_1 + A_2 + A_4 + A_{11}$
87	$2\,A_1 + A_5 + A_{11}$
88	$A_2 + A_5 + A_{11}$
89	$A_1 + A_6 + A_{11}$
90	$2\,A_1 + 2\,A_2 + A_{12}$
91	$A_1 + A_2 + A_3 + A_{12}$
92	$2\,A_1 + A_4 + A_{12}$
93	$A_2 + A_4 + A_{12}$
94	$A_1 + A_5 + A_{12}$
95	$A_6 + A_{12}$
96	$A_1 + 2\,A_2 + A_{13}$
97	$3\,A_1 + A_2 + A_{13}$
98	$2\,A_1 + A_3 + A_{13}$
99	$A_2 + A_3 + A_{13}$
100	$A_1 + A_4 + A_{13}$
101	$A_5 + A_{13}$
102	$2\,A_2 + A_{14}$
103	$2\,A_1 + A_2 + A_{14}$
104	$A_1 + A_3 + A_{14}$
105	$A_4 + A_{14}$
106	$3\,A_1 + A_{15}$
107	$A_1 + A_2 + A_{15}$
108	$A_3 + A_{15}$
109	$2\,A_1 + A_{16}$
110	$A_2 + A_{16}$
111	$A_1 + A_{17}$
112	A_{18}
113	$2\,A_4 + 2\,D_5$
114	$A_3 + 2\,A_5 + D_5$
115	$2\,A_4 + A_5 + D_5$
116	$A_1 + A_3 + A_4 + A_5 + D_5$
117	$A_1 + 2\,A_6 + D_5$
118	$2\,A_2 + A_3 + A_6 + D_5$
119	$A_1 + A_2 + A_4 + A_6 + D_5$
120	$A_2 + A_5 + A_6 + D_5$
121	$A_1 + A_7 + 2\,D_5$
122	$A_1 + A_2 + A_3 + A_7 + D_5$
123	$2\,A_1 + A_4 + A_7 + D_5$
124	$A_8 + 2\,D_5$
125	$A_1 + A_4 + A_8 + D_5$
126	$A_5 + A_8 + D_5$
127	$2\,A_2 + A_9 + D_5$
128	$2\,A_1 + A_2 + A_9 + D_5$
129	$A_1 + A_3 + A_9 + D_5$
130	$A_4 + A_9 + D_5$
131	$A_1 + A_2 + A_{10} + D_5$
132	$2\,A_1 + A_{11} + D_5$
133	$A_2 + A_{11} + D_5$
134	$A_1 + A_{12} + D_5$
135	$A_{13} + D_5$
136	$3\,D_6$
137	$2\,A_3 + 2\,D_6$
138	$2\,A_2 + 2\,A_4 + D_6$
139	$2\,A_1 + 2\,A_5 + D_6$
140	$A_1 + 2\,A_3 + A_5 + D_6$
141	$A_3 + A_4 + A_5 + D_6$
142	$2\,A_6 + D_6$
143	$A_2 + A_4 + A_6 + D_6$
144	$A_1 + 2\,A_2 + A_7 + D_6$
145	$A_2 + A_3 + A_7 + D_6$
146	$A_1 + A_4 + A_7 + D_6$
147	$A_4 + A_8 + D_6$
148	$A_1 + A_2 + A_9 + D_6$
149	$A_3 + A_9 + D_6$
150	$A_2 + A_{10} + D_6$
151	$A_1 + A_{11} + D_6$

$$E = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

$$A_1 = \tfrac{1}{4}(0,0,0,0,1,1,1,-3,0,0,0,0,1,1,1,-3)$$

$$A_2 = \tfrac{1}{4}(-3,1,1,1,0,0,0,0,-3,1,1,1,0,0,0,0)$$

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No	Σ
1	$6\,A_3$!!
2	$2\,A_1 + 4\,A_4$
3	$2\,A_2 + 2\,A_3 + 2\,A_4$
4	$3\,A_1 + 3\,A_5$
5	$4\,A_2 + 2\,A_5$
6	$A_3 + 3\,A_5$
7	$2\,A_1 + 2\,A_3 + 2\,A_5$
8	$A_1 + 2\,A_2 + A_3 + 2\,A_5$
9	$2\,A_4 + 2\,A_5$
10	$2\,A_2 + A_4 + 2\,A_5$
11	$A_1 + A_3 + A_4 + 2\,A_5$
12	$A_1 + A_2 + 2\,A_3 + A_4 + A_5$
13	$3\,A_6$!!
14	$2\,A_1 + 2\,A_2 + 2\,A_6$
15	$2\,A_3 + 2\,A_6$
16	$A_2 + A_4 + 2\,A_6$
17	$2\,A_1 + A_2 + 2\,A_4 + A_6$
18	$A_1 + A_3 + 2\,A_4 + A_6$
19	$A_2 + 2\,A_3 + A_4 + A_6$
20	$A_1 + 2\,A_2 + A_3 + A_4 + A_6$
21	$2\,A_1 + 2\,A_5 + A_6$
22	$A_1 + 2\,A_3 + A_5 + A_6$
23	$A_1 + A_2 + A_4 + A_5 + A_6$
24	$A_3 + A_4 + A_5 + A_6$
25	$4\,A_1 + 2\,A_7$
26	$2\,A_2 + 2\,A_7$
27	$A_1 + A_3 + 2\,A_7$
28	$2\,A_1 + 3\,A_3 + A_7$
29	$A_2 + 3\,A_3 + A_7$
30	$2\,A_2 + A_3 + A_4 + A_7$
31	$2\,A_1 + A_2 + A_3 + A_4 + A_7$
32	$A_1 + 2\,A_5 + A_7$
33	$3\,A_1 + A_3 + A_5 + A_7$
34	$A_1 + A_2 + A_3 + A_5 + A_7$
35	$2\,A_1 + A_4 + A_5 + A_7$
36	$A_2 + A_4 + A_5 + A_7$
37	$A_1 + 2\,A_2 + A_6 + A_7$
38	$2\,A_1 + A_3 + A_6 + A_7$
39	$A_2 + A_3 + A_6 + A_7$
40	$A_1 + A_4 + A_6 + A_7$
41	$A_5 + A_6 + A_7$
42	$2\,A_1 + 2\,A_8$
43	$A_1 + 3\,A_2 + A_3 + A_8$
44	$2\,A_1 + 2\,A_4 + A_8$
45	$3\,A_2 + A_4 + A_8$
46	$A_1 + A_2 + A_3 + A_4 + A_8$
47	$A_1 + 2\,A_2 + A_5 + A_8$
48	$A_2 + A_3 + A_5 + A_8$
49	$A_1 + A_4 + A_5 + A_8$
50	$2\,A_1 + A_2 + A_6 + A_8$
51	$A_1 + A_3 + A_6 + A_8$
52	$A_4 + A_6 + A_8$
53	$A_1 + A_2 + A_7 + A_8$
54	$2\,A_9$
55	$A_1 + A_2 + 2\,A_3 + A_9$
56	$2\,A_1 + 2\,A_2 + A_3 + A_9$
57	$A_1 + 2\,A_4 + A_9$
58	$3\,A_1 + A_2 + A_4 + A_9$
59	$2\,A_1 + A_3 + A_4 + A_9$
60	$2\,A_1 + A_2 + A_5 + A_9$
61	$A_1 + A_3 + A_5 + A_9$
62	$A_4 + A_5 + A_9$
63	$3\,A_1 + A_6 + A_9$
64	$A_1 + A_2 + A_6 + A_9$
65	$A_3 + A_6 + A_9$
66	$A_2 + A_7 + A_9$
67	$A_1 + A_8 + A_9$
68	$A_2 + 2\,A_3 + A_{10}$
69	$A_1 + 2\,A_2 + A_3 + A_{10}$
70	$2\,A_4 + A_{10}$
71	$2\,A_2 + A_4 + A_{10}$
72	$2\,A_1 + A_2 + A_4 + A_{10}$
73	$A_1 + A_3 + A_4 + A_{10}$
74	$A_1 + A_2 + A_5 + A_{10}$
75	$A_3 + A_5 + A_{10}$
76	$2\,A_1 + A_6 + A_{10}$
77	$A_2 + A_6 + A_{10}$
78	$A_1 + A_7 + A_{10}$
79	$A_8 + A_{10}$
80	$A_1 + 3\,A_2 + A_{11}$
81	$3\,A_1 + 2\,A_2 + A_{11}$
82	$A_1 + 2\,A_3 + A_{11}$
83	$2\,A_2 + A_3 + A_{11}$
84	$2\,A_1 + A_2 + A_3 + A_{11}$
85	$3\,A_1 + A_4 + A_{11}$
86	$A_1 + A_2 + A_4 + A_{11}$
87	$2\,A_1 + A_5 + A_{11}$
88	$A_2 + A_5 + A_{11}$
89	$A_1 + A_6 + A_{11}$
90	$2\,A_1 + 2\,A_2 + A_{12}$
91	$A_1 + A_2 + A_3 + A_{12}$
92	$2\,A_1 + A_4 + A_{12}$
93	$A_2 + A_4 + A_{12}$
94	$A_1 + A_5 + A_{12}$
95	$A_6 + A_{12}$
96	$A_1 + 2\,A_2 + A_{13}$
97	$3\,A_1 + A_2 + A_{13}$
98	$2\,A_1 + A_3 + A_{13}$
99	$A_2 + A_3 + A_{13}$
100	$A_1 + A_4 + A_{13}$
101	$A_5 + A_{13}$
102	$2\,A_2 + A_{14}$
103	$2\,A_1 + A_2 + A_{14}$
104	$A_1 + A_3 + A_{14}$
105	$A_4 + A_{14}$
106	$3\,A_1 + A_{15}$
107	$A_1 + A_2 + A_{15}$
108	$A_3 + A_{15}$
109	$2\,A_1 + A_{16}$
110	$A_2 + A_{16}$
111	$A_1 + A_{17}$
112	A_{18}
113	$2\,A_4 + 2\,D_5$
114	$A_3 + 2\,A_5 + D_5$
115	$2\,A_4 + A_5 + D_5$
116	$A_1 + A_3 + A_4 + A_5 + D_5$
117	$A_1 + 2\,A_6 + D_5$
118	$2\,A_2 + A_3 + A_6 + D_5$
119	$A_1 + A_2 + A_4 + A_6 + D_5$
120	$A_2 + A_5 + A_6 + D_5$
121	$A_1 + A_7 + 2\,D_5$
122	$A_1 + A_2 + A_3 + A_7 + D_5$
123	$2\,A_1 + A_4 + A_7 + D_5$
124	$A_8 + 2\,D_5$
125	$A_1 + A_4 + A_8 + D_5$
126	$A_5 + A_8 + D_5$
127	$2\,A_2 + A_9 + D_5$
128	$2\,A_1 + A_2 + A_9 + D_5$
129	$A_1 + A_3 + A_9 + D_5$
130	$A_4 + A_9 + D_5$
131	$A_1 + A_2 + A_{10} + D_5$
132	$2\,A_1 + A_{11} + D_5$
133	$A_2 + A_{11} + D_5$
134	$A_1 + A_{12} + D_5$
135	$A_{13} + D_5$
136	$3\,D_6$
137	$2\,A_3 + 2\,D_6$
138	$2\,A_2 + 2\,A_4 + D_6$
139	$2\,A_1 + 2\,A_5 + D_6$
140	$A_1 + 2\,A_3 + A_5 + D_6$
141	$A_3 + A_4 + A_5 + D_6$
142	$2\,A_6 + D_6$
143	$A_2 + A_4 + A_6 + D_6$
144	$A_1 + 2\,A_2 + A_7 + D_6$
145	$A_2 + A_3 + A_7 + D_6$
146	$A_1 + A_4 + A_7 + D_6$
147	$A_4 + A_8 + D_6$
148	$A_1 + A_2 + A_9 + D_6$
149	$A_3 + A_9 + D_6$
150	$A_2 + A_{10} + D_6$
151	$A_1 + A_{11} + D_6$

$$E = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \qquad \begin{aligned} A_1 &= \tfrac{1}{4}(0,0,0,0,1,1,1,-3,0,0,0,0,1,1,1,-3) \\ A_2 &= \tfrac{1}{4}(-3,1,1,1,0,0,0,0,-3,1,1,1,0,0,0,0) \end{aligned}$$

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No	Σ
1	$6\,A_3$!!
2	$2\,A_1+4\,A_4$
3	$2\,A_2+2\,A_3+2\,A_4$
4	$3\,A_1+3\,A_5$
5	$4\,A_2+2\,A_5$
6	$A_3+3\,A_5$
7	$2\,A_1+2\,A_3+2\,A_5$
8	$A_1+2\,A_2+A_3+2\,A_5$
9	$2\,A_4+2\,A_5$
10	$2\,A_2+A_4+2\,A_5$
11	$A_1+A_3+A_4+2\,A_5$
12	$A_1+A_2+2\,A_3+A_4+A_5$
13	$3\,A_6$!!
14	$2\,A_1+2\,A_2+2\,A_6$
15	$2\,A_3+2\,A_6$
16	$A_2+A_4+2\,A_6$
17	$2\,A_1+A_2+2\,A_4+A_6$
18	$A_1+A_3+2\,A_4+A_6$
19	$A_2+2\,A_3+A_4+A_6$
20	$A_1+2\,A_2+A_3+A_4+A_6$
21	$2\,A_1+2\,A_5+A_6$
22	$A_1+2\,A_3+A_5+A_6$
23	$A_1+A_2+A_4+A_5+A_6$
24	$A_3+A_4+A_5+A_6$
25	$4\,A_1+2\,A_7$
26	$2\,A_2+2\,A_7$
27	$A_1+A_3+2\,A_7$
28	$2\,A_1+3\,A_3+A_7$
29	$A_2+3\,A_3+A_7$
30	$2\,A_2+A_3+A_4+A_7$
31	$2\,A_1+A_2+A_3+A_4+A_7$
32	$A_1+2\,A_5+A_7$
33	$3\,A_1+A_3+A_5+A_7$
34	$A_1+A_2+A_3+A_5+A_7$
35	$2\,A_1+A_4+A_5+A_7$
36	$A_2+A_4+A_5+A_7$
37	$A_1+2\,A_2+A_6+A_7$
38	$2\,A_1+A_3+A_6+A_7$
39	$A_2+A_3+A_6+A_7$
40	$A_1+A_4+A_6+A_7$
41	$A_5+A_6+A_7$
42	$2\,A_1+2\,A_8$
43	$A_1+3\,A_2+A_3+A_8$
44	$2\,A_1+2\,A_4+A_8$
45	$3\,A_2+A_4+A_8$
46	$A_1+A_2+A_3+A_4+A_8$
47	$A_1+2\,A_2+A_5+A_8$
48	$A_2+A_3+A_5+A_8$
49	$A_1+A_4+A_5+A_8$
50	$2\,A_1+A_2+A_6+A_8$
51	$A_1+A_3+A_6+A_8$
52	$A_4+A_6+A_8$
53	$A_1+A_2+A_7+A_8$
54	$2\,A_9$
55	$A_1+A_2+2\,A_3+A_9$
56	$2\,A_1+2\,A_2+A_3+A_9$
57	$A_1+2\,A_4+A_9$
58	$3\,A_1+A_2+A_4+A_9$
59	$2\,A_1+A_3+A_4+A_9$
60	$2\,A_1+A_2+A_5+A_9$
61	$A_1+A_3+A_5+A_9$
62	$A_4+A_5+A_9$
63	$3\,A_1+A_6+A_9$
64	$A_1+A_2+A_6+A_9$
65	$A_3+A_6+A_9$
66	$A_2+A_7+A_9$
67	$A_1+A_8+A_9$
68	$A_2+2\,A_3+A_{10}$
69	$A_1+2\,A_2+A_3+A_{10}$
70	$2\,A_4+A_{10}$
71	$2\,A_2+A_4+A_{10}$
72	$2\,A_1+A_2+A_4+A_{10}$
73	$A_1+A_3+A_4+A_{10}$
74	$A_1+A_2+A_5+A_{10}$
75	$A_3+A_5+A_{10}$
76	$2\,A_1+A_6+A_{10}$
77	$A_2+A_6+A_{10}$
78	$A_1+A_7+A_{10}$
79	A_8+A_{10}
80	$A_1+3\,A_2+A_{11}$
81	$3\,A_1+2\,A_2+A_{11}$
82	$A_1+2\,A_3+A_{11}$
83	$2\,A_2+A_3+A_{11}$
84	$2\,A_1+A_2+A_3+A_{11}$
85	$3\,A_1+A_4+A_{11}$
86	$A_1+A_2+A_4+A_{11}$
87	$2\,A_1+A_5+A_{11}$
88	$A_2+A_5+A_{11}$
89	$A_1+A_6+A_{11}$
90	$2\,A_1+2\,A_2+A_{12}$
91	$A_1+A_2+A_3+A_{12}$
92	$2\,A_1+A_4+A_{12}$
93	$A_2+A_4+A_{12}$
94	$A_1+A_5+A_{12}$
95	A_6+A_{12}
96	$A_1+2\,A_2+A_{13}$
97	$3\,A_1+A_2+A_{13}$
98	$2\,A_1+A_3+A_{13}$
99	$A_2+A_3+A_{13}$
100	$A_1+A_4+A_{13}$
101	A_5+A_{13}
102	$2\,A_2+A_{14}$
103	$2\,A_1+A_2+A_{14}$
104	$A_1+A_3+A_{14}$
105	A_4+A_{14}
106	$3\,A_1+A_{15}$
107	$A_1+A_2+A_{15}$
108	A_3+A_{15}
109	$2\,A_1+A_{16}$
110	A_2+A_{16}
111	A_1+A_{17}
112	A_{18}
113	$2\,A_4+2\,D_5$
114	$A_3+2\,A_5+D_5$
115	$2\,A_4+A_5+D_5$
116	$A_1+A_3+A_4+A_5+D_5$
117	$A_1+2\,A_6+D_5$
118	$2\,A_2+A_3+A_6+D_5$
119	$A_1+A_2+A_4+A_6+D_5$
120	$A_2+A_5+A_6+D_5$
121	$A_1+A_7+2\,D_5$
122	$A_1+A_2+A_3+A_7+D_5$
123	$2\,A_1+A_4+A_7+D_5$
124	$A_8+2\,D_5$
125	$A_1+A_4+A_8+D_5$
126	$A_5+A_8+D_5$
127	$2\,A_2+A_9+D_5$
128	$2\,A_1+A_2+A_9+D_5$
129	$A_1+A_3+A_9+D_5$
130	$A_4+A_9+D_5$
131	$A_1+A_2+A_{10}+D_5$
132	$2\,A_1+A_{11}+D_5$
133	$A_2+A_{11}+D_5$
134	$A_1+A_{12}+D_5$
135	$A_{13}+D_5$
136	$3\,D_6$
137	$2\,A_3+2\,D_6$
138	$2\,A_2+2\,A_4+D_6$
139	$2\,A_1+2\,A_5+D_6$
140	$A_1+2\,A_3+A_5+D_6$
141	$A_3+A_4+A_5+D_6$
142	$2\,A_6+D_6$
143	$A_2+A_4+A_6+D_6$
144	$A_1+2\,A_2+A_7+D_6$
145	$A_2+A_3+A_7+D_6$
146	$A_1+A_4+A_7+D_6$
147	$A_4+A_8+D_6$
148	$A_1+A_2+A_9+D_6$
149	$A_3+A_9+D_6$
150	$A_2+A_{10}+D_6$
151	$A_1+A_{11}+D_6$

- We got the 325 groups!!

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- Is there a generalized Dynkin diagram for $\Gamma^{18,2}$?

Effective actions

Massless states bosonic string

$$\mathcal{M}_D \times T^d$$

$$U(1)^d \times U(1)^d \longrightarrow \overset{\substack{\text{rank d}\\ \text{dim n}}}{G} \times \overset{\substack{\text{rank d}\\ \text{dim n}}}{G}$$

$A^m \quad \bar{A}^m$

2d vectors

$A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$

2n vectors

$g_{\mu m} \pm B_{\mu m}$

M^{mn}

d² scalars

$g_{mn} + B_{mn}$

$$\underbrace{M^{mn} \quad M^{m\beta} \quad M^{\alpha n} \quad M^{\alpha\beta}}_{M^{ab}} \qquad a = 1, ..., n$$

n² scalars

0= $M^2 = 2(N + \bar{N} - 2) + \left(p_L^2 + p_R^2\right)$

LMC $\quad 0 = 2(N - \bar{N}) + \left(p_L^2 - p_R^2\right)$

$$p = EZ$$

Scalars

$\bar{N}_y = N_y = 1 \quad M^{mn} \quad p_L = p_R = 0$

$N_y = 1, \bar{N} = 0 \quad M^{m\beta} \quad p_L = 0, p_R^2 = 2$

$N = 0, \bar{N}_y = 1 \quad M^{\alpha n} \quad p_L^2 = 2, p_R = 0$

$N = \bar{N} = 0 \quad M^{\alpha\beta} \quad p_L^2 = p_R^2 = 2$

Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

rank d rank d
dim n dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

$A^m \quad \bar{A}^m$

$A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$

2d vectors

2n vectors

$g_{\mu m} \pm B_{\mu m}$

M^{mn}

$$\underbrace{M^{mn} \quad M^{\alpha n} \quad M^{m\beta} \quad M^{\alpha\beta}}_{M^{ab}}$$

$a = 1, ..., n$

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d² scalars

$g_{mn} + B_{mn}$

n² scalars

$$\boxed{\mathcal{H} \in \frac{O(d,d)}{O(d)\times O(d)}}$$

Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

$$U(1)^d \times U(1)^d \longrightarrow \overset{\substack{\text{rank d}\\ \text{dim n}}}{G} \times \overset{\substack{\text{rank d}\\ \text{dim n}}}{G}$$

$$A^m \qquad \bar{A}^m \qquad\qquad A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$$

2d vectors

2n vectors

$$g_{\mu m} \pm B_{\mu m}$$

$$M^{mn} \qquad\qquad \underbrace{M^{mn} \quad M^{\alpha n} \quad M^{m\beta} \quad M^{\alpha\beta}}_{M^{ab}}$$

d² scalars

$$g_{mn} + B_{mn}$$

n² scalars

$$a = 1, \ldots, n$$

$$\boxed{\mathcal{H} \in \frac{O(d,d)}{O(d)\times O(d)}} \qquad \boxed{\mathcal{H} \in \frac{O(n,n)}{O(n)\times O(n)}}$$

Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

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$A^m \quad \bar{A}^m$

2dD dof

2d vectors

$g_{\mu m} \pm B_{\mu m}$

M^{mn}

d² scalars

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$$\boxed{\mathcal{H} \in \frac{O(d,d)}{O(d) \times O(d)}}$$

$A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$

2n vectors

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n² scalars

$a = 1, ..., n$

$$\boxed{\mathcal{H} \in \frac{O(n,n)}{O(n) \times O(n)}}$$

Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

rank d rank d
dim n dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

D2
dof

tensors

$g_{\mu\nu}, B_{\mu\nu}$

tensors

$g_{\mu\nu}, B_{\mu\nu}$

$A^m \quad \bar{A}^m$

$A^m, A^\alpha, A^{-\alpha} \quad A^m, A^\alpha, A^{-\alpha}$

2dD
dof

2d vectors

$g_{\mu m} \pm B_{\mu m}$

2n vectors

M^{mn}

$$\underbrace{M^{mn} \quad M^{\alpha n} \quad M^{m\beta} \quad M^{\alpha\beta}}$$

d2 scalars

$g_{mn} + B_{mn}$

M^{ab}

n2 scalars

$a = 1, ..., n$

$$\boxed{\mathcal{H} \in \frac{O(d,d)}{O(d)\times O(d)}}$$

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Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

rank d dim n | rank d dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

D2 dof

tensors $g_{\mu\nu}, B_{\mu\nu}$

tensors $g_{\mu\nu}, B_{\mu\nu}$

$A^m \quad \bar{A}^m$

$A^m, A^\alpha, A^{-\alpha} \qquad A^m, A^\alpha, A^{-\alpha}$

2dD dof

2d vectors

$g_{\mu m} \pm B_{\mu m}$

2n vectors

M^{mn}

$$\underbrace{M^{mn} \quad M^{\alpha n} \quad M^{m\beta} \quad M^{\alpha\beta}}$$

d2 scalars

$g_{mn} + B_{mn}$

M^{ab}

n2 scalars

$a = 1, ..., n$

$$\mathcal{H} \in \frac{O(d,d)}{O(d)\times O(d)} \qquad \mathcal{H} \in \frac{O(n,n)}{O(n)\times O(n)}$$

$$\mathcal{H} \in \frac{O(D+d,D+d)}{O(D+d)\times O(D+d)} \qquad \mathcal{H} \in \frac{O(D+n,D+n)}{O(D+n)\times O(D+n)}$$

Fields of reduced theory for bosonic string

$\mathcal{M}_D \times T^d$

rank d rank d

dim n dim n

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

D2 dof

tensors

$g_{\mu\nu}, B_{\mu\nu}$

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2dD dof

2d vectors

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2n vectors

M^{mn}

$$\underbrace{M^{mn} \quad M^{\alpha n} \quad M^{m\beta} \quad M^{\alpha\beta}}$$

d2 scalars

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M^{ab}

n2 scalars

$a = 1, ..., n$

$$\mathcal{H} \in \frac{O(d,d)}{O(d)\times O(d)}$$

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$$\mathcal{H} \in \frac{O(D+d,D+d)}{O(D+d)\times O(D+d)}$$

$$\mathcal{H} \in \frac{O(D+n,D+n)}{O(D+n)\times O(D+n)}$$

DFT

Fields of reduced theory for heterotic string

$\mathcal{M}_D \times T^d$

rank d+16
dim q

$$U(1)^{d+16} \times U(1)^d \longrightarrow G \times U(1)^d$$

$\underbrace{A^m, A^I}_{A^{\hat{I}}}$ $\bar{A}^m$ $\hat{I} = 1, ..., d+16$

$\underbrace{A^m, A^I, A^\alpha, A^{-\alpha}}_{A^a}$ A^m $a = 1, ..., q$

2d+16 vectors

$g_{\mu m} \pm B_{\mu m}$, A^I_μ

q+d vectors

$M^{\hat{I}m}$

(d+16)x d scalars

g_{mn}, B_{mn}, A^I_m

$\underbrace{M^{\hat{I}m} \; M^{\alpha n}}$

M^{am}

q x d scalars

$\mathcal{H} \in \frac{O(d+16,d)}{O(d+16)\times O(d)}$

$\mathcal{H} \in \frac{O(q,d)}{O(q)\times O(d)}$

DFT

0= $M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + p_L^2 + p_R^2$

LMC $0 = 2\left(N - \bar{N} - \frac{1}{2}\right) + p_L^2 - p_R^2$

$p = EZ$

Extra vectors $N = 0, \bar{N}_\mathsf{x} = 1/2$

$p_L^2 - p_R^2 = 2$ LMC

$p_L^2 + p_R^2 = 2$ M2=0

$p_L^2 = 2, p_R = 0$

Extra scalars $N = 0, \bar{N}_\mathsf{y} = \frac{1}{2}$

$p_L^2 = 2, p_R = 0$

Effective action from string theory for bosonic string

Computing 3-point functions <VVV> at a point of enhancement we read off

$$\mathcal{L} \;=\; R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}_a^{\mu\nu}$$

$$+\frac{1}{4} M_{aa'} F^a_{\mu\nu} \bar{F}^{a'\mu\nu} + D_\mu M_{aa'} D^\mu M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}$$

$$H = dB + A^a \wedge F_a + f_{abc} A^a \wedge A^b \wedge A^c$$

$$- \bar{A}^a \wedge \bar{F}_a - \bar{f}_{abc} \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c$$

$$F^a = dA^a + f^a_{bc} A^b \wedge A^c$$

$$D_\mu M^{aa'} = \partial_\mu M^{aa'} + f^a_{bc} A^b_\mu M^{ca'} + f^{a'}_{b'c'} \bar{A}^{b'}_\mu M^{ac'}$$

G. Aldazabal, MG, S. Iguri, M. Mayo, C. Nuñez 15
Y. Cagnacci, MG, S. Iguri, C. Nuñez 17

Effective action from string theory for bosonic string

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$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}^{\mu\nu}_a$$

$$+ \frac{1}{4} M_{aa'} F^a_{\mu\nu} \bar{F}^{a'\mu\nu} + D_\mu M_{aa'} D^\mu M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}$$

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Higgs mechanism

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Effective action from string theory for bosonic string

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$$\begin{aligned}\mathcal{L} \quad = \quad & R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}_a^{\mu\nu} \\ & + \frac{1}{4} M_{aa'} F^a_{\mu\nu} \bar{F}^{a'\mu\nu} + D_\mu M_{aa'} D^\mu M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}\end{aligned}$$

$$\begin{aligned}H = dB + A^a \wedge F_a + f_{abc} A^a \wedge A^b \wedge A^c \\ - \bar{A}^a \wedge \bar{F}_a - \bar{f}_{abc} \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c\end{aligned}$$

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Higgs mechanism

$$M^{mn} = \underbrace{v^{mn}}_{\text{deviation from point of enhancement}} + M'^{mn}$$

$$\delta(g + B)_{mn}$$

G. Aldazabal, MG, S. Iguri, M. Mayo, C. Nuñez 15
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Effective action from string theory for bosonic string

Computing 3-point functions <VVV> at a point of enhancement we read off

$$
\begin{aligned}
\mathcal{L} \;=\;& R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}_a^{\mu\nu} \\
& + \frac{1}{4} M_{aa'} F^a_{\mu\nu} \bar{F}^{a'\mu\nu} + \boxed{D_\mu M_{aa'} D^\mu M^{aa'}} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}
\end{aligned}
$$

$$
\begin{aligned}
H =\;& dB + A^a \wedge F_a + f_{abc} A^a \wedge A^b \wedge A^c \\
& - \bar{A}^a \wedge \bar{F}_a - \bar{f}_{abc} \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c
\end{aligned}
$$

$$F^a = dA^a + f^a_{bc} A^b \wedge A^c$$

$$D_\mu M^{aa'} = \partial_\mu M^{aa'} + f^a_{bc} A^b_\mu M^{ca'} + \bar{f}^{a'}_{b'c'} \bar{A}^{b'}_\mu M^{ac'}$$

A^α
$\bar{A}^\alpha$ acquire mass² $\sim v v^t$

Higgs mechanism

$$M^{mn} = \underbrace{v^{mn}}_{\text{deviation from point of enhancement}} + M'^{mn}$$

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$$+ \frac{1}{4} M_{aa'} F^a_{\mu\nu} \bar{F}^{a'\mu\nu} + \boxed{D_\mu M_{aa'} D^\mu M^{aa'}} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}$$

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$$- \bar{A}^a \wedge \bar{F}_a - \bar{f}_{abc} \bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c$$

$$F^a = dA^a + f^a_{bc} A^b \wedge A^c$$

$$D_\mu M^{aa'} = \partial_\mu M^{aa'} + f^a_{bc} A^b_\mu M^{ca'} + f^{a'}_{b'c'} \bar{A}^{b'}_\mu M^{ac'}$$

A^α
$\bar{A}^\alpha$ acquire mass² $\sim vv^t$

Higgs mechanism

$$M^{mn} = \underbrace{v^{mn}}_{} + M'^{mn}$$

deviation from
point of enhancement $\delta(g+B)_{mn}$

G x G → U^d(1) x U^d(1)

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Effective action from string theory for bosonic string

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$$\mathcal{L} = R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F_{a}^{\mu\nu} + \frac{1}{4}\bar{F}^{a}_{\mu\nu}\bar{F}_{a}^{\mu\nu}$$

$$+\frac{1}{4}M_{aa'}F^{a}_{\mu\nu}\bar{F}^{a'\mu\nu} + D_{\mu}M_{aa'}D^{\mu}M^{aa'} - \frac{1}{12}f_{abc}\bar{f}_{a'b'c'}M^{aa'}M^{bb'}M^{cc'}$$

$M^{\alpha\beta}$

acquire mass² $\sim v$

- cubic potential, unbounded from below

$$H = dB + A^{a} \wedge F_{a} + f_{abc}A^{a} \wedge A^{b} \wedge A^{c}$$

$$- \bar{A}^{a} \wedge \bar{F}_{a} - \bar{f}_{abc}\bar{A}^{a} \wedge \bar{A}^{b} \wedge \bar{A}^{c}$$

$$F^{a} = dA^{a} + f^{a}_{bc}A^{b} \wedge A^{c}$$

$$D_{\mu}M^{aa'} = \partial_{\mu}M^{aa'} + f^{a}_{bc}A^{b}_{\mu}M^{ca'} + f^{a'}_{b'c'}\bar{A}^{b'}_{\mu}M^{ac'}$$

A^{α}
$\bar{A}^{\alpha}$ acquire mass² $\sim vv^{t}$

Higgs mechanism

$$M^{mn} = \underbrace{v^{mn}}_{} + M'^{mn}$$

deviation from
point of enhancement $\delta(g+B)_{mn}$

G x G → U^{d}(1) x U^{d}(1)

G. Aldazabal, MG, S. Iguri, M. Mayo, C. Nuñez 15
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Effective action from string theory for bosonic string

Computing 3-point functions <VVV> at a point of enhancement we read off

$$\mathcal{L} = R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}_a^{\mu\nu}$$

$$+\frac{1}{4}M_{aa'}F^a_{\mu\nu}\bar{F}^{a'\mu\nu} + D_\mu M_{aa'}D^\mu M^{aa'} - \frac{1}{12}f_{abc}\bar{f}_{a'b'c'}M^{aa'}M^{bb'}M^{cc'}$$

$M^{\alpha\beta}$ acquire mass$^2 \sim v$

- cubic potential, unbounded from below
- tachyonic masses

$$H = dB + A^a \wedge F_a + f_{abc}A^a \wedge A^b \wedge A^c$$

$$- \bar{A}^a \wedge \bar{F}_a - \bar{f}_{abc}\bar{A}^a \wedge \bar{A}^b \wedge \bar{A}^c$$

$$F^a = dA^a + f^a_{bc}A^b \wedge A^c$$

$$D_\mu M^{aa'} = \partial_\mu M^{aa'} + f^a_{bc}A^b_\mu M^{ca'} + f^{a'}_{b'c'}\bar{A}^{b'}_\mu M^{ac'}$$

A^α
$\bar{A}^\alpha$ acquire mass^2 $\sim vv^t$

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deviation from
point of enhancement $\delta(g+B)_{mn}$

G x G → U^d(1) x U^d(1)

G. Aldazabal, MG, S. Iguri, M. Mayo, C. Nuñez 15
Y. Cagnacci, MG, S. Iguri, C. Nuñez 17

Effective action from string theory for heterotic string

Computing 3-point functions <VVV> at a point of enhancement we read off

$$\mathcal{L} \;=\; R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{m}_{\mu\nu}\bar{F}^{\mu\nu}_{m} + \frac{1}{4}M_{am}F^{a}_{\mu\nu}\bar{F}^{m\mu\nu}$$

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$$\delta(g+B)_{mn} \,, \delta A^I_m$$

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We get these actions from DFT !!

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Conclusions

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Gives the point in moduli space where the enhancement occurs

$$E_{9-p} \times E_{9-q} \times SU(p+q)$$
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- Heterotic on T^2 : found all the groups that appear in F-theory on K3

Gives strong support to duality!

- There does not seem to exist a (unique) generalized Dynkin diagram

Conclusions

- Effective action:

$O(p,q)$-covariance

$p=q=n$ for bosonic string n=dimension of G (rank d simply-laced group)

$p=n$, $q=k$ for heterotic string n=dimension of G (rank d+16 simply-laced group)

- A^I, A^I n+n non-abelian vectors, M^{IJ} n^2 scalars in adj x adj for bosonic
- A^I, A^m n non-abelian+d abelian vectors, M^{Im} n x d scalars for heterotic
- M^3 potential in the bosonic theory
- M^4 potential in the heterotic theory
- Higgs mechanism describing symmetry breaking

DFT description

DFT description

DFT O(N,$\bar{\text{N}}$) action

Hull & Zweibach 09

$$S = \int dX \left(-\partial_{MN}\mathcal{H}^{MN} + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \right)$$

M=1,...,N+$\bar{\text{N}}$

Equivalent to

$$S = \int dX \, \mathbb{R}$$

generalized Ricci scalar

Coimbra, Strickland-Constable, Waldram 09

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$$\mathcal{H}^{MN}=\delta^{AB}E_A{}^M E_B{}^N \qquad E_A(x,y)=\ U_A{}^{A'}(x)\,E'_{A'}(y)$$

O(N,N̄) ⟶ O(D,D) x O(n-D,n̄-D)

external internal

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$$\partial_M = (\underbrace{\partial_\mu,}_{D}\ \underbrace{\overset{N-D\equiv n}{\downarrow}\!\!\partial_{\color{red}m},\ \overset{\bar{N}-D\equiv\bar{n}}{\downarrow}\!\!\partial_{\color{blue}m},}_{\substack{{\color{red}n}+{\color{blue}\bar{n}}\\ I}}\ \cancel{\partial^\mu})$$

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Claim: this action reproduces the string theory action for compactifications of bosonic and heterotic on T^d close to enhancement point

Action close to a generic point in moduli space (no enhancement) $\mathcal{M}_D \times T^d$

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Let's look at internal piece only : O(d,d(+16))

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(g_0, B_0, A_0)

$\parallel$

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so far indep of y

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Action close to a generic point in moduli space (no enhancement) $\mathcal{M}_D \times T^d$

Let's look at internal piece only : O(d,d(+16))

Frame $E_A{}^M = E$

M=1,...,d+d for bosonic

M=1,...,d+d+16 for heterotic

$$E = \underset{\uparrow}{E_0} + \delta E = \underbrace{(1 + \delta E\, E_0^{-1})}_{U(x)} \underbrace{E_0}_{E'(y)}$$

so far indep of y

$$E(x,y) = U(x) E'(y)$$

constant

(g_0, B_0, A_0)

||

$e_0^t e_0$

$$\mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$$

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so far indep of y

$$\mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$$

$\delta B + \frac{1}{2}(\delta A A_0^t - A_0 \delta A^t)$

we get $\mathcal{M} = \begin{pmatrix} I_d + \frac{1}{2}M^t M & M^t \\ M & I_{d(+16)} + \frac{1}{2}MM^t \end{pmatrix}$ where $M = \begin{pmatrix} \hat{e}_0(\delta G - \delta B')\hat{e}_0^t \\ \delta A\, \hat{e}_0^t \end{pmatrix}$

Action close to a generic point in moduli space (no enhancement)

$\mathcal{M}_D \times T^d$

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Plug that in

Action close to a generic point in moduli space (no enhancement)

$$\mathcal{M}_D \times T^d$$

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constant

(g_0, B_0, A_0)

||

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so far indep of y

$$\mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$$

$$\delta B + \tfrac{1}{2}(\delta A A_0^t - A_0 \delta A^t)$$
$$\downarrow$$

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Plug that in

$$\begin{aligned}\mathcal{L} \;&=\; R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^J_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ &\quad -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right)\end{aligned}$$

Action close to a generic point in moduli space (no enhancement)

$\mathcal{M}_D \times T^d$

Let's look at internal piece only : O(d,d(+16))

Frame $E_A{}^M = E$

M=1,...,d+d for bosonic
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$$E = \overset{\uparrow}{E_0} + \delta E = \underbrace{(1 + \delta E\, E_0^{-1})}_{U(x)}\underbrace{E_0}_{E'(y)} \qquad E(x,y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I{}_{JK}E'_K$$

constant
(g_0, B_0, A_0)
||
$e_0^t e_0$

so far
indep of y

$$\mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$$

$\delta B + \frac{1}{2}(\delta A A_0^t - A_0 \delta A^t)$
⇣

we get $\mathcal{M} = \begin{pmatrix} I_d + \frac{1}{2}M^t M & M^t \\ M & I_{d(+16)} + \frac{1}{2}MM^t \end{pmatrix}$ where $M = \begin{pmatrix} \hat{e}_0(\delta G - \delta B')\hat{e}_0^t \\ \delta A\, \hat{e}_0^t \end{pmatrix}$

Plug that in

$$\begin{aligned} \mathcal{L} \;=\; & R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^J_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ & -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right) \end{aligned}$$

Action close to a generic point in moduli space (no enhancement) $\mathcal{M}_D \times T^d$

Let's look at internal piece only : O(d,d(+16))

Frame $E_A{}^M = E$

M=1,...,d+d for bosonic

M=1,...,d+d+16 for heterotic

$$E = \underset{\substack{\uparrow \\ \text{constant}}}{E_0} + \delta E \;= \underbrace{(1+\delta E\, E_0^{-1})}_{U(x)}\underbrace{E_0}_{E'(y)} \qquad E(x,y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I{}_{JK}E'_K$$

(g_0, B_0, A_0)

||

$e_0^t e_0$

so far indep of y

$$\mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$$

we get
$$\mathcal{M} = \begin{pmatrix} I_d + \frac{1}{2}M^t M & M^t \\ M & I_{d(+16)} + \frac{1}{2}MM^t \end{pmatrix}$$
where
$$M = \begin{pmatrix} \hat{e}_0(\delta G - \overset{\substack{\delta B + \frac{1}{2}(\delta A A_0^t - A_0 \delta A^t) \\ \downarrow}}{\delta B'})\hat{e}_0^t \\ \delta A\, \hat{e}_0^t \end{pmatrix}$$

Plug that in

$$\mathcal{L} \;=\; R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^J_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(\overset{\substack{\partial_\mu M \\ \downarrow}}{D^\mu}\mathcal{M})^{IJ}$$
$$\cancel{-\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right)}$$

Action close to a generic point in moduli space (no enhancement) $\mathcal{M}_D \times T^d$

Let's look at internal piece only : O(d,d(+16))

Frame $E_A{}^M = E$ M=1,...,d+d for bosonic

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$$E = \overset{\uparrow}{E_0} + \delta E = \underbrace{(1+\delta E\, E_0^{-1})}_{U(x).}\underbrace{E_0}_{E'(y)} \qquad E(x,y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I{}_{JK}E'_K$$

constant

(g_0, B_0, A_0)

||

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so far indep of y

$$\mathcal{H} = E^tE = E'^tU^TUE' \equiv E'^t\mathcal{M}E'$$

we get $\mathcal{M} = \begin{pmatrix} I_d + \frac{1}{2}M^tM & M^t \\ M & I_{d(+16)} + \frac{1}{2}MM^t \end{pmatrix}$ where $M = \begin{pmatrix} \hat{e}_0(\delta G - \delta \overset{\delta B + \frac{1}{2}(\delta A A_0^t - A_0\delta A^t)}{\overset{\downarrow}{B'}})\hat{e}_0^t \\ \delta A\, \hat{e}_0^t \end{pmatrix}$

Plug that in

$$\mathcal{L} = R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^J_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(\overset{\partial_\mu M}{\overset{\downarrow}{D^\mu}}\mathcal{M})^{IJ}$$

$$\cancel{-\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right)}$$

No enhancement of symmetry

no double field theory

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

$$E(x,y) = U(x)E'(y)$$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

$\overset{d}{y_R^m}, \quad \overset{d(+16)}{y_L^m}$

"Double coordinates": only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

Action close to a special point in moduli space (enhancement)

$\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
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$$\overset{d}{y_R^m}\,, \quad \overset{d(+16)}{y_L^m}$$

“Double coordinates”: only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

$$\mathcal{H} = E^t E = E'^t(y)\mathcal{M}(x)E'(y)$$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

$\overset{d}{y_R^m}, \ \overset{d(+16)}{y_L^m}$ "Double coordinates": only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

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$$\Big\downarrow \, U^TU$$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
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$$\overset{d}{y_R^m},\ \overset{d(+16)}{y_L^m}$$

"Double coordinates": only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

$$\mathcal{H} = E^t E = E'^t(y)\mathcal{M}(x)E'(y)$$

$$\Big\downarrow U^T U$$

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

$$\uparrow$$

$$M^{aa'}$$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

d d(+16)

$y_R^m\,,\ \ y_L^m$

“Double coordinates”: only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

$$\mathcal{H} = E^t E = E'^t(y)\mathcal{M}(x)E'(y)$$

$U^T U$

$$[E'_J, E'_K]_C = f^I{}_{JK} E'_K$$

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

$M^{aa'}$

Action close to a special point in moduli space (enhancement) $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{\text{n}}$+n

- for bosonic $\bar{\text{n}}$=n= dim of group G$_{\text{bos}}$ of rank d
- for heterotic $\bar{\text{n}}$=d, n=dim of group G$_{\text{het}}$ of rank d+16

$$\overset{d}{y_R^m}, \quad \overset{d(+16)}{y_L^m}$$

"Double coordinates": only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

$$\mathcal{H} = E^t E = E'^t(y)\mathcal{M}(x)E'(y)$$

$$\Big\downarrow U^T U$$

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$$\uparrow$$

$$M^{aa'}$$

$$\dashrightarrow [E'_J, E'_K]_C = f^I{}_{JK}E'_K$$

E' such that

- for bosonic $f_{ab}{}^c$ = $f_{ab}{}^c$ = str const of G$_{\text{bos}}$

Action close to a special point in moduli space (enhancement)

$\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,...,$\bar{n}$+n

- for bosonic $\bar{n}$=n= dim of group G_{bos} of rank d
- for heterotic $\bar{n}$=d, n=dim of group G_{het} of rank d+16

d d(+16)

$y_R^m, \; y_L^m$

“Double coordinates”: only the Cartan directions: d+d(+16)

$$E(x,y) = U(x)E'(y)$$

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$U^T U$

$$[E'_J, E'_K]_C = f^I{}_{JK}E'_K$$

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^tM & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$M^{aa'}$

E’ such that

- for bosonic $f_{ab}{}^c$ = $f_{ab}{}^c$ = str const of G_{bos}
- for heterotic $f_{ab}{}^c$= 0, $f_{ab}{}^c$ =str const of G_{het}

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} &= R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^{J}_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ &\quad -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right) \end{aligned}$$

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

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And we get

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2} M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2} M M^t \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} \;&=\; R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_\mu \mathcal{M})_{IJ} (D^\mu \mathcal{M})^{IJ} \\ &\quad - \frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3\, \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2\, \eta^{IL} \eta^{JM} \eta^{KN} \right) \end{aligned}$$

And we get

$$\begin{aligned} \mathcal{L} \;&=\; R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F_{a}^{\mu\nu} + \frac{1}{4} \bar{F}^{a}_{\mu\nu} \bar{F}_{a}^{\mu\nu} + \frac{1}{4} M_{aa'} F^{a}_{\mu\nu} \bar{F}^{a'\mu\nu} \\ &\quad + D_\mu M_{aa'} D^\mu M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'} \end{aligned}$$

bosonic

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^tM & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} \;&=\; R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^J_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ &\quad -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right) \end{aligned}$$

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bosonic

$$\begin{aligned} \mathcal{L} \;&=\; R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{4}\bar{F}^m_{\mu\nu}\bar{F}^{\mu\nu}_m + \frac{1}{4}M_{am}F^a_{\mu\nu}\bar{F}^{m\mu\nu} \\ &\quad +D_\mu M_{am}D^\mu M^{am} - \frac{1}{12}f_{abc}f_{a'b'c}M^{am}M^{a'm}M^{bn}M^{b'n} \end{aligned}$$

heterotic

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} \;=\; & R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^{J}_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ & -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right) \end{aligned}$$

And we get

$$\begin{aligned} \mathcal{L} \;=\; & R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{a}_{\mu\nu}\bar{F}^{\mu\nu}_{a} + \frac{1}{4}M_{aa'}F^{a}_{\mu\nu}\bar{F}^{a'\mu\nu} \\ & +D_\mu M_{aa'}D^\mu M^{aa'} - \frac{1}{12}f_{abc}\bar{f}_{a'b'c'}M^{aa'}M^{bb'}M^{cc'} \end{aligned}$$

bosonic

$$\begin{aligned} \mathcal{L} \;=\; & R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{m}_{\mu\nu}\bar{F}^{\mu\nu}_{m} + \frac{1}{4}M_{am}F^{a}_{\mu\nu}\bar{F}^{m\mu\nu} \\ & +D_\mu M_{am}D^\mu M^{am} - \frac{1}{12}f_{abc}f_{a'b'c}M^{am}M^{a'm}M^{bn}M^{b'n} \end{aligned}$$

heterotic

Precisely the string theory actions !

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^tM & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$$\begin{aligned}\mathcal{L} \;=\;& R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^{J}_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ & -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right)\end{aligned}$$

And we get

$$\begin{aligned}\mathcal{L} \;=\;& R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{a}_{\mu\nu}\bar{F}^{\mu\nu}_{a} + \frac{1}{4}M_{aa'}F^{a}_{\mu\nu}\bar{F}^{a'\mu\nu} \\ & + D_\mu M_{aa'}D^\mu M^{aa'} - \boxed{\frac{1}{12}f_{abc}\bar{f}_{a'b'c'}M^{aa'}M^{bb'}M^{cc'}}\end{aligned}$$

bosonic

•cubic potential, unbounded from below

$$\begin{aligned}\mathcal{L} \;=\;& R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{m}_{\mu\nu}\bar{F}^{\mu\nu}_{m} + \frac{1}{4}M_{am}F^{a}_{\mu\nu}\bar{F}^{m\mu\nu} \\ & + D_\mu M_{am}D^\mu M^{am} - \frac{1}{12}f_{abc}f_{a'b'c}M^{am}M^{a'm}M^{bn}M^{b'n}\end{aligned}$$

heterotic

Precisely the string theory actions !

Plug that in

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^t M & M^t \\ M & I_{\bar{n}} + \frac{1}{2}M M^t \end{pmatrix}$$

$$\begin{aligned} \mathcal{L} \ &= \ R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}\mathcal{M}_{IJ}F^{I\mu\nu}F^{J}_{\mu\nu} + (D_\mu\mathcal{M})_{IJ}(D^\mu\mathcal{M})^{IJ} \\ &\quad -\frac{1}{12}f_{IJK}f_{LMN}\left(\mathcal{M}^{IL}\mathcal{M}^{JM}\mathcal{M}^{KN} - 3\,\mathcal{M}^{IL}\eta^{JM}\eta^{KN} + 2\,\eta^{IL}\eta^{JM}\eta^{KN}\right) \end{aligned}$$

And we get

$$\begin{aligned} \mathcal{L} \ &= \ R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a} + \frac{1}{4}\bar{F}^{a}_{\mu\nu}\bar{F}^{\mu\nu}_{a} + \frac{1}{4}M_{aa'}F^{a}_{\mu\nu}\bar{F}^{a'\mu\nu} \\ &\quad + D_\mu M_{aa'}D^\mu M^{aa'} - \boxed{\frac{1}{12}f_{abc}\bar{f}_{a'b'c'}M^{aa'}M^{bb'}M^{cc'}} \end{aligned}$$

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heterotic

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bosonic

- •cubic potential, unbounded from below
- •tachyonic masses

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heterotic

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bosonic

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- tachyonic masses

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heterotic

- quartic potential
- positive masses $m_\alpha^2 = |\alpha \cdot v|^2$

Precisely the string theory actions !

Higgs mechanism $M^{\hat{I}n} = v^{\hat{I}n} + M'^{\hat{I}n}$

What about E’?

–

What about E’?

$$[E'_J, E'_K]_C = f^I{}_{JK} E'_K \qquad -$$

What about E'? $[E'_J, E'_K]_C = f^I{}_{JK} E'_K$

$I=1,\ldots,\bar{n}+n$

left and right split
discuss the left part

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I=1,...,$\bar{n}$+n

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discuss the left part

$$[V_1, V_2]_C = \frac{1}{2}(\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1)$$ **C-bracket**

$$(\mathcal{L}_{V_1} V_2)^I = V_1^J \partial_J V_2^I + \ (\partial^I V_{1J} - \partial_J V_1^I) V_2^J$$ generalized Lie derivative

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$$n = d(+16) + 2p$$

Cartans Roots $\alpha^{\pm}$

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The following E' does the job for SU(2) algebra

$$E' = \begin{pmatrix} e^{\sqrt{2} i y^L} & 0 & 0 \\ 0 & e^{-\sqrt{2} i y^L} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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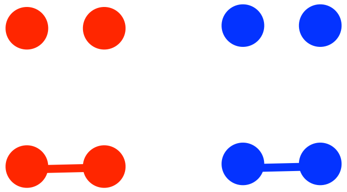
Straightforward generalization to SU(2)^d

What about other enhancement groups?

T^2
bosonic



$$SU(2)^2 \times SU(2)^2$$

$$SU(3) \times SU(3)$$

What about other enhancement groups?

T^2
bosonic

$$\color{red}{SU(2)^2} \times \color{blue}{SU(2)^2}$$

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3 positive roots : 2 simple, 1 non-simple

What about other enhancement groups?

T^2

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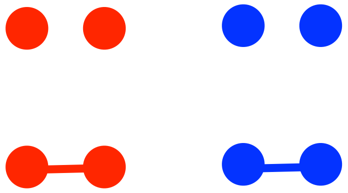

$SU(2)^2 \times SU(2)^2$

$SU(3) \times SU(3)$ 3 positive roots : 2 simple, 1 non-simple

for simple roots
same construction

$$[E'_\alpha, E'_{-\alpha}] = \alpha \cdot H$$

$$[E'_\alpha, H_m] = \alpha^m$$

$E_\alpha \sim e^{i\alpha \cdot y_L}$

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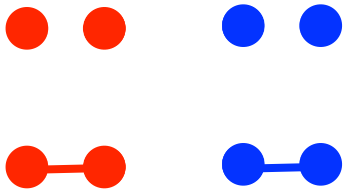

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√

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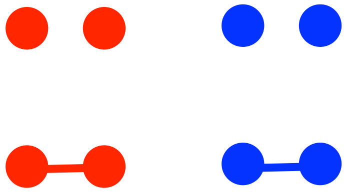

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for non simple roots no coordinate associated

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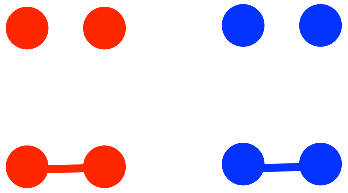

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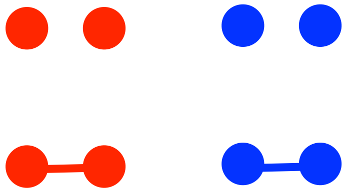
bosonic



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$$[E'_\alpha, E'_{-\alpha}] = \alpha\cdot H$$

$$[E'_\alpha, H_m] = \alpha^m$$ ✓

for non simple roots no coordinate associated

still $[E'_{\alpha_1+\alpha_2}, H_m] = (\alpha_1+\alpha_2)^m$ ✓

What about other enhancement groups?

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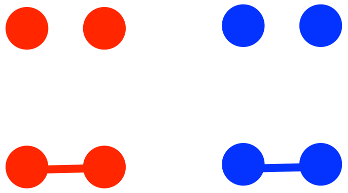
bosonic



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still $[E'_{\alpha_1+\alpha_2}, H_m] = (\alpha_1 + \alpha_2)^m$ ✓

but $[E'_{\alpha_1}, E'_{\alpha_2}] = E'_{\alpha_1+\alpha_2}$???

does not arise from any
obvious extension of the
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What about other enhancement groups?

T^2
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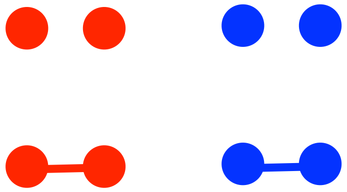

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$SU(3) \times SU(3)$ 3 positive roots : 2 simple, 1 non-simple

Deformed generalized Lie derivative

for simple roots
same construction

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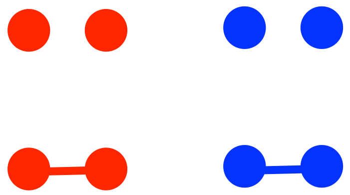

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$\surd$

Deformed generalized Lie derivative

$$\tilde{\mathcal{L}}_{E_I} E_J = \mathcal{L}_{E_I} E_J + \Omega_{IJ}{}^K E_K$$

for non simple roots no coordinate associated

still $[E'_{\alpha_1+\alpha_2}, H_m] = (\alpha_1 + \alpha_2)^m$ $\surd$

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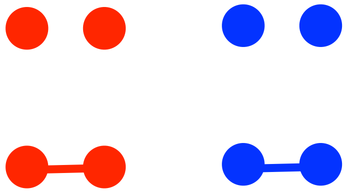

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Deformed generalized Lie derivative

$$\tilde{\mathcal{L}}_{E_I} E_J = \mathcal{L}_{E_I} E_J + \Omega_{IJ}{}^K E_K$$

Cocycle tensor

$$\underset{\alpha\beta\gamma}{\Omega_{IJK}} = \begin{cases} (-1)^{\alpha*\beta}\ \delta_{\alpha+\beta+\gamma} & \text{if two roots are positive} \\ -(-1)^{\alpha*\beta}\ \delta_{\alpha+\beta+\gamma} & \text{if two roots are negative} \end{cases}$$

for simple roots
same construction

$E_\alpha \sim e^{i\alpha\cdot y_L} \equiv e^{iy^\alpha}$

$[E'_\alpha, E'_{-\alpha}] = \alpha \cdot H$

$[E'_\alpha, H_m] = \alpha^m$ √

for non simple roots no coordinate associated

still $[E'_{\alpha_1+\alpha_2}, H_m] = (\alpha_1+\alpha_2)^m$ √

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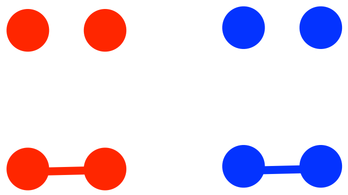

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Deformed generalized Lie derivative

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This reproduces $\boxed{[E'_J, E'_K]_{\tilde{C}} = f^I{}_{JK} E'_K}$ for any group

Heterotic SO(32) and E8 x E8 have exactly the same cocycles !

Supports idea that heterotic SO(32) and E8 x E8 different vacua of same theory Dixon, Harvey, Vafa, Witten 86

Conclusions

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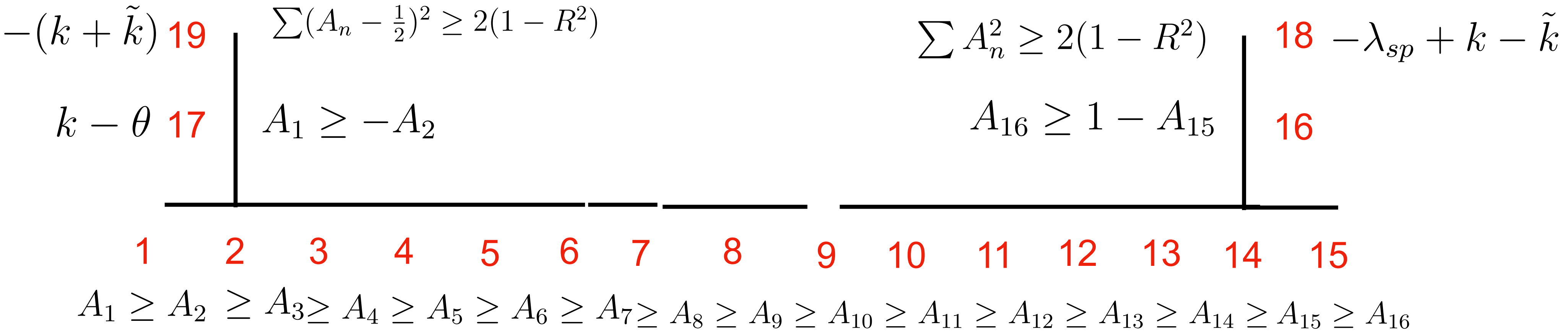
- For T^d, is there a vielbein depending on Cartan coordinates only that satisfies algebra under ordinary bracket?



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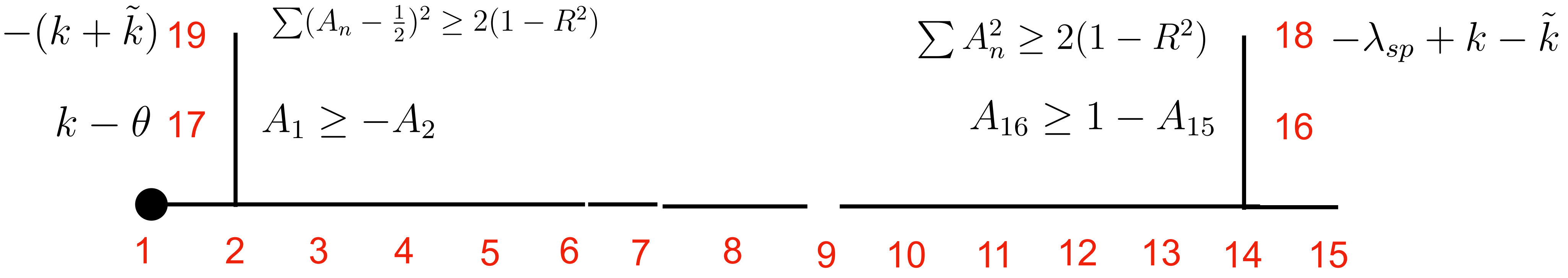

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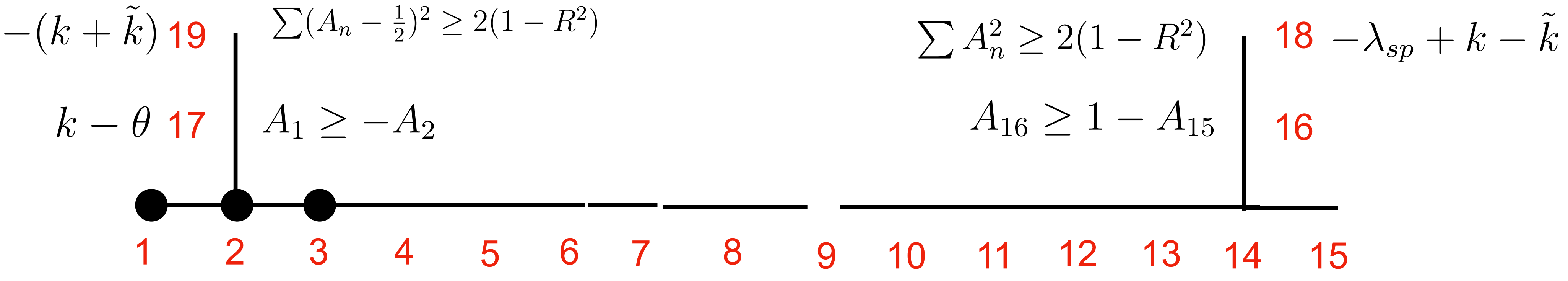

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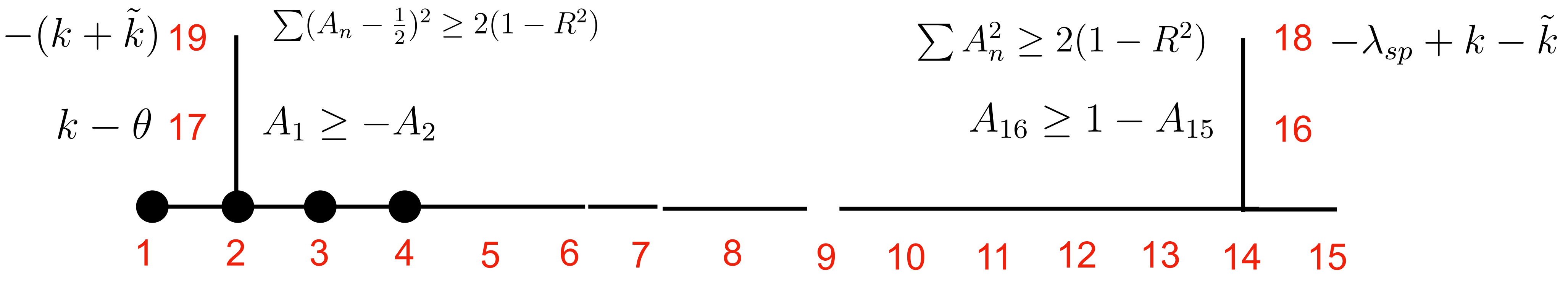

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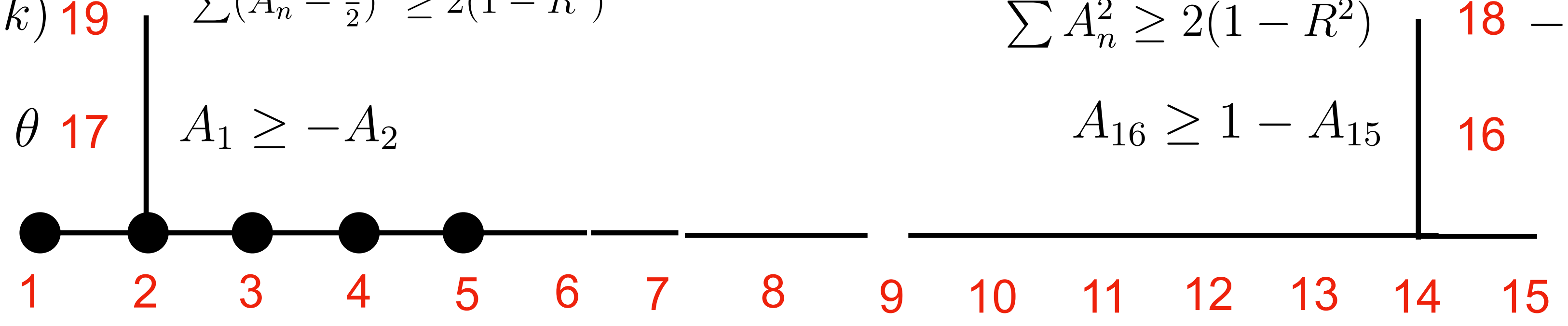

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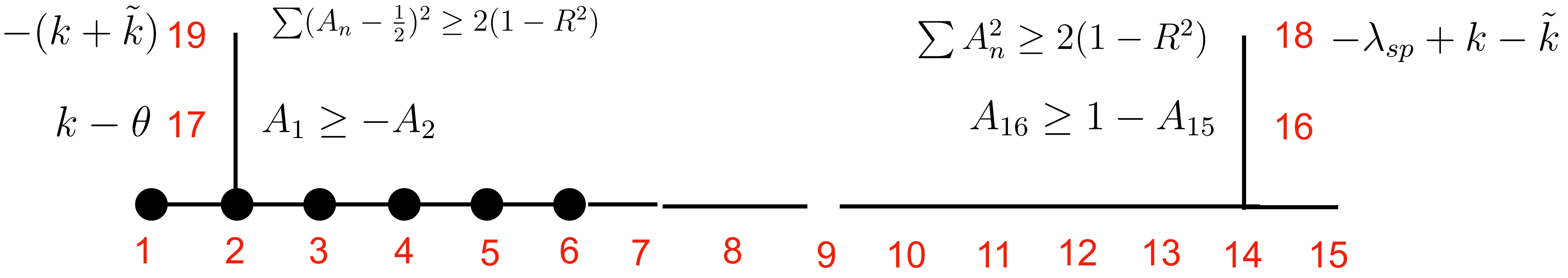

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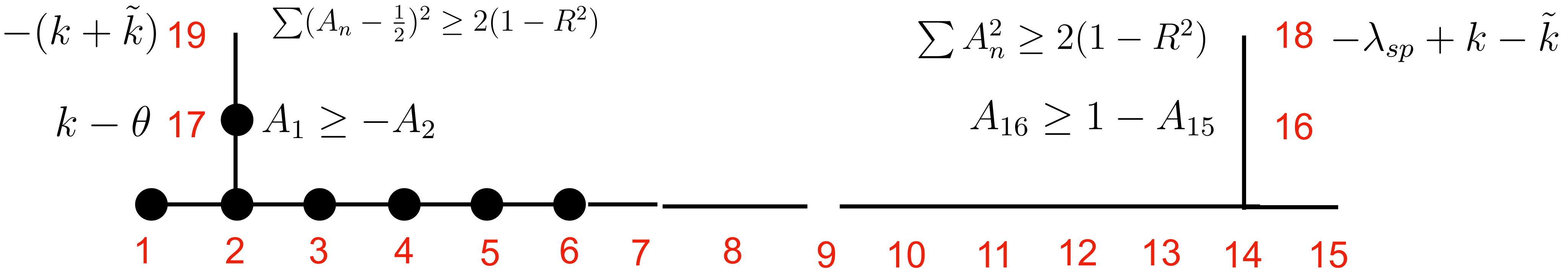

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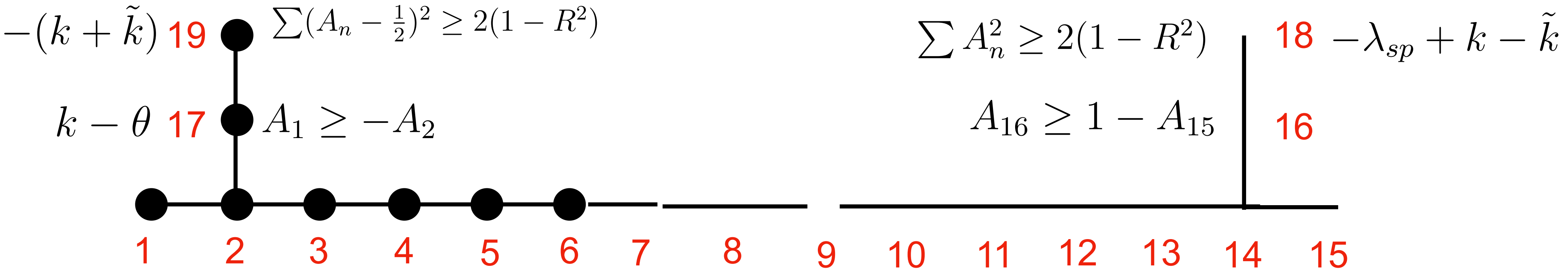

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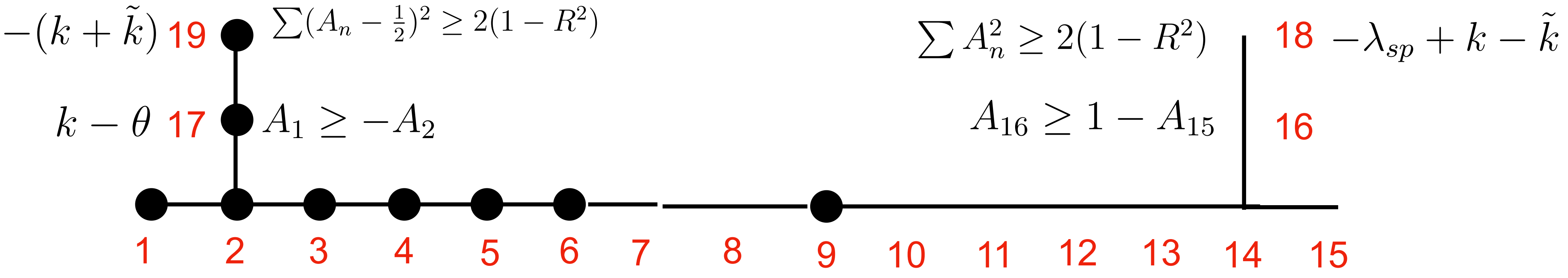

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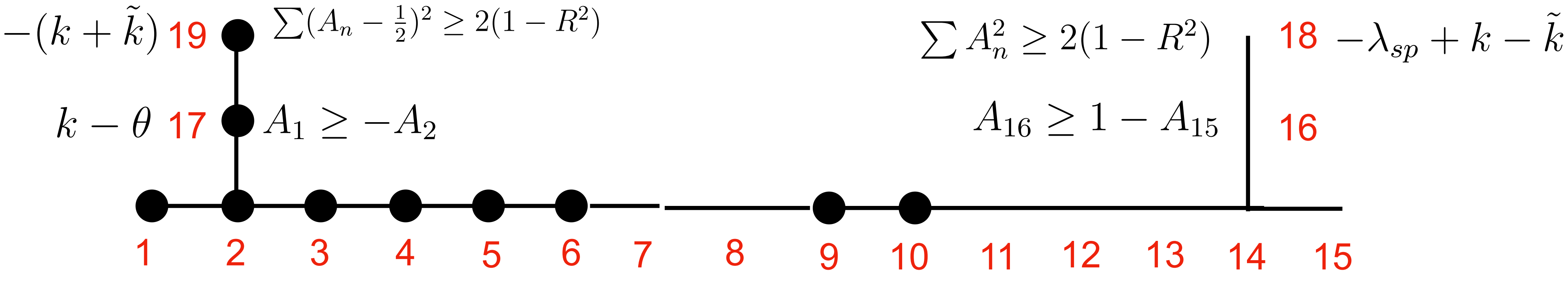

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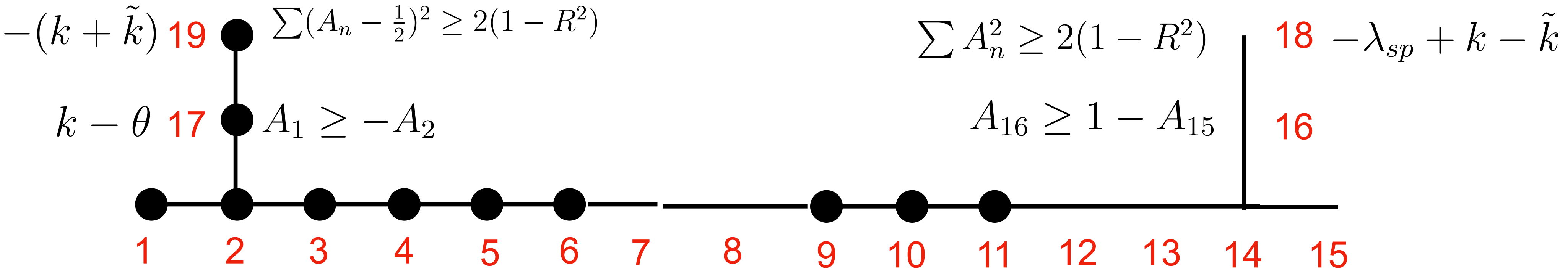

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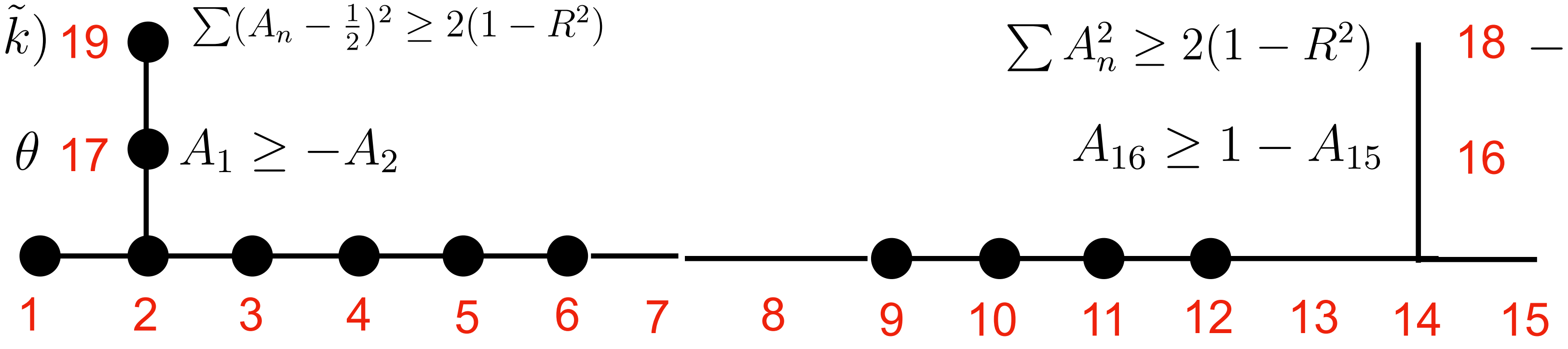

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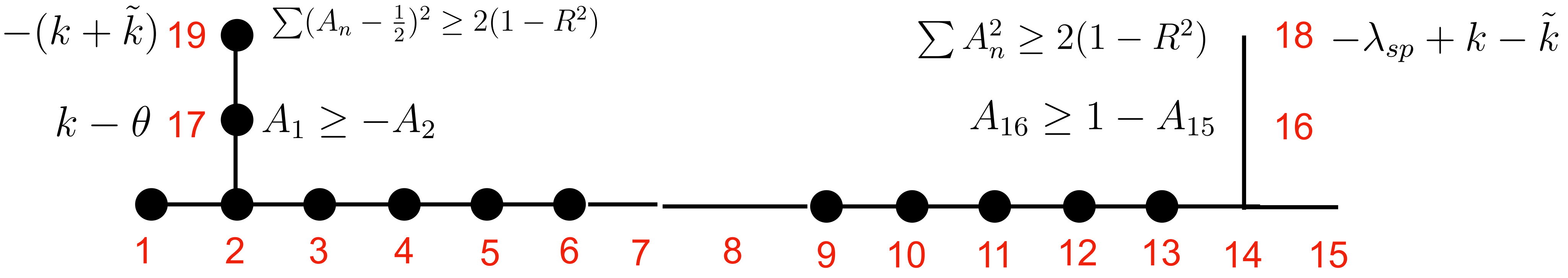

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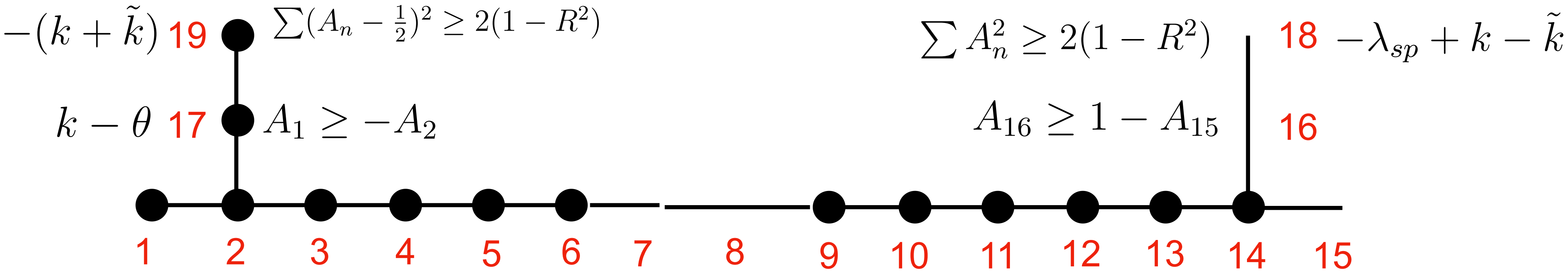

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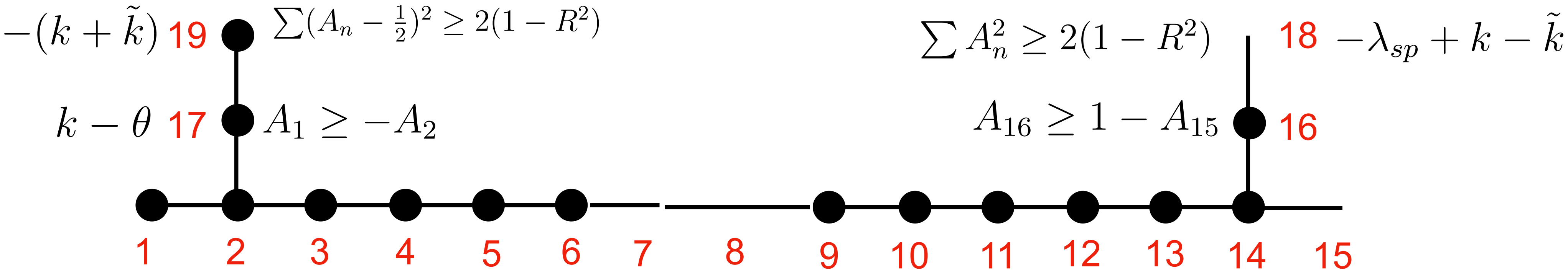

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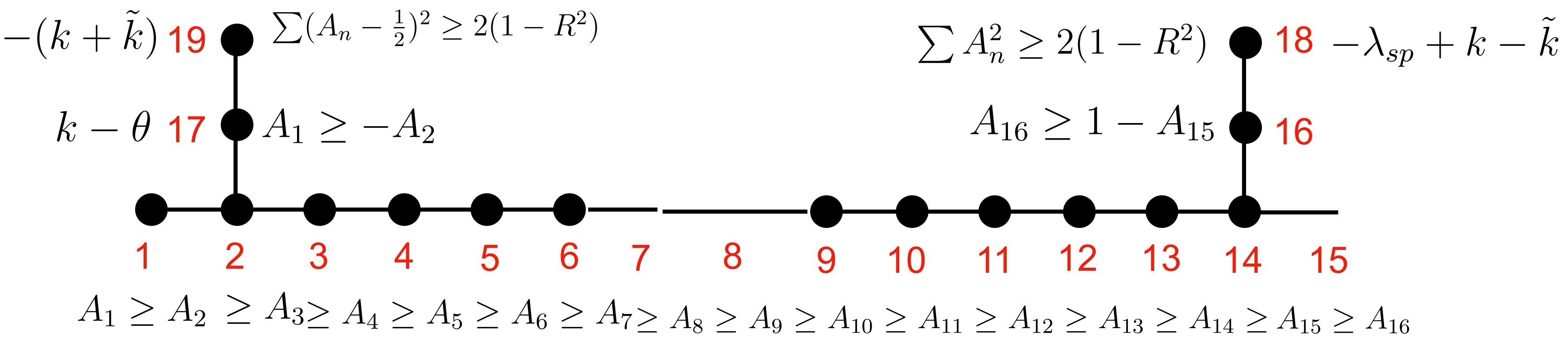

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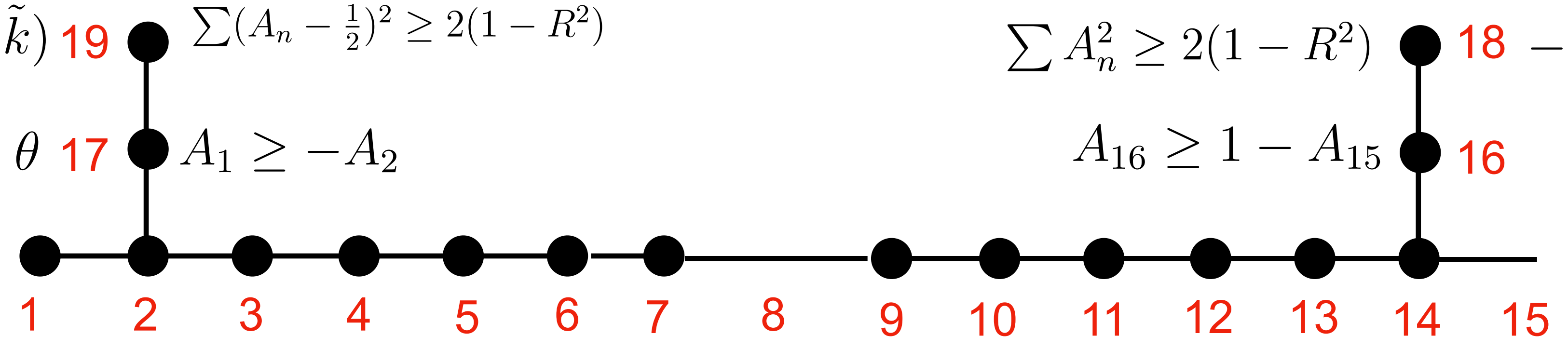

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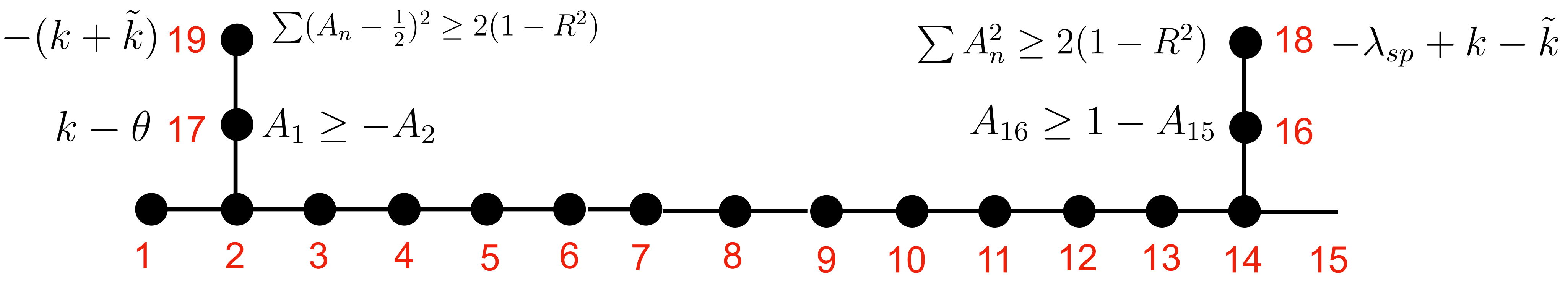

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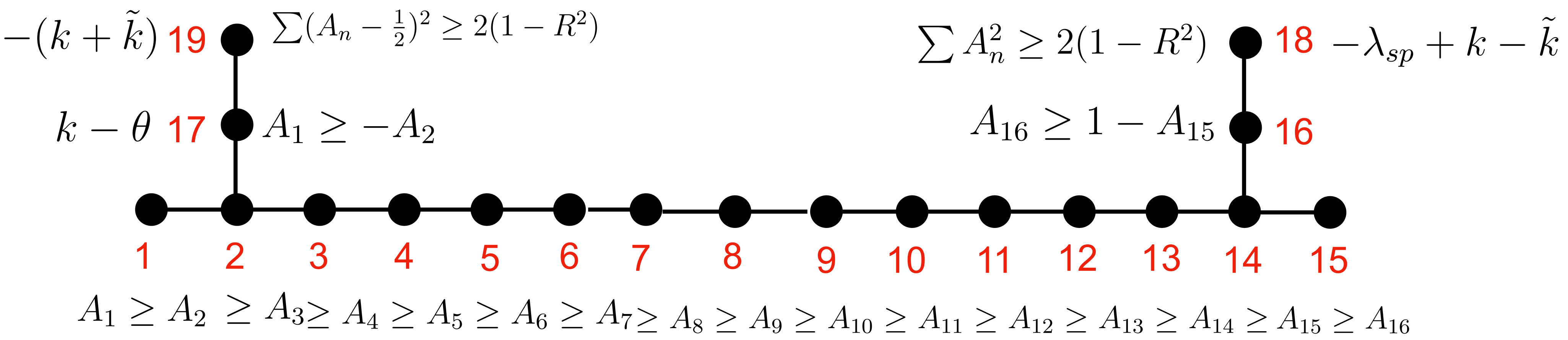

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