Exploring the moduli space of toroidal string compactifications

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In collaboration with

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Simplest: bosonic string on S¹:

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Level-matching $\bar{N} - N = p\tilde{p}$

Simplest: bosonic string on S¹:

Hamiltonian
$$M^{2} = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^{2}}{R^{2}} + \frac{\tilde{p}^{2}}{\tilde{R}^{2}} \qquad \qquad \tilde{R} = \frac{\alpha'}{R}$$

Level-matching $\bar{N} - N = p\tilde{p}$

Massless states: $N = \bar{N} = 1$: $g_{\mu\nu}, B_{\mu\nu}, \phi$

Simplest: bosonic string on S¹:



At $R = \tilde{R} = \sqrt{\alpha'}$ Extra massless states for ex: $\bar{N} = 1, N = 0$ $p = \tilde{p} = \pm 1$

Simplest: bosonic string on S¹:



• Fully understand compactifications on T^d at all points in moduli space

Simplest: bosonic string on S¹:



At $R = \tilde{R} = \sqrt{\alpha'}$ =1 Extra massless states for ex: $\bar{N} = 1, N = 0$ $p = \tilde{p} = \pm 1$

• Fully understand compactifications on T^d at all points in moduli space

• Get an effective description of the physics valid at $E << \frac{1}{R} \sim \frac{1}{\sqrt{\alpha'}} = 1$ $\alpha' = 1$

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At $R = \tilde{R} = \sqrt{\alpha'}$ =1 Extra massless states for ex: $\bar{N} = 1, N = 0$ $p = \tilde{p} = \pm 1$

• Fully understand compactifications on T^d at all points in moduli space

• Get an effective description of the physics valid at $E << \frac{1}{R} \sim \frac{1}{\sqrt{\alpha'}} = 1$ From string scattering amplitudes

Simplest: bosonic string on S¹:



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• Fully understand compactifications on T^d at all points in moduli space

• Get an effective description of the physics valid at $E << \frac{1}{R} \sim \frac{1}{\sqrt{\alpha'}} = 1$ \downarrow From string scattering amplitudes Extra massless states have momentum and/or winding on circle

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Massless states at $R = \tilde{R} = 1$

Mass
$$M^2 = 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2}$$

Level-matching $\bar{N} - N = p\tilde{p}$

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Level-matching $\bar{N} - N = p\tilde{p}$
 $k_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}}\right)$

-
$$N_y = 1$$
 $(g_{\mu y} + B_{\mu y})$

Vectors $\ ar{N}_x = 1$

-
$$N_y = 0$$
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Mass
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Level-matching $\bar{N} - N = p\tilde{p}$
 $k_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right)$
 $V \sim J^3(z) \cdot (\bar{\partial} X^\mu e^{ikX})$

$$V \sim J^{\pm}(z) \cdot (\bar{\partial} X^{\mu} e^{ikX})$$

$$J^{3}(z) = \partial Y^{L}(z)$$
$$J^{\pm}(z) = e^{\pm \sqrt{2}iY^{L}(z)}$$

Massless states at $R=\tilde{R}=1$

Vectors
$$\ \bar{N}_x = 1$$

-
$$N_y = 1$$
 $(g_{\mu y} + B_{\mu y})$

$$-N_y = 0$$
 $p = \tilde{p} = \pm 1$ $(k_L = \pm \sqrt{2})$

Mass
$$M^2 = 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2}$$

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Bosonic string on S¹ Mass $M^2 = 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2}$ Massless states at $R = \tilde{R} = 1$ Level-matching $\bar{N} - N = p\tilde{p}$ $k_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right)$ • SU(2) Vectors $\bar{N}_x = 1$ A_{μ}^{3} - $N_y = 1$ $(g_{\mu y} + B_{\mu y})$

$$-N_y = 0$$
 $p = \tilde{p} = \pm 1$ $(k_L = \pm \sqrt{2})$: A^{\pm}_{μ}

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Massless states at $~R=\tilde{R}=1$

• SU(2)_L Vectors $ar{N}_x=1$

-
$$N_y = 1$$
 $(g_{\mu y} + B_{\mu y})$: A_{μ}^3

-
$$N_y = 0$$
 $p = \tilde{p} = \pm 1$ $(k_L = \pm \sqrt{2})$: A^{\pm}_{μ}

• SU(2)_R Vectors $N_x = 1$ $A^i \to \bar{A}^i$

$$\begin{split} \text{Mass} \quad M^2 &= 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2} \\ \text{Level-matching} \quad \bar{N} - N &= p\tilde{p} \\ k_{L,R} &= \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right) \\ \text{A}^3_{\mu} \qquad V \sim J^3(z) \cdot (\bar{\partial} X^{\mu} e^{ikX}) \\ \text{A}^{\pm}_{\mu} \qquad V \sim J^{\pm}(z) \cdot (\bar{\partial} X^{\mu} e^{ikX}) \\ J^i(z) \to \bar{J}^i(\bar{z}) \qquad Y^L(z) \to Y^R(\bar{z}) \end{split}$$

$$J^{3}(z) = \partial Y^{L}(z)$$
$$J^{\pm}(z) = e^{\pm \sqrt{2}iY^{L}(z)}$$

$$J^{i}(z)J^{j}(0) \sim \frac{\delta^{ij}}{z^{2}} + \frac{i\,\epsilon^{ijk}}{z}J^{k}(0)$$

Massless states at $R = \tilde{R} = 1$

• SU(2)_L Vectors $\bar{N}_x=1$

-
$$N_y = 1$$
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-
$$N_y = 0$$
 $p = \tilde{p} = \pm 1$ $(k_L = \pm \sqrt{2})$: A^{\pm}_{μ}

• SU(2)_R Vectors $N_x = 1$ $A^i \to \bar{A}^i$

• Scalars $N_x = \bar{N}_x = 0$

$$\begin{split} \text{Mass} \quad M^2 &= 2(N + \bar{N} - 2) + \frac{p^2}{R^2} + \frac{\tilde{p}^2}{\tilde{R}^2} \\ \text{Level-matching} \quad \bar{N} - N &= p\tilde{p} \\ k_{L,R} &= \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right) \\ \text{A}^3_{\mu} \qquad V \sim J^3(z) \cdot (\bar{\partial} X^{\mu} e^{ikX}) \\ \text{A}^{\pm}_{\mu} \qquad V \sim J^{\pm}(z) \cdot (\bar{\partial} X^{\mu} e^{ikX}) \\ J^i(z) \to \bar{J}^i(\bar{z}) \qquad Y^L(z) \to Y^R(\bar{z}) \end{split}$$

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Bosonic string on S ¹	Mass	Λ
Massless states at $R={\tilde R}=1$	Level-r	nato
• SU(2)L Vectors $ar{N}_x=1$		
- $N_y = 1$ $(g_{\mu y} + B_{\mu y})$:	$\stackrel{3}{A_{\mu}}$	
- $N_y = 0$ $p = \tilde{p} = \pm 1$ $(k_L = \pm \sqrt{2})$:	A^{\pm}_{μ}	
• SU(2) _R Vectors $N_x = 1$ $A^i ightarrow ar{A}^i$		
• Scalars $N_x = \bar{N}_x = 0$		
$N_y = 1, \bar{N}_y = 1 \ (g_{yy})$		
$N_y = 1, p = -\tilde{p} = \pm 1 \ (k_R = \pm \sqrt{2})$		
$\bar{N}_y = 1, p = \tilde{p} = \pm 1 \ (k_L = \pm \sqrt{2})$		
$p = \pm 2, \tilde{p} = 0 \ (k_L = k_R = \pm \sqrt{2})$		
$p = 0, \tilde{p} = \pm 2 \ (k_{\!L} = -k_{\!R} = \pm \sqrt{2})$		

$M^2 = 2(N + \bar{N} - 2)$ atching $\bar{N} - N = p\tilde{p}$	$)+rac{p^2}{R^2}+rac{ ilde{p}^2}{ ilde{R}^2}$
k_{j}	$L_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right)$
$V \sim J^3(z) \cdot (\bar{\partial} X^\mu e$	$e^{ikX})$
$V \sim J^{\pm}(z) \cdot (\bar{\partial} X^{\mu} \epsilon)$	$e^{ikX})$
$J^i(z) o ar{J^i}(ar{z})$	$Y^L(z) \to Y^R(\bar{z})$

$$J^{3}(z) = \partial Y^{L}(z)$$
$$J^{\pm}(z) = e^{\pm \sqrt{2}iY^{L}(z)}$$

$$J^{i}(z)J^{j}(0) \sim \frac{\delta^{ij}}{z^{2}} + \frac{i\,\epsilon^{ijk}}{z}J^{k}(0)$$

Bosonic string on S ¹		Mass	N
Massless states at $R= ilde{R}=1$		Level-m	nato
• SU(2)L Vectors $ar{N}_x=1$			
- $N_y = 1$ $(g_{\mu y} + B_{\mu y})$		$: \overset{3}{A_{\mu}}$	
$-N_y = 0 p = \tilde{p} = \pm 1 (k_L = \pm 1)$	$\left(\sqrt{2}\right)$: A^{\pm}_{μ}	
• SU(2) _R Vectors $N_x = 1$ $A^i ightarrow 1$	$ar{A}^i$		
• Scalars (3,3) $N_x=ar{N}_x=0$			
$N_y = 1, \bar{N}_y = 1 \ (g_{yy})$	•	M^{33}	
$N_y = 1, p = -\tilde{p} = \pm 1 \ (k_R = \pm \sqrt{2})$	•	$M^{3\pm}$	
$ar{N}_y = 1, p = ilde{p} = \pm 1 \; (k_L = \pm \sqrt{2})$	•	$M^{\pm 3}$	
$p=\pm 2, {\widetilde p}=0 \; (k_{L}\!\!=k_{R}\!\!=\!\pm \sqrt{2})$	•	$M^{\pm\pm}$	
$p = 0, \tilde{p} = \pm 2 \ (k_{\!L} = -k_{\!R} = \pm \sqrt{2})$	•	$M^{\pm\mp}$	
$V^{ij} \sim J^i J^j e^{ikX}$			

$$M^{2} = 2(N + \bar{N} - 2) + \frac{p^{2}}{R^{2}} + \frac{\tilde{p}^{2}}{\tilde{R}^{2}}$$

tching $\bar{N} - N = p\tilde{p}$
 $k_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \frac{\tilde{p}}{\tilde{R}} \right)$
 $V \sim J^{3}(z) \cdot (\bar{\partial}X^{\mu}e^{ikX})$
 $V \sim J^{\pm}(z) \cdot (\bar{\partial}X^{\mu}e^{ikX})$
 $J^{i}(z) \rightarrow \bar{J}^{i}(\bar{z}) \quad Y^{L}(z) \rightarrow Y^{R}(\bar{z})$

$$J^{3}(z) = \partial Y^{L}(z)$$
$$J^{\pm}(z) = e^{\pm \sqrt{2}iY^{L}(z)}$$

$$J^{i}(z)J^{j}(0) \sim \frac{\delta^{ij}}{z^{2}} + \frac{i\epsilon^{ijk}}{z}J^{k}(0)$$

Symmetry enhancement (recap)

$$R \neq 1 (\neq \tilde{R})$$
$$U(1) \times U(1)$$
$$A \quad \bar{A}$$

2 vectors $(g_{\mu y} \pm B_{\mu y})$



Symmetry enhancement (recap)

$$R \neq 1 (\neq \tilde{R}) \qquad R = \tilde{R} = 1$$

$$U(1) \times U(1) \longrightarrow SU(2) \times SU(2)$$

$$A^{3} \quad \bar{A}^{3} \qquad A^{i} \qquad \bar{A}^{i} \qquad i = \pm, 3$$

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Symmetry enhancement (recap)



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Massless states:

 $g_{\mu m}, B_{\mu m}$ 2d vectors: U(I)^d x U(I)^d

 g_{mn}, B_{mn} d² scalars

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Massless states:

 $g_{\mu m}, B_{\mu m}$ 2d vectors: U(1)^d x U(1)^d

 g_{mn}, B_{mn} d² scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

Narain 86

$$\begin{split} \mathbf{S}^{1} \quad M^{2} &= \frac{2}{\alpha'}(N+\bar{N}-2) + \frac{p^{2}}{R^{2}} + \tilde{p}^{2}R^{2} \\ 0 &= N-\bar{N}+p\tilde{p} \end{split}$$

Massless states:

$$g_{\mu m}, B_{\mu m}$$

$$\mathcal{S}_{\mu m}$$
 2d vectors: U(1)^d x U(1)^d

 g_{mn}, B_{mn}

d² scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

Mass $M^2 = 2(N + \overline{N} - 2) + Z^t \mathcal{H} Z$

$$Z \qquad Z = \begin{pmatrix} p_m \\ \tilde{p}^m \end{pmatrix}$$

Level-matching $0 = (N - \overline{N}) + \frac{1}{2} Z^t \eta Z$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \in \frac{O(d,d)}{O(d) \times O(d)}$$
$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\eta^{LR} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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$$\begin{split} \mathbf{S}^{1} \quad M^{2} &= \frac{2}{\alpha'}(N+\bar{N}-2) + \frac{p^{2}}{R^{2}} + \tilde{p}^{2}R^{2} \\ 0 &= N-\bar{N} + p\tilde{p} \end{split}$$

Massless states:

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 2d vectors: U(1)^d x U(1)^d

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d² scalars

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lots of extra vectors & scalars with mom & winding at points of enhancement

Mass $M^2 = 2(N + \bar{N} - 2) + Z_{\vec{L}}^t \mathcal{H} Z = \begin{pmatrix} p_m \\ \tilde{p}^m \end{pmatrix}$

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Narain 86 S¹

S¹
$$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

 $0 = N - \bar{N} + p\tilde{p}$

Massless states:

$$g_{\mu m}, B_{\mu m}$$

$$\beta_{\mu m}$$
 2d vectors: U(1)^d x U(1)^d

 g_{mn}, B_{mn}

d² scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

Mass
$$M^2 = 2(N + \overline{N} -$$

$$N + \bar{N} - 2) + Z_{E^{T}E}^{t} \mathcal{H}Z \qquad Z = \begin{pmatrix} p_{m} \\ \tilde{p}^{m} \end{pmatrix}$$

p = EZ

Level-matching $0 = (N - \overline{N}) + \frac{1}{2} Z^t \eta Z$

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d² scalars

+

 $\eta^{\mathbf{LR}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

lots of extra vectors & scalars with mom & winding at points of enhancement

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$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

 $p = EZ \qquad \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m \left[p_m + (g_{mn} + B_{mn}) \tilde{p}^n \right] \\ e_a^m \left[p_m - (g_{mn} - B_{mn}) \tilde{p}^n \right] \end{pmatrix}$

Mass

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Level-matching $0 = (N - \overline{N}) + \frac{1}{2}Z^t \eta Z$

S¹
$$M^2 = \frac{2}{\alpha'}(N+\bar{N}-2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

 $0 = N - \bar{N} + p\tilde{p}$
 $p_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \tilde{p}R\right)$
 $M^2 = 2(N+\bar{N}-2) + Z_{E^TE}^t Z = \begin{pmatrix} p_m \\ \tilde{p}^m \end{pmatrix}$

Narain 86

Massless states:

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 2d vectors: U(1)^d x U(1)^d

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S¹
$$M^2 = \frac{2}{\alpha'}(N + \bar{N} - 2) + \frac{p^2}{R^2} + \tilde{p}^2 R^2$$

 $0 = N - \bar{N} + p\tilde{p}$
 $p_{L,R} = \frac{1}{\sqrt{2}} \left(\frac{p}{R} \pm \tilde{p}R\right)$
Mass $M^2 = 2(N + \bar{N} - 2) + Z^t \mathcal{H} Z$
 $E^T E$
 $p_L^2 + p_R^2$
 $Z = \begin{pmatrix} p_m \\ \bar{p}^m \end{pmatrix}$
 $p_L^2 + p_R^2$

Level-matching
$$0 = (N - \bar{N}) + \frac{1}{2} Z_{...}^{t} \eta Z_{...E^{T} \eta E}$$
$$p_{L}^{2} - p_{R}^{2}$$

p =

$$EZ \qquad \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m \left[p_m + (g_{mn} + B_{mn}) \tilde{p}^n \right] \\ e_a^m \left[p_m - (g_{mn} - B_{mn}) \tilde{p}^n \right] \end{pmatrix}$$

form a lattice: Lorentzian (d,d), even, self-dual

$\mathcal{M}_D \times T^d$

$U(1)^d \times U(1)^d$

 A^m \overline{A}^m

2d vectors

 $\mathcal{M}_D \times T^d$

0=
$$M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

$$p = EZ$$

Extra vectors

$$N=0,\;\bar{N}=1\quad\text{or}\ N=1,\bar{N}=0$$

 $U(1)^d \times U(1)^d$

 $A^m = \overline{A}^m$

2d vectors

 $U(1)^d \times U(1)^d$

 $\mathcal{M}_D \times T^d$

$$0= M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

p = EZ

Extra vectors

 $N=0,\ \bar{N}=1$ or $N=1, \bar{N}=0$ $p_L^2-p_R^2=\pm 2 \qquad {\rm LMC} \label{eq:mass_state}$

$$p_L^2 + p_R^2 = 2$$
 M²=0

 A^m \overline{A}^m

2d vectors

 $\mathcal{M}_D \times T^d$

0=
$$M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

p = EZ

Extra vectors

$$\begin{split} N = 0, \ \bar{N} = 1 \quad \text{or} \quad N = 1, \ \bar{N} = 0 \\ p_L^2 - p_R^2 = \pm 2 & \text{LMC} \\ p_L^2 + p_R^2 = 2 & \text{M}^2 \text{=} 0 \end{split}$$

$$p_L^2 = 2, p_R = 0$$

 $U(1)^d \times U(1)^d$

 $A^m \quad \bar{A}^m$

2d vectors

 $\mathcal{M}_D \times T^d$

0=
$$M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

p = EZ

Extra vectors

 $N=0, \ \bar{N}=1 \quad \text{or} \ \ N=1, \ \bar{N}=0$ $p_L^2-p_R^2=\pm 2 \qquad \text{LMC} \\ p_L^2+p_R^2=2 \qquad \text{M2=0}$

$$p_L^2 = 2, p_R = 0 \qquad p_L = 0, p_R^2 = 2$$

$$U(1)^d \times U(1)^d$$

$$A^m$$
 \overline{A}^m

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

 $\mathcal{M}_D \times T^d$

0=
$$M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

p = EZ

Extra vectors

$$\begin{split} N = 0, \ \bar{N} = 1 \quad \text{or} \quad N = 1, \ \bar{N} = 0 \\ p_L^2 - p_R^2 = \pm 2 \qquad \text{LMC} \\ p_L^2 + p_R^2 = 2 \qquad \text{M2=0} \end{split}$$

$$p_L^2 = 2, p_R = 0 \qquad p_L = 0, p_R^2 = 2$$

$$p = EZ = \binom{p_L}{p_R} = \frac{1}{\sqrt{2}} \binom{e_a^m \left[p_m + (g_{mn} + B_{mn})\tilde{p}^n\right]}{e_a^m \left[p_m - (g_{mn} - B_{mn})\tilde{p}^n\right]}$$

$$U(1)^d \times U(1)^d$$

$$A^m$$
 \overline{A}^m

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$
$\mathcal{M}_D \times T^d$

0=
$$M^2 = 2(N + \bar{N} - 2) + (p_L^2 + p_R^2)$$

LMC
$$0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$$

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at generic point in moduli space, no solution

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at generic point in moduli space, no solution

at special points:

solutions are roots of simply-laced gauge group

 $G \times G$

 $U(1)^d \times U(1)^d$

$$A^m$$
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2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

 $\mathcal{M}_D \times T^d$

$$U(1)^d \times U(1)^d \longrightarrow G \times G$$

 $A^m = \overline{A}^m$

2d vectors

$$g_{\mu m} \pm B_{\mu m}$$

$$0= M^{2} = 2(N + \bar{N} - 2) + (p_{L}^{2} + p_{R}^{2})$$

$$LMC \quad 0 = 2(N - \bar{N}) + (p_{L}^{2} - p_{R}^{2})$$

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Extra vectors

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at generic point in moduli space, no solution

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 $+ p_R^2$)

$$U(1)^{d} \times U(1)^{d} \longrightarrow G \times G$$

$$A^{m} \bar{A}^{m} A^{m}, A^{\alpha}, A^{-\alpha} A^{m}, A^{\alpha}, A^{-\alpha}$$

$$0 = M^{2} = 2(N + \bar{N} - 2) + (p_{L}^{2} + Q_{L}^{2}) + (p_{L}^{2} - p_{R}^{2}) + ($$

2d vectors

 $N = 0, \ \bar{N} = 1 \quad \text{or} \quad N = 1, \ \bar{N} = 0$ $p_L^2 - p_R^2 = \pm 2 \qquad \text{LMC}$ $p_L^2 + p_R^2 = 2 \qquad \text{M2=0}$

$$p_L^2 = 2, p_R = 0 \qquad p_L = 0, p_R^2 = 2$$

$$p = EZ = \begin{pmatrix} p_L \\ p_R \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_a^m \left[p_m + (g_{mn} + B_{mn}) \tilde{p}^n \right] \\ e_a^m \left[p_m - (g_{mn} - B_{mn}) \tilde{p}^n \right] \end{pmatrix}$$

at generic point in moduli space, no solution

at special points:

solutions are roots of simply-laced gauge group

 $G \times G$

$g_{\mu m} \pm B_{\mu m}$

 $\mathcal{M}_D \times T^d$

$$U(1)^{d} \times U(1)^{d} \longrightarrow G \times G$$

$$A^{m} \quad \bar{A}^{m} \qquad A^{m}, A^{\alpha}, A^{-\alpha} \qquad A^{m}, A^{\alpha}, A^{-\alpha} \qquad D = M^{2} = 2(N + \bar{N} - 2) + (p_{L}^{2} + p_{R}^{2})$$

$$LMC \quad 0 = 2(N - \bar{N}) + (p_{L}^{2} - p_{R}^{2})$$

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$$p = EZ$$

$$Extra vectors$$

$$p = EZ$$

$$Extra vectors$$

$$N = 0, \bar{N} = 1 \text{ or } N = 1, \bar{N} = 0$$

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at generic point in moduli space, no solution

at special points:

solutions are roots of simply-laced gauge group

 $G \times G$

Symmetry enhancement, bosonic string on T ^d			G	simply-laced Lie algebra	
	maximal enhancement			rank d, dim n	rank d, dim n
	$U(1)^d imes U(1)^d$	• $G \times G$			E = g + B
d=I		$SU(2) \times SU(2)$	$A_1 imes A_1$		1

Symmetry enhancement, bosonic string on T ^d				G	simply-laced Lie algebra rank d, dim n	
	$U(1)^d \times U(1)^d \longrightarrow$	$G \times G$			E = g + B	
d=I		SU(2) imes SU(2)	$A_1 imes A_1$		1	
d=2	• • • •	$SU(2)^2 \times SU(2)^2$			$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Symmetry enhancement, bosonic string on T ^d				G	simply-laced Lie algebra rank d, dim n	
	$U(1)^d \times U(1)^d \longrightarrow$	G imes G			E = g + B	
d=I	• •	SU(2) imes SU(2)	$A_1 imes A_1$		1	
d=2		$SU(2)^2 \times SU(2)^2$			$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
	— —	$SU(3) \times SU(3)$	$A_2 imes A_2$		$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	

Symmetry enhancement, bosonic string on T ^d				simply-laced Lie algebra rank d. dim n	
		maximal enhancement			
	$U(1)^d imes U(1)^d$ –	$\longrightarrow \qquad G \times G$		E = g + B	
d=I		SU(2) imes SU(2)	$A_1 imes A_1$	1	
d=2		$SU(2)^2 \times SU(2)^2$		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
	•••	SU(3) imes SU(3)	$A_2 imes A_2$	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	
d=3		$SU(2)^3 \times SU(2)^3$			
	$\bullet \bullet \bullet \bullet \bullet \bullet SU$	$V(2) \times SU(3) \times SU(3) \times SU(3) \times SU(3)$	SU(2)		
		SU(4) imes SU(4)	$A_3 imes A_3$	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$	
d=4 🛑		$SU(2)^4 imes SU(2)^4$			
•		SU(5) imes SU(5)	$A_4 imes A_4$		
($SO(8) \times SO(8)$	$D_4 imes D_4$	ADE series	

G

bosonic x superstring

Massless states:

 $g_{\mu m}, B_{\mu m}, A_{\mu}^{I} ~~ {\rm 2d+16~vectors:}~ {\rm U(I)^{d+16}~x~U(I)^{d}}$

 g_{mn}, B_{mn}, A_m^I (d+16) x d scalars

bosonic x superstring

Massless states:

 $g_{\mu m}, B_{\mu m}, A_{\mu}^{I} ~~ {\rm 2d+16~vectors:}~ {\rm U(1)^{d+16}~x~U(1)^{d}}$

 g_{mn}, B_{mn}, A_m^I (d+16) x d scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

bosonic x superstring

Massless states:

 $g_{\mu m}, B_{\mu m}, A^{I}_{\mu}$ 2d+16 vectors: U(1)^{d+16} x U(1)^d

 g_{mn}, B_{mn}, A_m^I (d+16) x d scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement $M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^t \mathcal{H} Z_1$

bosonic x superstring

Massless states:

 $g_{\mu m}, B_{\mu m}, A^I_\mu ~~ {\rm 2d+16~vectors:}~ {\rm U(I)^{d+16}~x~U(I)^d}$

 g_{mn}, B_{mn}, A_m^I (d+16) x d scalars

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lots of extra vectors & scalars with mom & winding at points of enhancement

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^{t}g^{-1} & g + C^{t}g^{-1}C + AA^{t} & (1 + C^{t}g^{-1})A \\ -A^{t}g^{-1} & A^{t}(1 + g^{-1}C) & 1 + A^{t}g^{-1}A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)}$$

$$I$$

$$C = B + \frac{1}{2}AA^{t}$$

$$M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^t \mathcal{H} Z$$

Massless states:

 $g_{\mu m}, B_{\mu m}, A^I_\mu ~~ {\rm 2d+16\ vectors:} ~~ {\rm U(I)^{d+16}\ x\ U(I)^d}$

$$g_{mn}, B_{mn}, A_m^I$$
 (d+16) x d scalars

╋

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$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^{t}g^{-1} & g + C^{t}g^{-1}C + AA^{t} & (1 + C^{t}g^{-1})A \\ -A^{t}g^{-1} & A^{t}(1 + g^{-1}C) & 1 + A^{t}g^{-1}A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)}$$

$$I$$

$$C = B + \frac{1}{2}AA^{t}$$

$$Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix} \bullet \dots \bullet \mathsf{quantized\ momenta} \\ \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8 \\ \mathsf{SO}(\mathsf{32}) \qquad \mathsf{E}_8 \times \mathsf{E}_8 \end{cases}$$

$$M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^t \mathcal{H} Z$$

bosonic x superstring

Massless states:

 $g_{\mu m}, B_{\mu m}, A_{\mu}^{I} ~~ \rm 2d+16~vectors:~ U(1)^{d+16}~x~U(1)^{d}$

 g_{mn}, B_{mn}, A_m^I (d+16) x d scalars

+

lots of extra vectors & scalars with mom & winding at points of enhancement

$$Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix} \leftarrow \qquad \text{quantized momenta} \\ \in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8 \\ \text{SO(32)} \qquad \text{E}_8 \times \text{E}_8 \end{cases}$$

$$M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^t \mathcal{H} Z$$

$$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^{t}g^{-1} & g + C^{t}g^{-1}C + AA^{t} & (1 + C^{t}g^{-1})A \\ -A^{t}g^{-1} & A^{t}(1 + g^{-1}C) & 1 + A^{t}g^{-1}A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)} \qquad \begin{array}{c} \text{Level-} \\ \text{matching} & 0 = 2\left(N - \bar{N} - \frac{1}{2}\right) + Z^{t}\eta Z \\ C = B - \frac{1}{2}AA^{t} \\ \eta = \begin{pmatrix} 0 & 1_{d} & 0 \\ 1_{d} & 0 & 0 \\ 0 & 0 & 1_{16} \end{pmatrix} \\ \eta_{LR} = \begin{pmatrix} -1_{d} & 0 & 0 \\ 0 & 1_{d} & 0 \\ 0 & 0 & 1_{16} \end{pmatrix}$$

bosonic x superstring

Massless states: $Z = \begin{pmatrix} \tilde{p}^m \\ p_m \\ \pi^I \end{pmatrix} \mathsf{\leftarrow} \mathsf{m} \mathsf{quantized momenta}$ $g_{\mu m}, B_{\mu m}, A^I_\mu ~~ {\rm 2d+16~vectors:}~ {\rm U(I)^{d+16}~x~U(I)^d}$ $\in \Gamma_{16}$ or $\Gamma_8 \times \Gamma_8$ SO(32) $E_8 \times E_8$ g_{mn}, B_{mn}, A_m^I (d+16) x d scalars + $M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^t \mathcal{H} Z_{E}$ lots of extra vectors & scalars with mom & winding at points of enhancement $p_{L}^{2} + p_{R}^{2}$ $\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^{t}g^{-1} & g + C^{t}g^{-1}C + AA^{t} & (1 + C^{t}g^{-1})A \\ -A^{t}g^{-1} & A^{t}(1 + g^{-1}C) & 1 + A^{t}g^{-1}A \end{pmatrix} \in \frac{O(d+16,d)}{O(d+16) \times O(d)} \qquad \begin{array}{c} \text{Level-} \\ \text{matching} \\ \text{matching} \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} = \left(N - \bar{N} - \frac{1}{2} \right) + Z^{t}\eta Z \\ \mathcal{H} =$ $E^T n E$ $p_{L}^{2} - p_{R}^{2}$ $C = B - \frac{1}{2}AA^t$ $\eta = \begin{pmatrix} 0 & 1_d & 0\\ 1_d & 0 & 0\\ 0 & 0 & 1_{16} \end{pmatrix}$ $\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0\\ 0 & 1_d & 0\\ 0 & 0 & 1_M \end{pmatrix}$

Heterotic string on T^d

<u>Massless states</u> :		$\left(\tilde{p}^{m}\right)$
I=1,,16 "chiral hete	erotic directions"	$Z = \begin{pmatrix} p_m \\ \pi^I \end{pmatrix} \leftarrow \qquad \text{quantized momenta}$
$g_{\mu m}, B_{\mu m}, A^{I}_{\mu}$ 2d+16 vectors: U(1)d+16 x	U(I) ^d	$\in \Gamma_{16} \text{ or } \Gamma_8 \times \Gamma_8$
g_{mn}, B_{mn}, A_m^I (d+16) x d scalars		SO(32) E ₈ x E ₈
+		
lots of extra vectors & scalars with mom & winding at points of enhancement		$M^{2} = 2\left(N + \bar{N} - \frac{3}{2}\right) + Z^{t}\mathcal{H}Z_{L}$ $E^{T}E$ $p_{L}^{2} + p_{R}^{2}$
$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}A \\ -C^{t}g^{-1} & g + C^{t}g^{-1}C + AA^{t} & (1 + C^{t}g^{-1}A^{t}g^{-1}A^{t}g^{-1}A^{t}A^{t}(1 + g^{-1}C) & 1 + A^{t}g^{-1}A^{t}A^{t}A^{t}A^{t}A^{t}A^{t}A^{t}A^{t$	$ \begin{pmatrix} A \\ -1 \end{pmatrix} A \\ -1 A \end{pmatrix} \in \frac{O(d+16)}{O(d+16)} $	$\begin{array}{ll} \frac{16,d)}{XO(d)} & \begin{array}{l} \text{Level-} \\ \text{matching} \end{array} & 0 = 2\left(N - \bar{N} - \frac{1}{2}\right) + Z^t \eta Z \\ E^T \eta E \\ n^2 - n^2 \end{array}$
$C = B + \frac{1}{2}AA^{\iota}$		$p_L - p_R$
$\eta = \begin{pmatrix} 0 & 1_d & 0\\ 1_d & 0 & 0\\ 0 & 0 & 1_{16} \end{pmatrix}$	p = EZ	$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} - \frac{1}{2} A_{m}^{I} A_{n}^{I} \tilde{p}^{n} \right] \\ e_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} - \frac{1}{2} A_{m}^{I} A_{n}^{I} \tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{cases}$
$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0\\ 0 & 1_d & 0\\ 0 & 0 & 1_{16} \end{pmatrix}$		

Heterotic string on T^d

<u>Massless states</u> :		$\langle \tilde{p}^m \rangle$	
I = 1,, 16 "chira	I heterotic directions"	$Z = \begin{pmatrix} p_m \\ \pi^I \end{pmatrix} \leftarrow m \text{quantized mome}$	nta
$g_{\mu m}, B_{\mu m}, A^{I}_{\mu}$ 2d+16 vectors: U(1)	⁺¹⁶ x U(1) ^d	$\in \Gamma_{16} \text{ or } \Gamma_8$	$\times \Gamma_8$
g_{mn}, B_{mn}, A_m^I (d+16) x d scalars		SO(32) E ₈ :	х Е ₈
+			
lots of extra vectors & scalars with mom & winding at points of enhancement		$M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) -$	$+ Z^{t} \mathcal{H} Z_{L}$ $E^{T} E$ $p_{L}^{2} + p_{R}^{2}$
$\mathcal{H} = \begin{pmatrix} g^{-1} & -g^{-1}C & -g^{-1}C & -g^{-1}C & -g^{-1}C & -g^{-1}G & -g$	$ \begin{pmatrix} g^{-1}A\\ C^tg^{-1}A\\ A^tg^{-1}A \end{pmatrix} \in \frac{O(d+16)}{O(d+16)\times} $	$\frac{\mathbf{Level-}}{\mathbf{O}(d)} \qquad \frac{\mathbf{Level-}}{\mathbf{matching}} 0 = 2\left(N - \bar{N} - \frac{1}{2}\right)$	$+ Z^t \eta Z$ $E^T \eta E$
$C = B + \frac{1}{2}AA^t$			$p_L^2 - p_R^2$
$\eta = \begin{pmatrix} 0 & 1_d & 0\\ 1_d & 0 & 0\\ 0 & 0 & 1_{16} \end{pmatrix}$	p = EZ	$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} \\ e_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix}$	$ \begin{array}{l} &-\frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \end{array} \\ &-\frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \end{array} \end{array} $
$\eta_{LR} = \begin{pmatrix} -1_d & 0 & 0 \\ 0 & 1_d & 0 \end{pmatrix}$		form a lattice Lorentzian (<mark>d+16,d</mark>), even, se	elf-dual
$\begin{pmatrix} 0 & 0 & \mathbf{l}_{16} \end{pmatrix}$		[16+d,d	

Heterotic string on T^d

$$\begin{array}{l} \begin{array}{l} \begin{array}{l} \mbox{Massless states:} \\ I = 1, ..., 16 \quad "chiral heterotic directions" \\ g_{\mu m}, B_{\mu m}, A_{\mu}^{I} \quad 2d+16 \mbox{ vectors: } U(1)^{d+16} \times U(1)^{d} \\ \end{array} \\ \begin{array}{l} \mathcal{F} = \left(\prod_{k=1}^{p^{m}} \sum_{k=1}^{p^{m}} \sum_{k=1}^{p^{m}}$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I}\tilde{p}^{m} \right]$$

quantized momenta

 $\pi^{I} \in \Gamma_{16} \text{ or } \Gamma_{8} \times \Gamma_{8}$ SO(32) $E_{8} \times E_{8}$

$$p_{aR} = 0$$

$$p_{aL}^2 = 2$$

quantized momenta

$$\pi^{I} \in \Gamma_{16} \text{ or } \Gamma_{8} \times \Gamma_{8}$$

SO(32) $\mathsf{E}_8 \times \mathsf{E}_8$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I}\tilde{p}^{m} \right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I}\tilde{p}^{m} \right] \end{pmatrix}$$

$$p_{aR} = 0$$

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quantized momenta

$$\pi^{I} \in \Gamma_{16} \text{ or } \Gamma_{8} \times \Gamma_{8}$$

SO(32) $\mathsf{E}_8 \times \mathsf{E}_8$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} - \frac{1}{2} A_{m}^{I} A_{n}^{I} \tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn}) \tilde{p}^{n} - \pi^{I} A_{m}^{I} - \frac{1}{2} A_{m}^{I} A_{n}^{I} \tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(g + B - \frac{1}{2} A A^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p + \left(-g + B - \frac{1}{2} A A^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix}$$

for S¹

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2R}} \begin{pmatrix} p + \left(R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ \sqrt{2R}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

 $p_{aR} = 0$

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quantized momenta

$$\pi^{I} \in \Gamma_{16} \text{ or } \Gamma_{8} \times \Gamma_{8}$$
SO(32) $\mathbf{E}_{8} \times \mathbf{E}_{8}$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix} \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - E^{t} \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - E^{t} \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix}$$

for S¹

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2R}} \begin{pmatrix} p + \left(R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ \sqrt{2R}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

 $p_{aR} = 0$

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SO(32) $\mathbf{E}_{8} \times \mathbf{E}_{8}$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix} \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - E^{t} \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix}$$

for S¹
$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2R}} \begin{pmatrix} p + \left(R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^2 - \frac{1}{2}|A|^2\right)\tilde{p} - \pi \cdot A \\ \sqrt{2R}\left[\pi + A\,\tilde{p}\right] \end{pmatrix}$$

$$R^{2} - \frac{1}{2}|A|^{2}$$

$$\prod_{\tilde{p}=0} p = \pi \cdot A - E \tilde{p} \in \mathbb{Z}$$

$$\tilde{p} = 0 \in \mathbb{Z}$$
symmetry breaking

 $p_{aL}^2 = 2$

quantized momenta

$$\pi^{I} \in \Gamma_{16} \text{ or } \Gamma_{8} \times \Gamma_{8}$$

SO(32) $\mathbf{E}_{8} \times \mathbf{E}_{8}$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e}_{a}^{m} \left[p_{m} + (g_{mn} + B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \hat{e}_{a}^{m} \left[p_{m} - (g_{mn} - B_{mn})\tilde{p}^{n} - \pi^{I}A_{m}^{I} - \frac{1}{2}A_{m}^{I}A_{n}^{I}\tilde{p}^{n} \right] \\ \sqrt{2} \left[\pi^{I} + A_{m}^{I} \tilde{p}^{m} \right] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix} \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{e} \left[p + \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - \left(-g + B - \frac{1}{2}AA^{t} \right) \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - E^{t} \tilde{p} - \pi \cdot A \right] \\ \hat{e} \left[p - E^{t} \tilde{p} - \pi \cdot A \right] \\ \sqrt{2} \left[\pi + A \tilde{p} \right] \end{pmatrix}$$

$$\begin{array}{l} \text{for S}^{1} \quad \begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2R}} \begin{pmatrix} p + \left(R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ p + \left(-R^{2} - \frac{1}{2}|A|^{2}\right)\tilde{p} - \pi \cdot A \\ \sqrt{2R}\left[\pi + A\,\tilde{p}\right] \end{pmatrix} \end{array}$$

$$R^{2} - \frac{1}{2}|A|^{2}$$

$$\prod_{\tilde{p}=0} p = \pi \cdot A - E \tilde{p} \in \mathbb{Z}$$

$$\tilde{p} = 0 \in \mathbb{Z}$$
symmetry breaking

 $p_{aL}^2 = 2 \implies |\pi + A\tilde{p}|^2 = 2(1 - R^2\tilde{p}^2)$ symmetry enhancement

Symmetry enhancement, heterotic string on S^I $U(1)^{17} \rightarrow G$ simply-laced Lie algebra rank 17 SO(32)



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^I $U(1)^{17} \rightarrow G$ simply-laced Lie algebra rank 17 SO(32)

$$A^{I} = 0$$
$$SO(32) \times U(1)$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^I $U(1)^{17} \rightarrow G$ simply-laced Lie algebra rank 17 SO(32)



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S[|] $U(1)^{17} \rightarrow G^{\text{simply-laced}}_{\text{Lie algebra}}$ rank 17 SO(32)



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$

Symmetry enhancement, heterotic string on S^I $U(1)^{17} \rightarrow G$ simply-laced Lie algebra rank 17 SO(32)



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$A = (1, 0_{15})$$

$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$\begin{pmatrix} p_{aR} \\ p_{aL} \\ p_{L} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$\begin{pmatrix} p_{aL} \\ p_{aL} \\ p_{L}^{I} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} R + \tilde{p}R - R(n-1) - \frac{1}{2}|A|^{2}\tilde{p} \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$



$$\begin{pmatrix} \boldsymbol{p_{aR}} \\ \boldsymbol{p_{aL}} \\ \boldsymbol{p_{L}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{p}{R} + \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \frac{p}{R} - \tilde{p}R - \frac{1}{R}(\pi^{I}A^{I} - \frac{1}{2}|A|^{2}\tilde{p}) \\ \sqrt{2}\left[\pi^{I} + A^{I}\tilde{p}\right] \end{pmatrix}$$






$$\left(p_{L}^{I} \right)$$









Same groups? What are all the groups one can get?







 $\tilde{k} - \theta' \quad k - \theta$ $-(k + \tilde{k})$ $k = \frac{1}{\sqrt{2}}(1, 1)$ $\tilde{k} = \frac{1}{\sqrt{2}}(1, -1)$

Goddard, Olive 85

Cachazo, Vafa 00

Goddard, Olive 85 Cachazo, Vafa 00



- 19 roots: $|\alpha|^2 = 2$
- Associated to Weyl reflections in $\Gamma^{17,1}$

 $SO(17, 1, \mathbb{Z})$ is generated by these 19 Weyl reflections



 $\mathbf{\hat{E}_8 \times \hat{E}_8}$ $k = \frac{1}{\sqrt{2}}(1,1)$

 $\tilde{k} = \frac{1}{\sqrt{2}}(1, -1)$

Goddard, Olive 85 Cachazo, Vafa 00

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• Weyl group divides the moduli space into chambers Fundamental chamber: fundamental region in mod space

Goddard, Olive 85 Cachazo, Vafa 00

 $\stackrel{\wedge}{\mathsf{E}_8} \times \stackrel{\wedge}{\mathsf{E}_8}$

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- Each root corresponds to a boundary in mod space

Goddard, Olive 85 Cachazo, Vafa 00

 $\stackrel{\wedge}{\mathsf{E}_8} \times \stackrel{\wedge}{\mathsf{E}_8}$

 $k = \frac{1}{\sqrt{2}}(1,1)$

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- Weyl group divides the moduli space into chambers Fundamental chamber: fundamental region in mod space
- Each root corresponds to a boundary in mod space

 $\begin{aligned} & \bigwedge_{8} \bigwedge_{8} \bigwedge_{8} \\ & k = \frac{1}{\sqrt{2}} (1, 1) \\ & \tilde{k} = \frac{1}{\sqrt{2}} (1, -1) \end{aligned} \in \Gamma^{1,1} \\ & \alpha_{0} = k - \theta \\ & \alpha'_{0} = \tilde{k} - \theta' \end{aligned}$

 $\alpha_B = -(k + \tilde{k})$



- 19 roots: $|\alpha|^2 = 2$
- Associated to Weyl reflections in $\Gamma^{17,1}$

 $SO(17, 1, \mathbb{Z})$ is generated by these 19 Weyl reflections

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 $\begin{aligned} & \bigwedge_{B} \bigwedge_{B} \bigwedge_{B} \overset{}{} \underset{k = \frac{1}{\sqrt{2}}(1, 1)}{k = \frac{1}{\sqrt{2}}(1, -1)} & \in \Gamma^{1, 1} \\ & \tilde{k} = \frac{1}{\sqrt{2}}(1, -1) \\ & \alpha_{0} = k - \theta \\ & \alpha'_{0} = \tilde{k} - \theta' \end{aligned}$

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$$\begin{split} & \bigwedge_{B} \bigwedge_{B} \bigwedge_{B} \bigwedge_{B} \\ & k = \frac{1}{\sqrt{2}}(1, 1) \\ & k = \frac{1}{\sqrt{2}}(1, -1) \\ & \tilde{k} = \frac{1}{\sqrt{2}}(1, -1) \\ & \alpha_{0} = k - \theta \\ & \alpha_{0}' = \tilde{k} - \theta' \\ & \alpha_{B} = -(k + \tilde{k}) \end{split}$$

8' 7' 6' 5' 4' 3' 2' 1' 0' B 0 1 2 3 4 5 6 $A_1 \ge A_2 \ge A_3 \ge A_4 \ge A_5 \ge A_6 \ge A_7$ $k = \frac{1}{\sqrt{2}}(1, 1)$

- 19 roots: $|\alpha|^2 = 2$
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 $\alpha_0 = k - \theta$ $\alpha'_0 = \tilde{k} - \theta'$ $\alpha_B = -(k + \tilde{k})$

$$\begin{split} \mathbf{\hat{E}}_{8} \times \mathbf{\hat{E}}_{8} \\ k &= \frac{1}{\sqrt{2}}(1, 1) \\ \tilde{k} &= \frac{1}{\sqrt{2}}(1, -1) \end{split} \in \Gamma^{1, 1} \end{split}$$



- 19 roots: $|\alpha|^2 = 2$
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Goddard, Olive 85 Cachazo, Vafa 00

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Wilson line	R^{-2}	Gauge group
$\left(\left(\frac{q}{2(p+q)}\right)_p, 0_{8-p}, \left(\frac{q}{2(p+q)}\right)_q, \left(\frac{1}{2}\right)_{8-q}\right)\right)$	$8\left(\frac{1}{p} + \frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(\left(\frac{q}{2(6+q)}\right)_{7}, -\frac{q}{2(6+q)}, \left(\frac{q}{2(6+q)}\right)_{q}, \left(\frac{1}{2}\right)_{8-q}\right)\right)$	$2 - \frac{2}{q+9} + \frac{8}{q}$	$SU(9+q) \times E_{9-q}$
$\left(\left(\frac{1}{4}\right)_7, -\frac{1}{4}, \left(\frac{1}{4}\right)_7, -\frac{1}{4}\right) + (0_{15}, 1)\right)$	4	SU(18)
$\left(0_{8+q}, (\frac{1}{2})_{8-q}\right)$	$\frac{8}{q}$	$SO(16+2q) \times E_{9-q}$
$(0_{15}, 1)$	2	SO(34)

Table 1: Maximal enhancements for the SO(32) theory.

Wilson line	R^2	Gauge group
$\left(\left(\frac{1}{p}\right)_p, 0_{8-p}, \left(\frac{1}{q}\right)_q, 0_{8-q}\right) - (1, 0_7, 1, 0_7)\right)$	$\frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$	$E_{9-p} \times E_{9-q} \times SU(p+q)$
$\left(\left(\frac{1}{6}\right)_7, -\frac{1}{6}, \left(\frac{1}{q}\right)_q, 0_{8-q}\right) - (1, 0_7, 1, 0_7)$	$\frac{1}{2}\left(\frac{1}{9} + \frac{1}{q}\right)$	$SU(9+q) \times E_{9-q}$
$\left(\left(\frac{1}{6}\right)_7, -\frac{1}{6}, \left(-\frac{1}{6}\right)_7\right) - (1, 0_7, 1, 0_7)\right)$	$\frac{1}{9}$	SU(18)
$\left(0_{8}, \left(\frac{1}{q}\right)_{q}, 0_{8-q}\right) - (1, 0_{7}, 1, 0_{7})$	$\frac{1}{2q}$	$SO(16+2q) \times E_{9-q}$
$(0_8, (\frac{1}{6})_7, -\frac{1}{6}) - (1, 0_7, 1, 0_7)$	$\frac{1}{18}$	SO(34)

Table 2: Maximal enhancements for the $E_8 \times E_8$ theory.

 $\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R}) \times O(2,\mathbb{R}) \times O(18,2,\mathbb{Z})}$

 $(p_L, p_R) \in \Gamma^{18,2}$

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• What are all the possible enhancement groups?

 $\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R}) \times O(2,\mathbb{R}) \times O(18,2,\mathbb{Z})}$

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• What are all the possible enhancement groups?

All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!
$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R}) \times O(2,\mathbb{R}) \times O(18,2,\mathbb{Z})}$

$$(p_L, p_R) \in \Gamma^{18,2}$$

- What are all the possible enhancement groups? All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!
- Just play and find them by hand...

$\mathcal{U} \subset$	$O(18,\!2,\!\mathbb{R})$
$n \in$	$\overline{O(18,\mathbb{R})} \times O(2,\mathbb{R}) \times O(18,2,\mathbb{Z})$

 $(p_L, p_R) \in \Gamma^{18,2}$

mod	space	of	F-theory	on	K3
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• What are all the possible enhancement groups?

All groups whose root lattice admits an embedding in $\Gamma^{18,2}$ not known!

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$\mathcal{H} \in \frac{O(18,2,\mathbb{R})}{O(18,\mathbb{R}) \times O(2,\mathbb{R}) \times O(18,2,\mathbb{Z})}$	$(p_L, p_R) \in$	$\Gamma^{18,2}$		
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 Search for all groups on K3 	Chabrol 19			
 Actually, have been all listed 	Shimada Zhang 00			

Shimada Zhang 00

Νο Σ	Νο Σ	No	Σ	No	Σ	No	Σ
$1 6 A_3$	$34 A_1 + A_2 + A_3 + A_5 + A_7$	65	$A_3 + A_6 + A_9$	92	$2A_1 + A_4 + A_{12}$	120	$A_2 + A_5 + A_6 + D_5$
2 2 $A_1 + 4 A_4$	$35 \ 2A_1 + A_4 + A_5 + A_7$	66	$A_2 + A_7 + A_9$				
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$36 A_2 + A_4 + A_5 + A_7$	67	$A_1 + A_8 + A_9$	93	$A_2 + A_4 + A_{12}$	121	$A_1 + A_7 + 2 D_5$
$4 3A_1 + 3A_5$		68	$A_2 + 2A_3 + A_{10}$	94	$A_1 + A_5 + A_{12}$	122	$A_1 + A_2 + A_3 + A_7 + D_5$
5 $4A_2 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	69	$A_1 + 2A_2 + A_3 + A_{10}$	95	$A_6 + A_{12}$	123	$2A_1 + A_4 + A_7 + D_5$
$6 A_3 + 3 A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	70	$2A_4 + A_{10}$	96	$A_1 + 2A_2 + A_{13}$	124	$A_8 + 2 D_5$
$7 2A_1 + 2A_3 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	71	$2A_2 + A_4 + A_{10}$			125	$A_1 + A_4 + A_8 + D_5$
$8 A_1 + 2A_2 + A_3 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	-		97	$3A_1 + A_2 + A_{13}$		
9 $2A_4 + 2A_5$	Ī	72	$2A_1 + A_2 + A_4 + A_{10}$	98	$2A_1 + A_3 + A_{13}$	126	$A_5 + A_8 + D_5$
$10 2A_2 + A_4 + 2A_5$	41 $A_5 + A_6 + A_7$	73	$A_1 + A_3 + A_4 + A_{10}$	99	$A_2 + A_3 + A_{13}$	127	$2A_2 + A_9 + D_5$
$11 A_1 + A_3 + A_4 + 2 A_5$	42 $2A_1 + 2A_8$			100	$A_1 + A_4 + A_{13}$	128	$2A_1 + A_2 + A_9 + D_5$
$12 A_1 + A_2 + 2A_3 + A_4 + A_5$	ī	74	$A_1 + A_2 + A_5 + A_{10}$		L	129	$A_1 + A_3 + A_9 + D_5$
$13 \ 3 A_6$	$43 A_1 + 3 A_2 + A_3 + A_8$	<u>-</u>				130	$A_4 + A_9 + D_5$
$14 2A_1 + 2A_2 + 2A_6$	$44 2A_1 + 2A_4 + A_8$	75	$A_3 + A_5 + A_{10}$	101	$A_5 + A_{13}$	131	$A_1 + A_2 + A_{10} + D_5$
$15 \ 2A_3 + 2A_6$	45 $3A_2 + A_4 + A_8$			102	$2A_2 + A_{14}$	132	$2A_1 + A_{11} + D_5$
$16 A_2 + A_4 + 2A_6$	$46 A_1 + A_2 + A_3 + A_4 + A_8$	76	$2A_1 + A_6 + A_{10}$	103	$2A_1 + A_2 + A_{14}$	133	$A_2 + A_{11} + D_5$
$17 2A_1 + A_2 + 2A_4 + A_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	77	$A_2 + A_6 + A_{10}$			134	$A_1 + A_{12} + D_5$
$18 A_1 + A_3 + 2A_4 + A_6$	$48 A_2 + A_3 + A_5 + A_8$			104	$A_1 + A_3 + A_{14}$		
	$49 A_1 + A_4 + A_5 + A_8$	78	$A_1 + A_7 + A_{10}$	105	$A_4 + A_{14}$	135	$A_{13} + D_5$
$19 A_2 + 2A_3 + A_4 + A_6$	$50 2A_1 + A_2 + A_6 + A_8$			106	$3A_1 + A_{15}$	136	$3 D_6$
$20 A_1 + 2A_2 + A_3 + A_4 + A_6$	$51 A_1 + A_3 + A_6 + A_8$	79	$A_8 + A_{10}$	107	$A_1 + A_2 + A_{15}$	137	$2A_3 + 2D_6$
21 $2A_1 + 2A_5 + A_6$	$52 A_4 + A_6 + A_8$	80	$A_1 + 3A_2 + A_{11}$			138	$2A_2 + 2A_4 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$53 A_1 + A_2 + A_7 + A_8$	81	$3A_1 + 2A_2 + A_{11}$	108	$A_3 + A_{15}$	139	$2A_1 + 2A_5 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	54 $2A_9$	82	$A_1 + 2A_3 + A_{11}$	109	$2A_1 + A_{16}$	140	$A_1 + 2A_3 + A_5 + D_6$
		83	$2A_2 + A_3 + A_{11}$			141	$A_3 + A_4 + A_5 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$55 A_1 + A_2 + 2A_3 + A_9$			110	$A_2 + A_{16}$	142	$2A_6 + D_6$
25 $4A_1 + 2A_7$	$56 2A_1 + 2A_2 + A_3 + A_9$	84	$2A_1 + A_2 + A_3 + A_{11}$	111	$A_1 + A_{17}$	143	$A_2 + A_4 + A_6 + D_6$
26 $2A_2 + 2A_7$	57 $A_1 + 2A_4 + A_9$					144	$A_1 + 2A_2 + A_7 + D_6$
	$58 3A_1 + A_2 + A_4 + A_9$	85	$3A_1 + A_4 + A_{11}$	112	A ₁₈	145	$A_2 + A_3 + A_7 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$59 2A_1 + A_3 + A_4 + A_9$	86	$A_1 + A_2 + A_4 + A_{11}$	113	$2A_4 + 2D_5$	146	$A_1 + A_4 + A_7 + D_6$
$28 2A_1 + 3A_3 + A_7$	$\begin{bmatrix} 60 & 2A_1 + A_2 + A_5 + A_9 \end{bmatrix}$	87	$2A_1 + A_5 + A_{11}$	114	$A_3 + 2A_5 + D_5$	147	$A_4 + A_8 + D_6$
$ 29 A_2 + 3A_3 + A_7 $	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$			115	$2A_4 + A_5 + D_5$	148	$A_1 + A_2 + A_9 + D_6$
$30 2A_2 + A_3 + A_4 + A_7$	$62 A_4 + A_5 + A_9$	88	$A_2 + A_5 + A_{11}$	116	$A_1 + A_3 + A_4 + A_5 + D_5$		
$ 31 2A_1 + A_2 + A_3 + A_4 + \overline{A_7} $		89	$A_1 + A_6 + A_{11}$	117	$A_1 + 2A_6 + D_5$	149	$A_3 + A_9 + D_6$
$32 A_1 + 2A_5 + A_7$	$63 \ 3A_1 + A_6 + A_9$	90	$2A_1 + 2A_2 + \overline{A_{12}}$	118	$2A_2 + A_3 + A_6 + D_5$	150	$A_2 + A_{10} + D_6$
$33 \ 3A_1 + A_3 + A_5 + A_7$	$64 A_1 + A_2 + A_6 + A_9$	91	$A_1 + A_2 + A_3 + A_{12}$	119	$A_1 + A_2 + A_4 + A_6 + D_5$	151	$A_1 + A_{11} + D_6$

Νο Σ	Νο Σ	Νο Σ	Νο Σ	Νο Σ
152 $A_{12} + D_6$	$188 2A_1 + 2A_3 + D_{10}$	$224 2A_2 + A_3 + A_5 + E_6$	258 $A_4 + 2E_7$	291 $D_{11} + E_7$
$153 A_2 + A_5 + D_5 + D_6$	189 $2A_4 + D_{10}$	225 $A_3 + A_4 + A_5 + E_6$	$\begin{array}{ $	292 $A_2 + A_3 + E_6 + E_7$
154 $A_7 + D_5 + D_6$	$190 A_1 + A_3 + A_4 + D_{10}$	226 $A_6 + 2E_6$	$260 2A_2 + A_3 + A_4 + E_7$	293 $A_1 + A_4 + E_6 + E_7$
155 $2A_2 + 2D_7$	$191 3A_1 + A_5 + D_{10}$	$227 A_1 + A_2 + A_3 + A_6 + E_6$	261 $2A_3 + A_5 + E_7$	294 $A_5 + E_6 + E_7$
156 $A_2 + 3A_3 + D_7$	192 $A_3 + A_5 + D_{10}$		$262 A_1 + A_2 + A_3 + A_5 + E_7 $	295 $D_5 + E_6 + E_7$
$157 A_1 + A_2 + 2 A_4 + D_7$	193 $A_2 + A_6 + D_{10}$	$228 2A_1 + A_4 + A_6 + E_6$	$263 \ 2A_1 + A_4 + A_5 + E_7$	296 $2A_1 + 2E_8$
$158 A_2 + A_3 + A_6 + D_7$	194 $A_8 + D_{10}$	$229 A_2 + A_4 + A_6 + E_6$	$264 A_2 + A_4 + A_5 + E_7$	297 $A_2 + 2 E_8$
159 $A_1 + A_4 + A_6 + D_7$	195 $A_1 + A_2 + D_5 + D_{10}$	$230 A_1 + A_5 + A_6 + E_6$	$265 A_1 + 2 A_2 + A_6 + E_7$	298 $2A_2 + 2A_3 + E_8$
160 $A_5 + A_6 + D_7$	196 $A_2 + D_6 + D_{10}$	$231 A_1 + A_4 + A_7 + E_6$	$266 A_2 + A_3 + A_6 + E_7$	$299 2A_1 + 2A_4 + E_8$
161 $2A_1 + A_2 + A_7 + D_7$	197 $A_1 + D_7 + D_{10}$	232 $A_5 + A_7 + E_6$	267 $A_1 + A_4 + A_6 + E_7$	$300 A_1 + A_2 + A_3 + A_4 + E_8$
162 $A_1 + A_3 + A_7 + D_7$	198 $2A_2 + A_3 + D_{11}$	233 $2A_2 + A_8 + E_6$		$301 \ 2A_5 + E_8$
163 $2A_1 + A_9 + D_7$	$199 A_1 + A_2 + A_4 + D_{11}$	$234 2A_1 + A_2 + A_8 + E_6$	268 $A_5 + A_6 + E_7$	$-302 A_2 + A_3 + A_5 + E_8$
164 $A_2 + A_9 + D_7$	$200 A_2 + A_5 + D_{11}$	$235 A_1 + A_3 + A_8 + E_6$	269 $2A_2 + A_7 + E_7$	$303 A_1 + A_4 + A_5 + E_8$
165 $A_1 + A_{10} + D_7$	$201 A_1 + A_6 + D_{11}$	236 $A_4 + A_8 + E_6$	$270 2A_1 + A_2 + A_7 + E_7$	$304 2A_2 + A_6 + E_8$
166 $A_{11} + D_7$	$202 2A_1 + 2A_2 + D_{12}$	$237 A_1 + A_2 + A_9 + E_6$	$ [271 A_1 + A_3 + A_7 + E_7] $	$-305 + 2A_1 + A_2 + A_6 + E_8$
$167 A_1 + A_5 + D_5 + D_7$	$203 A_1 + A_2 + A_3 + D_{12}$	238 $A_3 + A_9 + E_6$	$ [272 A_4 + A_7 + E_7] $	$-306 A_1 + A_2 + A_6 + E_8$
168 $A_5 + D_6 + D_7$	$204 2A_1 + A_4 + D_{12}$	$239 2A_1 + A_{10} + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$-\frac{307}{307}$ $A_4 + A_6 + E_8$
169 $2A_1 + 2D_8$	$205 A_1 + D_5 + D_{12}$	$240 A_2 + A_{10} + E_6$	$\boxed{274 A_3 + A_8 + E_7}$	$-\frac{308}{308} \frac{A_1 + A_2 + A_7 + E_9}{A_1 + A_2 + A_7 + E_9}$
$170 \ 2 A_2 + 2 A_3 + D_8$	$206 D_6 + D_{12}$	241 $A_1 + A_{11} + E_6$	275 $2A_1 + A_9 + E_7$	$-\frac{309}{309} \frac{2}{2} \frac{A_1 + A_2 + E_8}{A_1 + A_2 + E_8}$
171 $2A_5 + D_8$	$207 A_1 + A_4 + D_{13}$		276 $A_2 + A_9 + E_7$	$-\frac{310}{40} + \frac{40}{40} + \frac{100}{40} + 1$
$172 \ 2A_1 + A_3 + A_5 + D_8$	208 $A_5 + D_{13}$	$242 A_{12} + E_6$		$= 311 A_1 + A_2 + E_2$
$173 A_1 + A_4 + A_5 + D_8$	$209 D_5 + D_{13}$	$243 A_3 + A_4 + D_5 + E_6$	$ 277 A_1 + A_{10} + E_7 $	$\begin{array}{c c} 311 & A_1 + A_9 + B_8 \\ \hline 312 & A_{10} + F_2 \\ \hline \end{array}$
$174 \ 2A_2 + A_6 + D_8$	$210 2A_2 + D_{14}$	$244 A_1 + A_6 + D_5 + E_6$		$312 A_{10} + E_8$
$175 A_1 + A_2 + A_7 + D_8$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	245 $A_7 + D_5 + E_6$	$278 A_{11} + E_7$	$ \begin{array}{c c} 313 & 2D5 + D8 \\ \hline 214 & A + A + D + F \\ \hline \end{array} $
176 $A_1 + A_9 + D_8$	$212 A_1 + A_3 + D_{14}$	246 $D_6 + 2E_6$	$279 D_4 + 2E_7$	$= \frac{314}{215} + \frac{A_1 + A_4 + D_5 + L_8}{A_1 + D_2 + E_3}$
177 $2D_5 + D_8$	213 $A_4 + D_{14}$	$247 A_2 + A_4 + D_6 + E_6$	$280 A_2 + A_4 + D_5 + E_7$	$= \frac{313}{210} + \frac{1}{200} + $
178 $A_1 + A_3 + D_6 + D_8$	$214 A_1 + A_2 + D_{15}$	248 $A_6 + D_6 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= \frac{310}{2A_2 + D_6 + E_8}$
179 $2 D_9$	215 $2A_1 + D_{16}$	$249 A_1 + A_4 + D_7 + E_6$	$282 A_6 + D_5 + E_7$	$\frac{317}{A_4 + D_6 + E_8}$
$180 A_1 + 2A_2 + A_4 + D_9$	216 $A_2 + D_{16}$	$250 \mid D_5 + D_7 + E_6$	$283 A_2 + A_3 + D_6 + E_7$	$\begin{array}{c c} 318 & A_1 + A_2 + D_7 + E_8 \\ \hline \end{array}$
$181 A_1 + A_3 + A_5 + D_9$	217 $A_1 + D_{17}$	$251 A_4 + D_8 + E_6$	$284 A_5 + D_6 + E_7$	$319 A_1 + D_9 + E_8$
$1\overline{82} \overline{A_4 + A_5 + D_9}$	218 D ₁₈	$252 A_1 + A_2 + D_9 + E_6$	$285 D_5 + D_6 + E_7$	$320 D_{10} + E_8$
$183 A_1 + A_2 + A_6 + D_9$	219 3 E_6	253 $A_3 + D_9 + E_6$		$321 A_1 + A_3 + E_6 + E_8$
184 $2A_1 + A_7 + D_9$	220 $2A_3 + 2E_6$	$254 A_1 + D_{11} + E_6$	$\boxed{\begin{array}{c c} 287 \\ \hline \\ 44 \\ \hline \\ \hline \\ 44 \\ \hline \\ \hline \\ \hline \\ \hline \\$	$322 A_4 + E_6 + E_8$
$185 A_1 + A_8 + \overline{D_9}$	$221 A_1 + A_3 + 2A_4 + E_6 $	$255 D_{12} + E_6$	$ 288 A_1 + A_2 + D_8 + E_7 $	$323 D_4 + E_6 + E_8$
186 $A_9 + D_9$	$222 A_1 + A_5 + 2 E_6$	256 $2A_2 + 2E_7$	$289 A_2 + D_9 + E_7$	$324 A_1 + A_2 + E_7 + E_8$
$187 A_4 + D_5 + \overline{D_9}$	$223 A_2 + 2 A_5 + E_6$	257 $A_1 + A_3 + 2 E_7$	$ 290 A_1 + D_{10} + \overline{E_7} $	$ 325 A_3 + E_7 + E_8 $

Νο Σ	Νο Σ	Νο Σ	<u>Νο</u> Σ	Νο Σ
152 $A_{12} + D_6$	$188 2A_1 + 2A_3 + D_{10}$	$224 2A_2 + A_3 + A_5 + E_6$	$258 A_4 + 2E_7$	291 $D_{11} + E_7$
$153 A_2 + A_5 + D_5 + D_6$	189 $2A_4 + D_{10}$	225 $A_3 + A_4 + A_5 + E_6$	$\boxed{259 A_1 + 2A_3 + A_4 + E_7}$	292 $A_2 + A_3 + E_6 + E_7$
154 $A_7 + D_5 + D_6$	$190 A_1 + A_3 + A_4 + D_{10}$	226 $A_6 + 2E_6$	$\begin{array}{ $	293 $A_1 + A_4 + E_6 + E_7$
155 $2A_2 + 2D_7$	$191 3A_1 + A_5 + D_{10}$	$227 A_1 + A_2 + A_3 + A_6 + E_6$	$\boxed{261 \ 2A_3 + A_5 + E_7}$	294 $A_5 + E_6 + E_7$
156 $A_2 + 3A_3 + D_7$	192 $A_3 + A_5 + D_{10}$		$262 A_1 + A_2 + A_3 + A_5 + E_7 $	295 $D_5 + E_6 + E_7$
$157 A_1 + A_2 + 2 A_4 + D_7$	193 $A_2 + A_6 + D_{10}$	$228 2A_1 + A_4 + A_6 + E_6$	$263 \ 2A_1 + A_4 + A_5 + E_7$	$296 2A_1 + 2E_8$
$158 A_2 + A_3 + A_6 + D_7$	194 $A_8 + D_{10}$	$229 A_2 + A_4 + A_6 + E_6$	$264 A_2 + A_4 + A_5 + E_7$	297 $A_2 + 2E_8$
$159 A_1 + A_4 + A_6 + D_7$	$195 A_1 + A_2 + D_5 + D_{10}$	$230 A_1 + A_5 + A_6 + E_6$	$265 A_1 + 2A_2 + A_6 + E_7$	298 $2A_2 + 2A_3 + E_8$
160 $A_5 + A_6 + D_7$	196 $A_2 + D_6 + D_{10}$	$231 A_1 + A_4 + A_7 + E_6$	$ 266 A_2 + A_3 + A_6 + E_7 $	299 2 A_1 + 2 A_4 + E_8
$161 2A_1 + A_2 + A_7 + D_7$	197 $A_1 + D_7 + D_{10}$	232 $A_5 + A_7 + E_6$	$267 A_1 + A_4 + A_6 + E_7$	$300 A_1 + A_2 + A_3 + A_4 + E_8$
162 $A_1 + A_3 + A_7 + D_7$	$198 2A_2 + A_3 + D_{11}$	233 $2A_2 + A_8 + E_6$		$301 \ 2A_5 + E_8$
163 $2A_1 + A_9 + D_7$	$199 A_1 + A_2 + A_4 + D_{11}$	$234 \ 2A_1 + A_2 + A_8 + E_6$	268 $A_5 + A_6 + E_7$	$302 A_2 + A_3 + A_5 + E_8$
164 $A_2 + A_9 + D_7$	$200 A_2 + A_5 + D_{11}$	$235 A_1 + A_3 + A_8 + E_6$	269 $2A_2 + A_7 + E_7$	$303 A_1 + A_4 + A_5 + E_8$
165 $A_1 + A_{10} + D_7$	$201 A_1 + A_6 + D_{11}$	236 $A_4 + A_8 + E_6$	$270 2A_1 + A_2 + A_7 + E_7$	$304 2A_2 + A_6 + E_8$
166 $A_{11} + D_7$	$202 2A_1 + 2A_2 + D_{12}$	$237 A_1 + A_2 + A_9 + E_6$	$ [271 A_1 + A_3 + A_7 + E_7] $	$305 2A_1 + A_2 + A_6 + E_8$
167 $A_1 + A_5 + D_5 + D_7$	$203 A_1 + A_2 + A_3 + D_{12}$	238 $A_3 + A_9 + E_6$	$\boxed{272 \mid A_4 + A_7 + E_7}$	$\frac{306}{306} \frac{A_1 + A_2 + A_6 + E_9}{A_1 + A_2 + A_6 + E_9}$
168 $A_5 + D_6 + D_7$	$204 2A_1 + A_4 + D_{12}$	$239 2A_1 + A_{10} + E_6$	$ [273 A_1 + A_2 + A_8 + E_7] $	$307 A_4 + A_6 + E_8$
169 $2A_1 + 2D_8$	$205 A_1 + D_5 + D_{12}$	240 $A_2 + A_{10} + E_6$	$\boxed{274 \mid A_3 + A_8 + E_7}$	$308 A_1 + A_2 + A_7 + E_2$
$170 2A_2 + 2A_3 + D_8$	206 $D_6 + D_{12}$	241 $A_1 + A_{11} + E_6$	$275 2A_1 + A_9 + E_7$	$309 2 A_1 + A_2 + E_2$
171 $2A_5 + D_8$	$207 A_1 + A_4 + D_{13}$		$276 A_2 + A_9 + E_7$	$310 A_2 + A_2 + E_2$
$172 2A_1 + A_3 + A_5 + D_8$	208 $A_5 + D_{13}$	$242 A_{12} + E_6$		310 12 18 18
173 $A_1 + A_4 + A_5 + D_8$	209 $D_5 + D_{13}$	$243 A_3 + A_4 + D_5 + E_6$	$ 277 A_1 + A_{10} + E_7 $	$\begin{array}{c c} 311 & A_1 + A_9 + E_8 \\ \hline 212 & A_1 + E_2 \\ \hline \end{array}$
174 $2A_2 + A_6 + D_8$	210 $2A_2 + D_{14}$	$244 A_1 + A_6 + D_5 + E_6$		$312 A_{10} + E_8$
175 $A_1 + A_2 + A_7 + D_8$	$211 2A_1 + A_2 + D_{14}$	245 $A_7 + D_5 + E_6$	$278 A_{11} + E_7$	$313 2D_5 + E_8$
176 $A_1 + A_9 + D_8$	212 $A_1 + A_3 + D_{14}$	246 $D_6 + 2E_6$	$279 D_4 + 2E_7$	314 $A_1 + A_4 + D_5 + E_8$
177 $2 D_5 + D_8$	213 $A_4 + D_{14}$	$247 A_2 + A_4 + D_6 + E_6$	$280 A_2 + A_4 + D_5 + E_7$	$315 A_5 + D_5 + E_8$
178 $A_1 + A_3 + D_6 + D_8$	$214 A_1 + A_2 + D_{15}$	$248 \mid A_6 + D_6 + E_6$	$ 281 A_1 + A_5 + D_5 + E_7 $	$316 \ 2A_2 + D_6 + E_8$
179 2 D_9	215 $2A_1 + D_{16}$	$249 A_1 + A_4 + D_7 + E_6$	$282 A_6 + D_5 + E_7$	$317 A_4 + D_6 + E_8$
$180 A_1 + 2 A_2 + A_4 + D_9$	216 $A_2 + D_{16}$	$250 D_5 + D_7 + E_6$	$\begin{array}{ c } \hline 283 & A_2 + A_3 + D_6 + E_7 \\ \hline \end{array}$	$318 A_1 + A_2 + D_7 + E_8$
$181 A_1 + A_3 + A_5 + D_9$	217 $A_1 + D_{17}$	251 $A_4 + D_8 + E_6$	$284 A_5 + D_6 + E_7$	$319 A_1 + D_9 + E_8$
182 $A_4 + A_5 + D_9$	218 D_{18}	$252 A_1 + A_2 + D_9 + E_6 $	$285 D_5 + D_6 + E_7$	$320 D_{10} + E_8$
$183 A_1 + A_2 + A_6 + D_9$	219 3 E_6	253 $A_3 + D_9 + E_6$	$286 A_1 + A_3 + D_7 + E_7$	$321 A_1 + A_3 + E_6 + E_8$
$184 2A_1 + A_7 + D_9$	220 $2A_3 + 2E_6$	254 $A_1 + D_{11} + E_6$	$287 A_4 + D_7 + E_7$	$322 A_4 + E_6 + E_8$
$185 A_1 + A_8 + D_9$	$221 A_1 + A_3 + 2 A_4 + E_6$	255 $D_{12} + E_6$	$288 A_1 + A_2 + D_8 + E_7$	323 $D_4 + E_6 + E_8$
$186 A_9 + D_9$	$222 A_1 + A_5 + 2 E_6$	256 $2A_2 + 2E_7$	289 $A_2 + D_9 + E_7$	$324 A_1 + A_2 + E_7 + E_8$
187 $A_4 + D_5 + D_9$	$223 A_2 + 2A_5 + E_6$	257 $A_1 + A_3 + 2 E_7$	$1290 A_1 + D_{10} + E_7$	$325 A_3 + E_7 + E_8$

			$F = \begin{pmatrix} 1 & 0 \end{pmatrix}$	
			$L = \begin{pmatrix} 0 & 1 \end{pmatrix}$	
Νο Σ	Νο Σ	Νο Σ		Νο Σ
$152 A_{12} + D_6$	$188 \ 2A_1 + 2A_3 + D_{10}$	$224 \mid 2A_2 + A_3 + A_5 + E_6$	$258 A_4 + 2 E_7$	$291 D_{11} + E_7$
$\begin{array}{c c} \hline 153 & A_2 + A_5 + D_5 + D_6 \\ \hline \end{array}$	$189 2A_4 + D_{10}$	$225 A_2 + A_4 + A_5 + E_6$	$259 A_1 + 2A_2 + A_4 + N_7$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$
$\begin{array}{c c} \hline 154 & A_7 + D_5 + D_6 \\ \hline \end{array}$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$226 A_6 + 2 E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$
$155 2A_2 + 2D_7$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c} 227 & A_1 + A_2 + A_2 + A_6 + E_6 \\ \hline \end{array}$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$200 11 + 114 + E_6 + E_7$
$156 A_2 + 3A_3 + D_7$	$192 A_3 + A_5 + D_{10} $		$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
$157 A_1 + A_2 + 2 A_4 + D_7$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	228 $2A_1 + A_4 + A_6 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{206 + 2}{206 + 2} \frac{2}{4} + \frac{2}{5} \frac{1}{5} \frac{1}{5$
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$194 A_8 + D_{10}$	229 $A_2 + A_4 + A_6 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$2907 2A_1 + 2E_8$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$230 A_1 + A_5 + A_6 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$297 A_2 + 2 E_8$
$\frac{160}{160} \frac{A_5 + A_6 + D_7}{A_5 + A_6 + D_7}$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$231 A_1 + A_4 + A_7 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c} \hline 2 & - & - & - & - & - & - \\ \hline 197 & A_1 + D_7 + D_{10} \\ \hline \end{array}$	$232 A_5 + A_7 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{299}{2} \frac{2}{A_1} + \frac{2}{A_4} + \frac{1}{E_8}$
$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$233 2A_2 + A_8 + E_6$	$\begin{bmatrix} 207 & A_1 + A_4 + A_6 + D_7 \\ \end{bmatrix}$	$300 A_1 + A_2 + A_3 + A_4 + E_8$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$234 2A_1 + A_2 + A_8 + E_6$	$268 A_5 + A_6 + E_7$	$301 2A_5 + E_8$
$\frac{164}{164} + \frac{A_2 + A_0 + D_7}{A_2 + A_0 + D_7}$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$235 A_1 + A_3 + A_8 + E_6$	$269 + 2A_2 + A_7 + E_7$	$= 302 A_2 + A_3 + A_5 + E_8$
$\begin{array}{c c} \hline 165 & A_1 + A_{10} + D_7 \\ \hline \end{array}$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	270 $2A_1 + A_2 + A_7 + E_7$	$=$ 303 $A_1 + A_4 + A_5 + E_8$
$166 A_{11} + D_7$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c} 237 & A_1 + A_2 + A_0 + E_6 \end{array}$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$304 2A_2 + A_6 + E_8$
$167 A_1 + A_5 + D_5 + D_7$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= 305 2A_1 + A_2 + A_6 + E_8$
$168 A_5 + D_6 + D_7$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= 306 A_1 + A_3 + A_6 + E_8$
$169 2A_1 + 2D_2$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$307 A_4 + A_6 + E_8$
$170 2A_2 + 2A_2 + D_8$	$\frac{206}{206} \frac{D_6 + D_{12}}{D_6 + D_{12}}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= 308 A_1 + A_2 + A_7 + E_8$
$\frac{110}{171} \frac{2}{2} A_{z} + D_{o}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c} \mathcal{L} \mathbf{H} \\ \mathcal{L} \\ L$	$\frac{210}{276} \frac{2A_1 + A_2 + E_7}{276}$	$= 309 2A_1 + A_8 + E_8$
$172 2A_1 + A_2 + A_7 + D_8$	$\frac{201}{11} + \frac{114}{14} + \frac{213}{13}$ $\frac{208}{11} + \frac{4}{14} + \frac{2}{13}$	$242 A_{12} + E_6$	$A_2 + A_9 + L_7$	- 310 $A_2 + A_8 + E_8$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\frac{200}{209} \frac{D_5 + D_{13}}{D_5 + D_{12}}$	$243 A_3 + A_4 + D_5 + E_6$	$277 A_1 + A_{10} + E_7$	$311 A_1 + A_9 + E_8$
$174 2 A_2 + A_c + D_8$	200 2.3 + 2.13 210 $2.4_2 + D_{14}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		312 $A_{10} + E_8$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$278 A_{11} + E_7$	$313 \mid 2D_5 + E_8$
$\frac{110}{176} \frac{A_1 + A_2 + A_3}{A_1 + A_2}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$246 D_e + 2E_e$	$279 D_4 + 2E_7$	$314 A_1 + A_4 + D_5 + E_8$
$177 2D_r + D_s$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c} 247 & A_2 + A_4 + D_6 + E_6 \end{array}$	$280 A_2 + A_4 + D_5 + E_7$	$315 A_5 + D_5 + E_8$
$178 A_1 + A_2 + D_6 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c} \hline 281 & A_1 + A_5 + D_5 + E_7 \\ \hline \end{array}$	$316 \ 2 A_2 + D_6 + E_8$
$179 \ 2D_0$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$282 A_6 + D_5 + E_7$	$317 A_4 + D_6 + E_8$
$180 A_1 + 2 A_2 + A_4 + D_0$	$216 A_2 + D_{16}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$283 A_2 + A_3 + D_6 + E_7$	318 $A_1 + A_2 + D_7 + E_8$
$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$217 A_1 + D_{17}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$284 A_5 + D_6 + E_7$	$319 A_1 + D_9 + E_8$
$\frac{101}{182} \frac{1}{44} + \frac{1}{45} + \frac{1}{15} + \frac{1}{25} \frac{1}{9}$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$285 D_5 + D_6 + E_7$	$320 D_{10} + E_8$
$183 A_1 + A_2 + A_4 + D_0$	$219 3E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{bmatrix} 286 & A_1 + A_3 + D_7 + E_7 \end{bmatrix}$	$321 A_1 + A_3 + E_6 + E_8$
$184 2A_1 + A_7 + D_0$	$220 \ 2 \ A_3 + 2 \ E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{bmatrix} 287 & A_4 + D_7 + E_7 \end{bmatrix}$	$322 A_4 + E_6 + E_8$
$185 A_1 + A_0 + D_0$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{bmatrix} 288 & A_1 + A_2 + D_8 + E_7 \end{bmatrix}$	$323 \mid D_4 + E_6 + E_8$
$186 A_0 + D_0$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	256 + 2 + 2 = 2 = 2 = 2 = 2 = 2 = 2 = 2 = 2	$\begin{bmatrix} 289 & A_2 + D_0 + E_7 \end{bmatrix}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
$187 A_4 + D_5 + D_9$	$223 A_2 + 2A_5 + E_6$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$290 A_1 + D_{10} + E_7$	$325 A_3 + E_7 + E_8$
	2 0 0 0			- 0 0

			$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
Νο Σ	Νο Σ	Νο Σ	No 2	No 2
152 $A_{12} + D_6$	$188 2A_1 + 2A_3 + D_{10}$	$224 2A_2 + A_3 + A_5 + E_6$	258 $A_4 + 2E_7$	291 $D_{11} + E_7$
$153 A_2 + A_5 + D_5 + D_6$	189 $2A_4 + D_{10}$	$225 A_3 + A_4 + A_5 + E_6$	$\begin{array}{ $	$292 A_2 + A_3 + E_6 + E_7$
154 $A_7 + D_5 + D_6$	$190 A_1 + A_3 + A_4 + D_{10}$	226 $A_6 + 2E_6$	$\begin{array}{ $	293 $A_1 + A_4 + E_6 + E_7$
155 $2A_2 + 2D_7$	191 $3A_1 + A_5 + D_{10}$	$227 A_1 + A_2 + A_3 + A_6 + E_6$	$261 \ 2A_3 + A_5 + E_7$	294 $A_5 + E_6 + E_7$
156 $A_2 + 3A_3 + D_7$	192 $A_3 + A_5 + D_{10}$		$262 A_1 + A_2 + A_3 + A_5 + E_7 $	$295 D_5 + E_6 + E_7$
$157 A_1 + A_2 + 2 A_4 + D_7$	193 $A_2 + A_6 + D_{10}$	$228 2A_1 + A_4 + A_6 + E_6$	$263 2A_1 + A_4 + A_5 + E_7$	$296 2A_1 + 2E_8$
158 $A_2 + A_3 + A_6 + D_7$	194 $A_8 + D_{10}$	$229 A_2 + A_4 + A_6 + E_6$	$264 A_2 + A_4 + A_5 + E_7$	$297 A_2 + 2 E_8$
159 $A_1 + A_4 + A_6 + D_7$	$195 A_1 + A_2 + D_5 + D_{10}$	$230 A_1 + A_5 + A_6 + E_6$	$265 A_1 + 2A_2 + A_6 + E_7$	298 $2A_2 + 2A_3 + E_8$
160 $A_5 + A_6 + D_7$	$196 A_2 + D_6 + D_{10}$	$231 A_1 + A_4 + A_7 + E_6$	$266 A_2 + A_3 + A_6 + E_7$	299 $2A_1 + 2A_4 + E_8$
$161 2A_1 + A_2 + A_7 + D_7$	197 $A_1 + D_7 + D_{10}$	232 $A_5 + A_7 + E_6$	$267 A_1 + A_4 + A_6 + E_7$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$
$162 A_1 + A_3 + A_7 + D_7$	$198 2A_2 + A_3 + D_{11}$	233 $2A_2 + A_8 + E_6$		$301 2A_5 + E_8$
$163 2A_1 + A_9 + D_7$	$199 A_1 + A_2 + A_4 + D_{11}$	$234 2A_1 + A_2 + A_8 + E_6$	268 $A_5 + A_6 + E_7$	$302 A_2 + A_3 + A_5 + E_8$
164 $A_2 + A_9 + D_7$	$200 A_2 + A_5 + D_{11}$	$235 A_1 + A_3 + A_8 + E_6$	$269 2A_2 + A_7 + E_7$	$303 A_1 + A_4 + A_5 + E_8$
$165 A_1 + A_{10} + D_7$	$201 A_1 + A_6 + D_{11}$	236 $A_4 + A_8 + E_6$	$270 2A_1 + A_2 + A_7 + E_7$	$304 2A_2 + A_6 + E_8$
166 $A_{11} + D_7$	$202 2A_1 + 2A_2 + D_{12}$	$237 A_1 + A_2 + A_9 + E_6$	$271 A_1 + A_3 + A_7 + E_7$	$305 2A_1 + A_2 + A_6 + E_8$
$167 A_1 + A_5 + D_5 + D_7$	$203 A_1 + A_2 + A_3 + D_{12}$	238 $A_3 + A_9 + E_6$	272 $A_4 + A_7 + E_7$	$306 A_1 + A_3 + A_6 + E_8$
168 $A_5 + D_6 + D_7$	$204 2A_1 + A_4 + D_{12}$	$239 2A_1 + A_{10} + E_6$	$273 A_1 + A_2 + A_8 + E_7$	$307 A_4 + A_6 + E_8$
169 $2A_1 + 2D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	240 $A_2 + A_{10} + E_6$	274 $A_3 + A_8 + E_7$	$308 A_1 + A_2 + A_7 + E_8$
$170 2A_2 + 2A_3 + D_8$	$206 D_6 + D_{12}$	241 $A_1 + A_{11} + E_6$	275 $2A_1 + A_9 + E_7$	$= 309 + 2A_1 + A_8 + E_8$
171 $2A_5 + D_8$	$207 A_1 + A_4 + D_{13}$		276 $A_2 + A_9 + E_7$	$-310 A_2 + A_8 + E_8$
$172 2A_1 + A_3 + A_5 + D_8$	$208 A_5 + D_{13}$	$242 A_{12} + E_6$		$= 311 + A_0 + E_0$
$173 A_1 + A_4 + A_5 + D_8$	$209 D_5 + D_{13}$	$243 A_3 + A_4 + D_5 + E_6$	$ 277 A_1 + A_{10} + E_7 $	$312 A_{10} + E_0$
$174 \ 2A_2 + A_6 + D_8$	$210 2A_2 + D_{14}$	$244 A_1 + A_6 + D_5 + E_6$		$= \frac{312}{313} \frac{2}{2} D_r + E_0$
$175 A_1 + A_2 + A_7 + D_8$	$211 2A_1 + A_2 + D_{14}$	$245 A_7 + D_5 + E_6$	$278 A_{11} + E_7$	$= \frac{310}{2} \frac{2}{2} \frac{2}{5} + \frac{1}{28} \frac{1}{5} + \frac{1}{28} \frac{1}{5} + \frac{1}{28} \frac{1}{5} \frac{1}{5}$
$176 A_1 + A_9 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$246 D_6 + 2E_6$	$219 D_4 + 2E_7$	$= \frac{314}{315} \frac{A_1 + A_4 + D_5 + D_8}{A_7 + D_7 + E_9}$
$177 \ 2D_5 + D_8$	$213 A_4 + D_{14}$	$247 A_2 + A_4 + D_6 + E_6$	$280 A_2 + A_4 + D_5 + E_7$	$= \frac{316}{240} + \frac{2}{200} + $
$178 A_1 + A_3 + D_6 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$248 A_6 + D_6 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= \frac{310 \ 2A_2 + D_6 + D_8}{317 \ A_4 + D_6 + E_8}$
$179 2D_9$	$215 \ 2A_1 + D_{16}$	$249 A_1 + A_4 + D_7 + E_6$	$\frac{262}{282} + \frac{A_6 + D_5 + L_7}{282}$	$= \frac{317 + 14 + D_0 + D_8}{218 + 4_0 + D_7 + F_0}$
$180 A_1 + 2 A_2 + A_4 + D_9$	$216 A_2 + D_{16}$	$250 D_5 + D_7 + E_6$	$\frac{263}{284} + \frac{A_2 + A_3 + D_6 + L_7}{284}$	$= \frac{310 A_1 + A_2 + D_7 + E_8}{310 A_1 + D_2 + E_2}$
$181 A_1 + A_3 + A_5 + D_9$	$217 A_1 + D_{17}$	$251 A_4 + D_8 + E_6$	$284 A_5 + D_6 + E_7$	$= \frac{319}{220} + \frac{1}{20} + \frac{1}$
$182 A_4 + A_5 + D_9$	$218 D_{18}$	$252 A_1 + A_2 + D_9 + E_6$	$285 D_5 + D_6 + E_7$	$= \frac{320}{221} \frac{D_{10} + L_8}{A + A + E + E}$
$ 183 A_1 + A_2 + A_6 + D_9 $	$219 3E_6$	$253 A_3 + D_9 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$= \frac{\begin{vmatrix} 321 & A_1 + A_3 + E_6 + E_8 \end{vmatrix}}{\begin{vmatrix} 222 & A_1 + E_2 + E_2 \end{vmatrix}}$
$184 2A_1 + A_7 + D_9$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$254 A_1 + D_{11} + E_6$	$\frac{ 201 A_4 + D_7 + E_7}{ 200 A_4 + A_7 + D_7 + E_7}$	$= \frac{322 A_4 + E_6 + E_8}{222 D_4 + E_4 + E_5}$
$185 A_1 + A_8 + D_9$	$\begin{array}{ $	$255 D_{12} + E_6$	$208 A_1 + A_2 + D_8 + E_7$	$= \frac{\begin{array}{ c c c c c c c c c c c c c c c c c c $
$180 A_9 + D_9$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$256 2A_2 + 2E_7$	$289 A_2 + D_9 + E_7$	$= \frac{324 A_1 + A_2 + E_7 + E_8}{225 A + E + E}$
$181 A_4 + D_5 + D_9$	$223 A_2 + 2A_5 + E_6$	$ 257 A_1 + A_3 + 2E_7 $	$290 A_1 + D_{10} + E_7$	$323 \mid A_3 + E_7 + E_8$

			$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
Νο Σ	Νο Σ	Νο Σ	No Ž	No 2
152 $A_{12} + D_6$	$188 2A_1 + 2A_3 + D_{10}$	$224 2A_2 + A_3 + A_5 + E_6 $	258 $A_4 + 2E_7$	291 $D_{11} + E_7$
153 $A_2 + A_5 + D_5 + D_6$	189 $2A_4 + D_{10}$	225 $A_3 + A_4 + A_5 + E_6$	259 $A_1 + 2A_3 + A_4 + E_7$	292 $A_2 + A_3 + E_6 + E_7$
154 $A_7 + D_5 + D_6$	$190 A_1 + A_3 + A_4 + D_{10}$	226 $A_6 + 2E_6$	$260 2A_2 + A_3 + A_4 + E_7$	293 $A_1 + A_4 + E_6 + E_7$
155 $2A_2 + 2D_7$	191 $3A_1 + A_5 + D_{10}$	$227 A_1 + A_2 + A_3 + A_6 + E_6 $	261 $2A_3 + A_5 + E_7$	$294 A_5 + E_6 + E_7$
156 $A_2 + 3A_3 + D_7$	192 $A_3 + A_5 + D_{10}$		$262 A_1 + A_2 + A_3 + A_5 + E_7 $	$295 D_5 + E_6 + E_7$
$157 A_1 + A_2 + 2 A_4 + D_7$	193 $A_2 + A_6 + D_{10}$	$228 2A_1 + A_4 + A_6 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$296 2A_1 + 2E_8$
$158 A_2 + A_3 + A_6 + D_7$	194 $A_8 + D_{10}$	$229 A_2 + A_4 + A_6 + E_6$	$264 A_2 + A_4 + A_5 + E_7$	297 $A_2 + 2 E_8$
$159 A_1 + A_4 + A_6 + D_7$	$195 A_1 + A_2 + D_5 + D_{10}$	$230 A_1 + A_5 + A_6 + E_6$	$265 A_1 + 2 A_2 + A_6 + E_7$	298 $2A_2 + 2A_3 + E_8$
160 $A_5 + A_6 + D_7$	196 $A_2 + D_6 + D_{10}$	$231 A_1 + A_4 + A_7 + E_6$	$266 A_2 + A_3 + A_6 + E_7 $	299 $2A_1 + 2A_4 + E_8$
161 $2A_1 + A_2 + A_7 + D_7$	197 $A_1 + D_7 + D_{10}$	232 $A_5 + A_7 + E_6$	$267 A_1 + A_4 + A_6 + E_7$	$300 A_1 + A_2 + A_3 + A_4 + E_8$
$162 A_1 + A_3 + A_7 + D_7$	$198 2A_2 + A_3 + D_{11}$	233 $2A_2 + A_8 + E_6$	_	$301 \ 2 A_5 + E_8$
$163 2A_1 + A_9 + D_7$	$199 A_1 + A_2 + A_4 + D_{11}$	$234 2A_1 + A_2 + A_8 + E_6$	268 $A_5 + A_6 + E_7$	$302 A_2 + A_3 + A_5 + I_8$
164 $A_2 + A_9 + D_7$	$200 A_2 + A_5 + D_{11}$	$235 A_1 + A_3 + A_8 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$303 A_1 + A_4 + A_5 + E_8$
$165 A_1 + A_{10} + D_7$	$201 A_1 + A_6 + D_{11}$	236 $A_4 + A_8 + E_6$	$270 2A_1 + A_2 + A_7 + E_7$	$304 2A_2 + A_6 + E_8$
166 $A_{11} + D_7$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$237 A_1 + A_2 + A_9 + E_6$	$271 A_1 + A_3 + A_7 + E_7$	$305 2A_1 + A_2 + A_6 + E_8$
$167 A_1 + A_5 + D_5 + D_7$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	238 $A_3 + A_9 + E_6$	272 $A_4 + A_7 + E_7$	$306 A_1 + A_3 + A_6 + E_8$
168 $A_5 + D_6 + D_7$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$239 2A_1 + A_{10} + E_6$	273 $A_1 + A_2 + A_8 + E_7$	$307 A_4 + A_6 + E_8$
$169 2A_1 + 2D_8$	$\begin{array}{c c} 205 & A_1 + D_5 + D_{12} \\ \hline \end{array}$	$240 A_2 + A_{10} + E_6$	$274 A_3 + A_8 + E_7$	$308 A_1 + A_2 + A_7 + E_8$
$170 2A_2 + 2A_3 + D_8$	$206 D_6 + D_{12}$	241 $A_1 + A_{11} + E_6$	275 $2A_1 + A_9 + E_7$	$= 309 2A_1 + A_8 + E_8$
$171 \ 2A_5 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$ 276 A_2 + A_9 + E_7 $	$-310 A_2 + A_8 + E_8$
$172 2A_1 + A_3 + A_5 + D_8$	$208 A_5 + D_{13}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$		$= 311 + A_0 + E_0$
$173 A_1 + A_4 + A_5 + D_8$	$209 D_5 + D_{13}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$= \begin{vmatrix} 277 & A_1 + A_{10} + E_7 \end{vmatrix}$	$312 A_{10} + E_8$
$174 \ 2A_2 + A_6 + D_8$	$210 2A_2 + D_{14}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$= \begin{array}{ c c } \hline \hline$	$313 2 D_{5} + E_{9}$
$175 A_1 + A_2 + A_7 + D_8$	$211 2A_1 + A_2 + D_{14}$	245 $A_7 + D_5 + E_6$	$= \frac{270}{270} \frac{D_1 + 2F_2}{D_2 + 2F_2}$	$\frac{314}{314} + A_1 + A_4 + D_5 + E_8$
$176 A_1 + A_9 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$246 D_6 + 2E_6$	$= \frac{279}{280} \frac{D_4 + 2E_7}{A_7 + A_7 + D_7 + E_7}$	$= \frac{311}{315} \frac{A_{5} + D_{5}}{A_{5} + D_{5}} + E_{6}$
$177 2D_5 + D_8$	$213 A_4 + D_{14}$	$247 A_2 + A_4 + D_6 + E_6$	$= \frac{250}{200} + \frac{A_2 + A_4 + D_5 + E_7}{A_2 + A_5 + D_5 + E_7}$	$\frac{1010 + 13 + 23 + 90}{316 + 242 + D_6 + E_8}$
$178 A_1 + A_3 + D_6 + D_8$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$248 A_6 + D_6 + E_6$	$= \frac{281}{282} \frac{A_1 + A_5 + D_5 + D_7}{A_1 + A_5 + D_7 + E_7}$	$= \frac{317}{317} \frac{A_4 + D_6}{A_4 + D_6} + \frac{E_8}{E_8}$
$\frac{179}{180} \frac{2D_9}{4} + 2A + A + D$	$213 \ 2A_1 + D_{16}$	$249 A_1 + A_4 + D_7 + E_6$	$= \frac{262}{262} \frac{A_6 + D_5 + D_7}{A_6 + D_5 + D_7}$	$= \frac{318}{318} \frac{A_1 + A_2 + D_7 + E_8}{A_1 + A_2 + D_7 + E_8}$
$180 A_1 + 2A_2 + A_4 + D_9$	$210 A_2 + D_{16}$	$250 D_5 + D_7 + E_6$	$= \frac{263}{203} \frac{A_2 + A_3 + D_6 + D_7}{A_2 + A_3 + D_6 + D_7}$	$= \frac{310}{319} \frac{A_1 + D_2}{A_1 + D_2} \frac{D_1 + D_8}{E_2}$
$181 A_1 + A_3 + A_5 + D_9$	$217 A_1 + D_{17}$	251 $A_4 + D_8 + E_6$	$= \frac{264}{285} \frac{D_5 + D_6 + E_7}{D_5 + D_6 + E_7}$	$\frac{320}{D_{10} + E_0} \wedge -(1 \land 1)$
$102 A_4 + A_5 + D_9$	$210 D_{18}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$= \frac{200 D_5 + D_6 + D_7}{286 A_7 + A_5 + D_7 + F_7}$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
$ 100 A_1 + A_2 + A_6 + D_9 $	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$253 A_3 + D_9 + E_6$	$= \frac{200 A_1 + A_3 + D_7 + D_7}{287 A_2 + D_2 + F_2}$	$= \frac{321}{322} \frac{1}{4} \pm \frac{1}{5} \pm $
$104 \ 2 \ A_1 + A_7 + D_9$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$254 A_1 + D_{11} + E_6$	$= \frac{201}{288} \frac{A_4 + D_7 + D_7}{A_4 + A_5 + D_5 + F_5}$	$= \frac{322}{323} \frac{A_4 + E_6 + E_8}{D_4 + E_6 + E_6}$
$100 A_1 + A_8 + D_9$ $186 A_2 + D_2$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$255 D_{12} + E_6$	$= \frac{200 A_1 + A_2 + D_8 + E_7}{280 A_2 + D_2 + E_7}$	$= \frac{223}{224} + \frac{D_4 + D_6 + D_8}{A_1 + A_2 + F_2 + F_2}$
$ \begin{array}{c cccccccccccccccccccccccccccccccccc$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$230 2A_2 + 2E_7$	$= \frac{209 A_2 + D_9 + D_7}{290 A_1 + D_{10} + F_7}$	$\frac{324 + A_1 + A_2 + E_7 + E_8}{325 + A_2 + E_7 + E_2}$
101 $\Lambda_4 \pm \nu_5 \pm \nu_9$	220 $12 \pm 275 \pm 126$	$ 201 A_1 + A_3 + 2E_7 $	$250 \Lambda_1 + D_{10} + D_7$	520 A3 + E7 + E8

			$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
Νο Σ	Νο Σ	Νο Σ	No Ž	No 2
152 $A_{12} + D_6$	$188 2A_1 + 2A_3 + D_{10}$	$224 2A_2 + A_3 + A_5 + E_6$	258 $A_4 + 2E_7$	291 $D_{11} + E_7$
153 $A_2 + A_5 + D_5 + D_6$	189 $2A_4 + D_{10}$	225 $A_3 + A_4 + A_5 + E_6$	259 $A_1 + 2A_3 + A_4 + E_7$	292 $A_2 + A_3 + E_6 + E_7$
154 $A_7 + D_5 + D_6$	$190 A_1 + A_3 + A_4 + D_{10}$	226 $A_6 + 2E_6$	$260 2A_2 + A_3 + A_4 + E_7 $	293 $A_1 + A_4 + E_6 + E_7$
155 $2A_2 + 2D_7$	$191 3A_1 + A_5 + D_{10}$	$227 A_1 + A_2 + A_3 + A_6 + E_6 $	$\boxed{261 \ 2A_3 + A_5 + E_7}$	294 $A_5 + E_6 + E_7$
156 $A_2 + 3A_3 + D_7$	192 $A_3 + A_5 + D_{10}$		$262 A_1 + A_2 + A_3 + A_5 + E_7 $	295 $D_5 + E_6 + E_7$
$157 A_1 + A_2 + 2 A_4 + D_7$	193 $A_2 + A_6 + D_{10}$	$228 2A_1 + A_4 + A_6 + E_6$	$263 2A_1 + A_4 + A_5 + E_7$	296 $2A_1 + 2E_8$
$158 A_2 + A_3 + A_6 + D_7$	194 $A_8 + D_{10}$	229 $A_2 + A_4 + A_6 + E_6$	$264 A_2 + A_4 + A_5 + E_7$	297 $A_2 + 2E_8$
159 $A_1 + A_4 + A_6 + D_7$	$195 A_1 + A_2 + D_5 + D_{10} $	230 $A_1 + A_5 + A_6 + E_6$	$265 A_1 + 2 A_2 + A_6 + E_7$	298 $2A_2 + 2A_3 + E_8$
160 $A_5 + A_6 + D_7$	196 $A_2 + D_6 + D_{10}$	231 $A_1 + A_4 + A_7 + E_6$	$266 A_2 + A_3 + A_6 + E_7$	$299 \ 2 A_1 + 2 A_4 + E_8$
$161 2A_1 + A_2 + A_7 + D_7$	197 $A_1 + D_7 + D_{10}$	232 $A_5 + A_7 + E_6$	267 $A_1 + A_4 + A_6 + E_7$	$300 A_1 + A_2 + A_3 + A_4 + E_8$
$162 A_1 + A_3 + A_7 + D_7$	198 $2A_2 + A_3 + D_{11}$	233 $2A_2 + A_8 + E_6$		$301 2A_5 + E_8$
163 $2A_1 + A_9 + D_7$	199 $A_1 + A_2 + A_4 + D_{11}$	$234 2A_1 + A_2 + A_8 + E_6$	268 $A_5 + A_6 + E_7$	$302 A_2 + A_3 + A_5 + L_8$
164 $A_2 + A_9 + D_7$	200 $A_2 + A_5 + D_{11}$	235 $A_1 + A_3 + A_8 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$303 A_1 + A_4 + A_5 + E_8$
165 $A_1 + A_{10} + D_7$	201 $A_1 + A_6 + D_{11}$	236 $A_4 + A_8 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$304 2A_2 + A_6 + E_8$
166 $A_{11} + D_7$	$202 2A_1 + 2A_2 + D_{12}$	237 $A_1 + A_2 + A_9 + E_6$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$305 2 A_1 + A_2 + A_6 + E_8$
167 $A_1 + A_5 + D_5 + D_7$	$203 A_1 + A_2 + A_3 + D_{12} $	238 $A_3 + A_9 + E_6$	272 $A_4 + A_7 + E_7$	$\begin{array}{c c} 306 & 2111 + 112 + 110 & 128 \\ \hline 306 & A_1 + A_2 + A_4 + E_2 \\ \hline \end{array}$
168 $A_5 + D_6 + D_7$	$204 2A_1 + A_4 + D_{12}$	239 $2A_1 + A_{10} + E_6$	273 $A_1 + A_2 + A_8 + E_7$	$307 A_4 + A_6 + E_9$
169 $2A_1 + 2D_8$	205 $A_1 + D_5 + D_{12}$	240 $A_2 + A_{10} + E_6$	274 $A_3 + A_8 + E_7$	$\begin{array}{c c} 301 & 11_4 + 11_6 + 12_8 \\ \hline 308 & A_1 + A_2 + A_7 + E_2 \end{array}$
170 $2A_2 + 2A_3 + D_8$	206 $D_6 + D_{12}$	241 $A_1 + A_{11} + E_6$	275 $2A_1 + A_9 + E_7$	$309 2 4_1 + 4_2 + F_1$
171 $2A_5 + D_8$	$207 A_1 + A_4 + D_{13}$		$276 A_2 + A_9 + E_7$	$310 A_2 + A_2 + E_2$
172 $2A_1 + A_3 + A_5 + D_8$	208 $A_5 + D_{13}$	242 $A_{12} + E_6$	_	$\begin{array}{c c} 310 & A_2 + A_8 + D_8 \\ \hline 211 & A_1 + A_2 + F_2 \\ \hline \end{array}$
$173 A_1 + A_4 + A_5 + D_8 $	209 $D_5 + D_{13}$	243 $A_3 + A_4 + D_5 + E_6$	277 $A_1 + A_{10} + E_7$	$\begin{array}{c c} 311 & A_1 + A_9 + L_8 \\ \hline 212 & A + E \\ \hline \end{array}$
174 $2A_2 + A_6 + D_8$	210 $2A_2 + D_{14}$	$244 A_1 + A_6 + D_5 + E_6$		$312 A_{10} + E_8$
$175 A_1 + A_2 + A_7 + D_8$	$211 2A_1 + A_2 + D_{14}$	245 $A_7 + D_5 + E_6$	$278 A_{11} + E_7$	$313 2D_5 + E_8$
176 $A_1 + A_9 + D_8$	$212 A_1 + A_3 + D_{14}$	246 $D_6 + 2E_6$	279 $D_4 + 2E_7$	$314 A_1 + A_4 + \mu_5 + E_8$
177 $2 D_5 + D_8$	213 $A_4 + D_{14}$	247 $A_2 + A_4 + D_6 + E_6$	$280 A_2 + A_4 + D_5 + E_7$	$315 A_5 + D_5 + E_8$
$178 A_1 + A_3 + D_6 + D_8 $	214 $A_1 + A_2 + D_{15}$	248 $A_6 + D_6 + E_6$	$281 A_1 + A_5 + D_5 + E_7$	$316 \ 2A_2 + D_6 + E_8$
179 2 D_9	215 $2A_1 + D_{16}$	249 $A_1 + A_4 + D_7 + E_6$	$282 A_6 + D_5 + E_7$	$317 A_4 + D_6 + E_8$
$180 A_1 + 2 A_2 + A_4 + D_9 \ $	216 $A_2 + D_{16}$	$250 D_5 + D_7 + E_6$	$283 A_2 + A_3 + D_6 + E_7$	$318 A_1 + A_2 + D_7 + E_8$
$181 A_1 + A_3 + A_5 + D_9$	217 $A_1 + D_{17}$	251 $A_4 + D_8 + E_6$	$284 A_5 + D_6 + E_7$	$\begin{array}{c c c c c c c c c c c c c c c c c c c $
182 $A_4 + A_5 + D_9$	218 D_{18}	$252 A_1 + A_2 + D_9 + E_6$	$285 D_5 + D_6 + E_7$	$320 (D_{10} + E_8) A = (1,015)$
$183 A_1 + A_2 + A_6 + D_9$	219 3 E_6	253 $A_3 + D_9 + E_6$	$286 A_1 + A_3 + D_7 + E_7$	$321 A_1 + A_3 + E_6 + E_8 \qquad ($
$184 2A_1 + A_7 + D_9$	$2\overline{20}$ $2A_3 + 2E_6$	254 $A_1 + D_{11} + E_6$	287 $A_4 + D_7 + E_7$	$322 A_4 + E_6 + E_8$
185 $A_1 + A_8 + D_9$	$221 A_1 + A_3 + 2A_4 + E_6 $	255 $D_{12} + E_6$	$288 A_1 + A_2 + D_8 + E_7$	$323 \mid D_4 + E_6 + E_8$
186 $A_9 + D_9$	$2\overline{22}$ $A_1 + A_5 + 2E_6$	256 $2A_2 + 2E_7$	289 $A_2 + D_9 + E_7$	$324 A_1 + A_2 + E_7 + E_8$
187 $A_4 + D_5 + D_9$	223 $A_2 + 2A_5 + E_6$	257 $A_1 + A_3 + 2E_7$	$290 A_1 + D_{10} + E_7$	325 $A_3 + E_7 + E_8$

No	Σ	Νο Σ	No	Σ	No	Σ	No	Σ
1	$6A_3$	$\boxed{34 A_1 + A_2 + A_3 + A_5 + A_7}$	65	$A_3 + A_6 + A_9$	92	$2A_1 + A_4 + A_{12}$	120	$A_2 + A_5 + A_6 + D_5$
2	$2A_1 + 4A_4$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	66	$A_2 + A_7 + A_9$			101	
3	$2A_2 + 2A_3 + 2A_4$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	67	$A_1 + A_8 + A_9$	93	$A_2 + A_4 + A_{12}$	121	$A_1 + A_7 + 2 D_5$
4	$3A_1 + 3A_5$		68	$A_2 + 2A_3 + A_{10}$	94	$A_1 + A_5 + A_{12}$	122	$A_1 + A_2 + A_3 + A_7 + D_5$
5	$4A_2 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	69	$A_1 + 2A_2 + A_3 + A_{10}$	95	$A_6 + A_{12}$	123	$2A_1 + A_4 + A_7 + D_5$
6	$A_3 + 3 A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	70	$2A_4 + A_{10}$	96	$A_1 + 2A_2 + A_{13}$	124	$A_8 + 2 D_5$
7	$2A_1 + 2A_3 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	71	$2A_2 + A_4 + A_{10}$			125	$A_1 + A_4 + A_8 + D_5$
8	$A_1 + 2A_2 + A_3 + 2A_5$	$40 A_1 + A_4 + A_6 + A_7$			97	$3A_1 + A_2 + A_{13}$		
9	$2A_4 + 2A_5$		72	$2A_1 + A_2 + A_4 + A_{10}$	98	$2A_1 + A_3 + A_{13}$	126	$A_5 + A_8 + D_5$
10	$2A_2 + A_4 + 2A_5$	41 $A_5 + A_6 + A_7$	73	$A_1 + A_3 + A_4 + A_{10}$	99	$A_2 + A_3 + A_{13}$	127	$2A_2 + A_9 + D_5$
11	$A_1 + A_3 + A_4 + 2A_5$	42 $2A_1 + 2A_8$			100	$A_1 + A_4 + A_{13}$	128	$2A_1 + A_2 + A_9 + D_5$
12	$A_1 + A_2 + 2A_3 + A_4 + A_5$		74	$A_1 + A_2 + A_5 + A_{10}$			129	$A_1 + A_3 + A_9 + D_5$
13		$43 A_1 + 3A_2 + A_3 + A_8$					130	$A_4 + A_9 + D_5$
14	$2A_1 + 2A_2 + 2A_6$	$44 \ 2A_1 + 2A_4 + A_8$	75	$A_3 + A_5 + A_{10}$	101	$A_5 + A_{13}$	131	$A_1 + A_2 + A_{10} + D_5$
15	$2A_3 + 2A_6$	45 $3A_2 + A_4 + A_8$			102	$2A_2 + A_{14}$	132	$2A_1 + A_{11} + D_5$
16	$A_2 + A_4 + 2 A_6$	$46 A_1 + A_2 + A_3 + A_4 + A_8$	76	$2A_1 + A_6 + A_{10}$	103	$2A_1 + A_2 + A_{14}$	133	$A_2 + A_{11} + D_5$
17	$2A_1 + A_2 + 2A_4 + A_6$	$47 A_1 + 2A_2 + A_5 + A_8$	77	$A_2 + A_6 + A_{10}$			134	$A_1 + A_{12} + D_5$
18	$A_1 + A_3 + 2A_4 + A_6$	$48 A_2 + A_3 + A_5 + A_8$			104	$A_1 + A_3 + A_{14}$		
		$49 A_1 + A_4 + A_5 + A_8$	78	$A_1 + A_7 + A_{10}$	105	$A_4 + A_{14}$	135	$A_{13} + D_5$
19	$A_2 + 2A_3 + A_4 + A_6$	50 $2A_1 + A_2 + A_6 + A_8$			106	$3A_1 + A_{15}$	136	$3 D_6$
20	$A_1 + 2 A_2 + A_3 + A_4 + A_6$	51 $A_1 + A_3 + A_6 + A_8$	79	$A_8 + A_{10}$	107	$A_1 + A_2 + A_{15}$	137	$2A_3 + 2D_6$
21	$2A_1 + 2A_5 + A_6$	$52 A_4 + A_6 + A_8$	80	$A_1 + 3A_2 + A_{11}$			138	$2A_2 + 2A_4 + D_6$
22	$A_1 + 2A_3 + A_5 + A_6$	53 $A_1 + A_2 + A_7 + A_8$	81	$3A_1 + 2A_2 + A_{11}$	108	$A_3 + A_{15}$	139	$2A_1 + 2A_5 + D_6$
23	$A_1 + A_2 + A_4 + A_5 + A_6$	54 $2A_9$	82	$A_1 + 2A_3 + A_{11}$	109	$2A_1 + A_{16}$	140	$A_1 + 2A_3 + A_5 + D_6$
			83	$2A_2 + A_3 + A_{11}$			141	$A_3 + A_4 + A_5 + D_6$
24	$A_3 + A_4 + A_5 + A_6$	55 $A_1 + A_2 + 2A_3 + A_9$			110	$A_2 + A_{16}$	142	$2A_6 + D_6$
25	$4A_1 + 2A_7$	$56 2A_1 + 2A_2 + A_3 + A_9$	84	$2A_1 + A_2 + A_3 + A_{11}$	111	$A_1 + A_{17}$	143	$A_2 + A_4 + A_6 + D_6$
26	$2A_2 + 2A_7$	57 $A_1 + 2A_4 + A_9$					144	$A_1 + 2A_2 + \overline{A_7 + D_6}$
		$58 3A_1 + A_2 + A_4 + A_9$	85	$3A_1 + A_4 + A_{11}$	112	A ₁₈	145	$A_2 + A_3 + A_7 + D_6$
27	$A_1 + A_3 + 2A_7$	$59 2A_1 + A_3 + A_4 + A_9$	86	$A_1 + A_2 + A_4 + A_{11}$	113	$2A_4 + 2D_5$	146	$A_1 + A_4 + A_7 + D_6$
28	$2A_1 + 3A_3 + A_7$	$\begin{bmatrix} 60 & 2A_1 + A_2 + A_5 + A_9 \end{bmatrix}$	87	$2A_1 + A_5 + A_{11}$	114	$A_3 + 2A_5 + D_5$	147	$A_4 + A_8 + D_6$
29	$A_2 + 3A_3 + A_7$	$61 A_1 + A_3 + A_5 + A_9$			115	$2A_4 + A_5 + D_5$	148	$A_1 + A_2 + A_9 + D_6$
30	$2A_2 + A_3 + A_4 + A_7$	$62 A_4 + \overline{A_5 + A_9}$	88	$A_2 + A_5 + A_{11}$	116	$A_1 + A_3 + A_4 + A_5 + D_5$		
31	$2A_1 + A_2 + A_3 + A_4 + A_7$		89	$A_1 + A_6 + A_{11}$	117	$A_1 + 2A_6 + D_5$	149	$A_3 + A_9 + D_6$
32	$A_1 + 2A_5 + A_7$	$63 3A_1 + A_6 + A_9$	90	$2A_1 + 2A_2 + \overline{A_{12}}$	118	$2A_2 + A_3 + A_6 + D_5$	150	$A_2 + A_{10} + D_6$
33	$3A_1 + A_3 + A_5 + \overline{A_7}$	$64 A_1 + A_2 + A_6 + A_9$	91	$A_1 + A_2 + A_3 + A_{12}$	119	$A_1 + A_2 + A_4 + A_6 + D_5$	151	$A_1 + A_{11} + D_6$

 $E = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \qquad A_1 = \frac{1}{4}(0, 0, 0, 0, 1, 1, 1, -3, 0, 0, 0, 0, 1, 1, 1, -3) \\ A_2 = \frac{1}{4}(-3, 1, 1, 1, 0, 0, 0, 0, -3, 1, 1, 1, 0, 0, 0, 0)$

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Νο Σ	Νο Σ	No	Σ	No	Σ	No	Σ
$1 \left(6 A_3 \right)$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$65 A_3$	$+A_6+A_9$	92	$2A_1 + A_4 + A_{12}$	120	$A_2 + A_5 + A_6 + D_5$
$2 2A_1 + 4A_4$	$\boxed{35 2A_1 + A_4 + A_5 + A_7}$	$66 A_2$	$+A_7+A_9$				
$3 2A_2 + 2A_3 + 2A_4$	$36 A_2 + A_4 + A_5 + A_7$	67 A_1	$+A_{8}+A_{9}$	93	$A_2 + A_4 + A_{12}$	121	$A_1 + A_7 + 2 D_5$
$4 3A_1 + 3A_5$		$68 A_2$	$+2A_3 + A_{10}$	94	$A_1 + A_5 + A_{12}$	122	$A_1 + A_2 + A_3 + A_7 + D_5$
5 $4A_2 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$69 A_1$	$+2A_2 + A_3 + A_{10}$	95	$A_6 + A_{12}$	123	$2A_1 + A_4 + A_7 + D_5$
$6 A_3 + 3 A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	70 2 A	$A_4 + A_{10}$	96	$A_1 + 2A_2 + A_{13}$	124	$A_8 + 2D_5$
7 $2A_1 + 2A_3 + 2A_5$	$39 A_2 + A_3 + A_6 + A_7$	71 2 A	$A_2 + A_4 + A_{10}$			125	$A_1 + A_4 + A_8 + D_5$
8 $A_1 + 2A_2 + A_3 + 2A_5$	$\begin{array}{ c c c c c c c c c c c c c c c c c c c$			97	$3A_1 + A_2 + A_{13}$		
9 $2A_4 + 2A_5$		72 2 A	$A_1 + A_2 + A_4 + A_{10}$	98	$2A_1 + A_3 + A_{13}$	126	$A_5 + A_8 + D_5$
$10 2A_2 + A_4 + 2A_5$	41 $A_5 + A_6 + A_7$	$73 A_1$	$+A_3 + A_4 + A_{10}$	99	$A_2 + A_3 + A_{13}$	127	$2A_2 + A_9 + D_5$
$11 A_1 + A_3 + A_4 + 2 A_5$	42 $2A_1 + 2A_8$.		100	$A_1 + A_4 + A_{13}$	128	$2A_1 + A_2 + A_9 + D_5$
$12 A_1 + A_2 + 2A_3 + A_4 + A_5$		$74 A_1$	$+A_2 + A_5 + A_{10}$			129	$A_1 + A_3 + A_9 + D_5$
$13(3A_6)$	$43 A_1 + 3 A_2 + A_3 + A_8$					130	$A_4 + A_9 + D_5$
$14 2A_1 + 2A_2 + 2A_6$	$44 2A_1 + 2A_4 + A_8$	$75 A_3$	$+A_5 + A_{10}$	101	$A_5 + A_{13}$	131	$A_1 + A_2 + A_{10} + D_5$
$15 \ 2A_3 + 2A_6$	$45 3A_2 + A_4 + A_8$			102	$2A_2 + A_{14}$	132	$2A_1 + A_{11} + D_5$
$16 A_2 + A_4 + 2 A_6$	$46 A_1 + A_2 + A_3 + A_4 + A_8$	76 2A	$A_1 + A_6 + A_{10}$	103	$2A_1 + A_2 + A_{14}$	133	$A_2 + A_{11} + D_5$
$17 2A_1 + A_2 + 2A_4 + A_6$	$47 A_1 + 2A_2 + A_5 + A_8$	77 A_2	$+A_6 + A_{10}$			134	$A_1 + A_{12} + D_5$
$18 A_1 + A_3 + 2 A_4 + A_6$	$48 A_2 + A_3 + A_5 + A_8$			104	$A_1 + A_3 + A_{14}$		
	$49 A_1 + A_4 + A_5 + A_8$	$78 A_1$	$+A_7 + A_{10}$	105	$A_4 + A_{14}$	135	$A_{13} + D_5$
19 $A_2 + 2A_3 + A_4 + A_6$	$50 2A_1 + A_2 + A_6 + A_8$			106	$3A_1 + A_{15}$	136	$3 D_6$
$20 A_1 + 2A_2 + A_3 + A_4 + A_6$	$51 A_1 + A_3 + A_6 + A_8$	$79 A_8$	$+A_{10}$	107	$A_1 + A_2 + A_{15}$	137	$2A_3 + 2D_6$
21 $2A_1 + 2A_5 + A_6$	$52 A_4 + A_6 + A_8$	$80 A_1$	$+3A_2 + A_{11}$			138	$2A_2 + 2A_4 + D_6$
22 $A_1 + 2A_3 + A_5 + A_6$	53 $A_1 + A_2 + A_7 + A_8$	81 3 A	$A_1 + 2A_2 + A_{11}$	108	$A_3 + A_{15}$	139	$2A_1 + 2A_5 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$54 \ 2A_9$	82 A_1	$+2A_3 + A_{11}$	109	$2A_1 + A_{16}$	140	$A_1 + 2A_3 + A_5 + D_6$
		83 2 A	$A_2 + A_3 + A_{11}$			141	$A_3 + A_4 + A_5 + D_6$
24 $A_3 + A_4 + A_5 + A_6$	$55 A_1 + A_2 + 2A_3 + A_9$			110	$A_2 + A_{16}$	142	$2A_6 + D_6$
25 $4A_1 + 2A_7$	$56 2A_1 + 2A_2 + A_3 + A_9$	84 2 A	$A_1 + A_2 + A_3 + A_{11}$	111	$A_1 + A_{17}$	143	$A_2 + A_4 + A_6 + D_6$
26 $2A_2 + 2A_7$	57 $A_1 + 2A_4 + A_9$					144	$A_1 + 2A_2 + A_7 + D_6$
	$58 3A_1 + A_2 + A_4 + A_9$	85 3 A	$A_1 + A_4 + A_{11}$	112	A ₁₈	145	$A_2 + A_3 + A_7 + D_6$
27 $A_1 + A_3 + 2A_7$	$59 2A_1 + A_3 + A_4 + A_9$	86 A_1	$+A_2 + A_4 + A_{11}$	113	$2A_4 + 2D_5$	146	$A_1 + A_4 + A_7 + D_6$
$28 2A_1 + 3A_3 + A_7$	$60 2A_1 + A_2 + A_5 + A_9$	87 2 A	$A_1 + A_5 + A_{11}$	114	$A_3 + 2A_5 + D_5$	147	$A_4 + A_8 + D_6$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$61 A_1 + A_3 + A_5 + A_9$			115	$2A_4 + A_5 + D_5$	148	$A_1 + A_2 + A_9 + D_6$
$30 2A_2 + A_3 + A_4 + A_7$	$62 A_4 + A_5 + A_9$	88 A ₂	$+A_5 + A_{11}$	116	$A_1 + A_3 + A_4 + A_5 + D_5$		
$\begin{array}{ } 31 & 2A_1 + A_2 + A_3 + A_4 + A_7 \\ \hline \end{array}$		89 A_1	$+A_6 + A_{11}$	117	$A_1 + 2A_6 + D_5$	149	$A_3 + A_9 + D_6$
$32 A_1 + 2A_5 + A_7$	$ 63 \ 3A_1 + A_6 + A_9 $	90 2 A	$A_1 + 2A_2 + A_{12}$	118	$2A_2 + A_3 + A_6 + D_5$	150	$A_2 + A_{10} + \overline{D_6}$
$\begin{array}{ c c c c c c c c c c c c c c c c c c c$	$64 A_1 + A_2 + A_6 + A_9$	91 A_1	$+A_2 + A_3 + A_{12}$	119	$A_1 + A_2 + A_4 + A_6 + D_5$	151	$A_1 + A_{11} + D_6$

 $E = \begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \qquad A_1 = \frac{1}{4}(0, 0, 0, 0, 1, 1, 1, -3, 0, 0, 0, 0, 1, 1, 1, -3) \\ A_2 = \frac{1}{4}(-3, 1, 1, 1, 0, 0, 0, 0, -3, 1, 1, 1, 0, 0, 0, 0)$

No	Σ	No	\sum	No	Σ	No	Σ	No	Σ
1	$6A_3$	34	$A_1 + A_2 + A_3 + A_5 + A_7$	65	$A_3 + A_6 + A_9$	92	$2A_1 + A_4 + A_{12}$	120	$A_2 + A_5 + A_6 + D_5$
2	$2A_1 + 4A_4$	35	$2A_1 + A_4 + A_5 + A_7$	66	$A_2 + A_7 + A_9$				
3	$2A_2 + 2A_3 + 2A_4$	36	$A_2 + A_4 + A_5 + A_7$	67	$A_1 + A_8 + A_9$	93	$A_2 + A_4 + A_{12}$	121	$A_1 + A_7 + 2D_5$
4	$3A_1 + 3A_5$			68	$A_2 + 2A_3 + A_{10}$	94	$A_1 + A_5 + A_{12}$	122	$A_1 + A_2 + A_3 + A_7 + D_5$
5	$4A_2 + 2A_5$	37	$A_1 + 2A_2 + A_6 + A_7$	69	$A_1 + 2A_2 + A_3 + A_{10}$	95	$A_6 + A_{12}$	123	$2A_1 + A_4 + A_7 + D_5$
6	$A_3 + 3 A_5$	38	$2A_1 + A_3 + A_6 + A_7$	70	$2A_4 + A_{10}$	96	$A_1 + 2A_2 + A_{13}$	124	$A_8 + 2 D_5$
7	$2A_1 + 2A_3 + 2A_5$	39	$A_2 + A_3 + A_6 + A_7$	71	$2A_2 + A_4 + A_{10}$			125	$A_1 + A_4 + A_8 + D_5$
8	$A_1 + 2A_2 + A_3 + 2A_5$	40	$A_1 + A_4 + A_6 + A_7$			97	$3A_1 + A_2 + A_{13}$	100	
9	$2A_4 + 2A_5$			72	$2A_1 + A_2 + A_4 + A_{10}$	98	$2A_1 + A_3 + A_{13}$	126	$A_5 + A_8 + D_5$
10	$2A_2 + A_4 + 2A_5$	41	$A_5 + A_6 + A_7$	73	$A_1 + A_3 + A_4 + A_{10}$	99	$A_2 + A_3 + A_{13}$	127	$2A_2 + A_9 + D_5$
11	$A_1 + A_3 + A_4 + 2A_5$	42	$2A_1 + 2A_8$			100	$A_1 + A_4 + A_{13}$	128	$2A_1 + A_2 + A_9 + D_5$
12	$A_1 + A_2 + 2A_3 + A_4 + A_5$			74	$A_1 + A_2 + A_5 + A_{10}$			129	$A_1 + A_3 + A_9 + D_5$
13	$3A_6$	43	$A_1 + 3A_2 + A_3 + A_8$					130	$A_4 + A_9 + D_5$
14	$2A_1 + 2A_2 + 2A_6$	44	$2A_1 + 2A_4 + A_8$	75	$A_3 + A_5 + A_{10}$	101	$A_5 + A_{13}$	131	$A_1 + A_2 + A_{10} + D_5$
15	$2A_3 + 2A_6$	45	$3A_2 + A_4 + A_8$			102	$2A_2 + A_{14}$	132	$2A_1 + A_{11} + D_5$
16	$A_2 + A_4 + 2A_6$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	76	$2A_1 + A_6 + A_{10}$	103	$2A_1 + A_2 + A_{14}$	133	$A_2 + A_{11} + D_5$
17	$2A_1 + A_2 + 2A_4 + A_6$	47	$A_1 + 2A_2 + A_5 + A_8$	77	$A_2 + A_6 + A_{10}$			134	$A_1 + A_{12} + D_5$
18	$A_1 + A_3 + 2A_4 + A_6$	48	$A_2 + A_3 + A_5 + A_8$			104	$A_1 + A_3 + A_{14}$	105	
		49	$A_1 + A_4 + A_5 + A_8$	78	$A_1 + A_7 + A_{10}$	105	$A_4 + A_{14}$	135	$A_{13} + D_5$
19	$A_2 + 2A_3 + A_4 + A_6$	50	$2A_1 + A_2 + A_6 + A_8$			106	$3A_1 + A_{15}$	136	$3D_6$
20	$A_1 + 2A_2 + A_3 + A_4 + A_6$	51	$A_1 + A_3 + A_6 + A_8$	79	$A_8 + A_{10}$	107	$A_1 + A_2 + A_{15}$	137	$2A_3 + 2D_6$
21	$2A_1 + 2A_5 + A_6$	52	$A_4 + A_6 + A_8$	80	$A_1 + 3A_2 + A_{11}$			138	$2A_2 + 2A_4 + D_6$
22	$A_1 + 2A_3 + A_5 + A_6$	53	$A_1 + A_2 + A_7 + A_8$	81	$3A_1 + 2A_2 + A_{11}$	108	$A_3 + A_{15}$	139	$2A_1 + 2A_5 + D_6$
23	$A_1 + A_2 + A_4 + A_5 + A_6$	54	$2A_9$	82	$A_1 + 2A_3 + A_{11}$	109	$2A_1 + A_{16}$	140	$A_1 + 2A_3 + A_5 + D_6$
				83	$2A_2 + A_3 + A_{11}$			141	$A_3 + A_4 + A_5 + D_6$
24	$A_3 + A_4 + A_5 + A_6$	55	$A_1 + A_2 + 2A_3 + A_9$			110	$A_2 + A_{16}$	142	$2A_6 + D_6$
25	$4A_1 + 2A_7$	56	$2A_1 + 2A_2 + A_3 + A_9$	84	$2A_1 + A_2 + A_3 + A_{11}$		$A_1 + A_{17}$	143	$A_2 + A_4 + A_6 + D_6$
26	$2A_2 + 2A_7$	57	$A_1 + 2A_4 + A_9$					144	$A_1 + 2A_2 + A_7 + D_6$
97	$A \rightarrow A \rightarrow Q A$	58	$3A_1 + A_2 + A_4 + A_9$	85	$3A_1 + A_4 + A_{11}$	112	A_{18}	145	$A_2 + A_3 + A_7 + D_6$
$\frac{21}{28}$	$A_1 + A_3 + 2A_7$	59	$2A_1 + A_3 + A_4 + A_9$	86	$A_1 + A_2 + A_4 + A_{11}$	113	$2A_4 + 2D_5$	146	$A_1 + A_4 + A_7 + D_6$
$\frac{28}{20}$	$2A_1 + 3A_3 + A_7$	60	$2A_1 + A_2 + A_5 + A_9$	87	$2A_1 + A_5 + A_{11}$	114	$A_3 + 2A_5 + D_5$	147	$A_4 + A_8 + D_6$
29	$A_2 + 5A_3 + A_7$	61	$A_1 + A_3 + A_5 + A_9$			115	$2A_4 + A_5 + D_5$	148	$A_1 + A_2 + A_9 + D_6$
ას 91	$\frac{2A_2 + A_3 + A_4 + A_7}{2A_4 + A_4 + A_4 + A_4 + A_4 + A_4}$	62	$A_4 + A_5 + A_9$	88	$A_2 + A_5 + A_{11}$	116	$A_1 + A_3 + A_4 + A_5 + D_5$		
<u>ა</u> 1	$4A_1 + A_2 + A_3 + A_4 + A_7$	<u> </u>		89	$A_1 + A_6 + A_{11}$	117	$A_1 + 2A_6 + D_5$	149	$A_3 + A_9 + D_6$
5 <u>7</u>	$\frac{A_1 + 2A_5 + A_7}{2A + A + A + A}$	63	$3A_1 + A_6 + A_9$	90	$2A_1 + 2A_2 + A_{12}$	118	$2A_2 + A_3 + A_6 + D_5$	150	$A_2 + A_{10} + D_6$
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- Is there a generalized Dynkin diagram for $\ \Gamma^{18,2}$?

Effective actions

Massless states bosonic string $\mathcal{M}_D \times T^d$ rank d rank d dim n dim n **0=** $M^2 = 2(N + \overline{N} - 2) + (p_L^2 + p_R^2)$ $U(1)^d \times U(1)^d \longrightarrow G \times G$ LMC $0 = 2(N - \bar{N}) + (p_L^2 - p_R^2)$ $A^m = \overline{A}^m$ $A^m, A^{\alpha}, A^{-\alpha}$ $A^m, A^{\alpha}, A^{-\alpha}$ p = EZ2n vectors 2d vectors **Scalars** $g_{\mu m} \pm B_{\mu m}$ $\bar{N}_{y} = N_{y} = 1 \quad M^{mn} \quad p_{L} = p_{R} = 0$ $M^{mn} M^{m\beta} M^{\alpha n} M^{\alpha \beta}$ M^{mn} $N_u = 1, \bar{N} = 0$ $M^{m\beta} p_L = 0, p_B^2 = 2$ d² scalars $M^{\boldsymbol{ab}}$ a = 1, ..., n $N = 0, \bar{N}_{u} = 1$ $M^{\alpha n} p_{L}^{2} = 2, p_{R} = 0$ $g_{mn} + B_{mn}$ n² scalars $N = \bar{N} = 0 \qquad M^{\alpha\beta} \quad p_L^2 = p_R^2 = 2$















 $\mathcal{M}_D \times T^d$



Fields of reduced theory for heterotic string

rank d+16 **0=** $M^2 = 2\left(N + \bar{N} - \frac{3}{2}\right) + p_L^2 + p_R^2$ dim q $U(1)^{d+16} \times U(1)^d \longrightarrow \quad \overset{\checkmark}{G} \times U(1)^d$ $\underbrace{A^{m}, A^{I}}_{A^{\hat{I}}} \quad \widehat{I} = 1, \dots, d+16 \qquad \underbrace{A^{m}, A^{I}, A^{\alpha}, A^{-\alpha}}_{A^{a}} \quad A^{m}$ $A^a \quad a = 1, ..., q$ 2d+16 vectors q+d vectors $g_{\mu m} \pm B_{\mu m}$, A^I_μ $M^{\tilde{I}m} M^{\alpha n}$ $M^{\hat{I}m}$ (d+16)x d scalars M^{am} g_{mn}, B_{mn}, A_m^I q x d scalars

LMC $0 = 2(N - \bar{N} - \frac{1}{2}) + p_L^2 - p_R^2$ p = EZExtra vectors $N = 0, N_x = 1/2$ $p_L^2 - p_R^2 = 2$ LMC $p_L^2 + p_R^2 = 2$ M²=0 $p_L^2 = 2, p_R = 0$

 $\mathcal{M}_D \times T^d$

Extra scalars $N = 0, \bar{N}_{\mathbf{y}} = \frac{1}{2}$ $p_I^2 = 2, p_B = 0$

DFT

$$\mathcal{H} \in \frac{O(d+16,d)}{O(d+16) \times O(d)} \qquad \qquad \mathcal{H} \in \frac{O(q,d)}{O(q) \times O(d)}$$

Effective action from string theory for bosonic string

Computing 3-point functions <VVV> at a point of enhancement we read off

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{a}_{\mu\nu} \bar{F}^{\mu\nu}_{a}$$
$$+ \frac{1}{4} M_{aa'} F^{a}_{\mu\nu} \bar{F}^{a'\mu\nu} + D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}$$

$$H = dB + A^{a} \wedge F_{a} + f_{abc}A^{a} \wedge A^{b} \wedge A^{c}$$
$$- \bar{A}^{a} \wedge \bar{F}_{a} - \bar{f}_{abc}\bar{A}^{a} \wedge \bar{A}^{b} \wedge \bar{A}^{c}$$

 $F^a = dA^a + f^a_{bc} A^b \wedge A^c$

 $D_{\mu}M^{aa'} = \partial_{\mu}M^{aa'} + f^{a}_{bc}A^{b}_{\mu}M^{ca'} + f^{a'}_{b'c'}\bar{A}^{b'}_{\mu}M^{ac'}$

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Higgs mechanism

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Higgs mechanism

$$M^{mn} = \underbrace{v_{\mathcal{I}}^{mn}}_{\mathcal{I}} + M'^{mn}$$

deviation from point of enhancement $\,\delta(g+B)_{mn}$

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Computing 3-point functions <VVV> at a point of enhancement we read off

$$\begin{split} \mathcal{L} &= R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{a}_{\mu\nu} \bar{F}^{\mu\nu}_{a} \\ &+ \frac{1}{4} M_{aa'} F^{a}_{\mu\nu} \bar{F}^{a'\mu\nu} + D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'} \\ H &= dB + A^{a} \wedge F_{a} + f_{abc} A^{a} \wedge A^{b} \wedge A^{c} \\ &- \bar{A}^{a} \wedge \bar{F}_{a} - \bar{f}_{abc} \bar{A}^{a} \wedge \bar{A}^{b} \wedge \bar{A}^{c} \\ F^{a} &= dA^{a} + f^{a}_{bc} A^{b} \wedge A^{c} \\ D_{\mu} M^{aa'} &= \partial_{\mu} M^{aa'} + f^{a}_{bc} A^{b}_{\mu} M^{ca'} + f^{a'}_{b'c'} \bar{A}^{b'}_{\mu} M^{ac'} \\ \end{split}$$

Higgs mechanism

$$M^{mn} = \underbrace{v_{j}^{mn}}_{\gamma} + M'^{mn}$$

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$$H = dB + A^{a} \wedge F_{a} + f_{abc} A^{a} \wedge A^{b} \wedge A^{c}$$

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$$Higgs mechanism$$

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deviation from point of enhancement $\,\delta(g+B)_{mn}\,$

 $G \times G \rightarrow U^{d}(1) \times U^{d}(1)$

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$$\begin{array}{c} M^{\alpha\beta} \\ A^{\alpha} \\ A^{\alpha} \\ A^{\alpha} \\ M^{bb'} \\ M^{cc'} \end{array}$$

$$\begin{array}{c} H = dB + A^{a} \land F_{a} + f_{abc} A^{a} \land A^{b} \land A^{c} \\ - \overline{A}^{a} \land \overline{F}_{a} - \overline{f}_{abc} \overline{A}^{a} \land \overline{A}^{b} \land \overline{A}^{c} \end{array}$$

$$\begin{array}{c} F^{a} = dA^{a} + f^{a}_{bc} \\ A^{b} \\ A^{\alpha} \\ A^{\alpha}$$

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deviation from point of enhancement $\ \delta(g+B)_{mn}$

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$$+D_{\mu}M_{am}D^{\mu}M^{am} - \frac{1}{12}f_{abc}f_{a'b'c}M^{am}M^{a'm}M^{bn}M^{b'n}$$

$$H = dB + A^a \wedge F_a + f_{abc}A^a \wedge A^b \wedge A^c - \bar{A}^m \wedge \bar{F}_m$$
$$F^a = dA^a + f^a_{bc}A^b \wedge A^c$$

 $D_{\mu}M^{am} = \partial_{\mu}M^{am} + f^{a}_{bc}A^{b}_{\mu}M^{cm}$

Computing 3-point functions <VVV> at a point of enhancement we read off

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{m}_{\mu\nu} \bar{F}^{\mu\nu}_{m} + \frac{1}{4} M_{am} F^{a}_{\mu\nu} \bar{F}^{m\mu\nu}$$

$$+D_{\mu}M_{am}D^{\mu}M^{am} - \frac{1}{12}f_{abc}f_{a'b'c}M^{am}M^{a'm}M^{bn}M^{b'n}$$

$$H = dB + A^a \wedge F_a + f_{abc}A^a \wedge A^b \wedge A^c - \bar{A}^m \wedge \bar{F}_m$$
$$F^a = dA^a + f^a_{bc}A^b \wedge A^c$$

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Higgs mechanism

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$$M^{\hat{I}n} = \underbrace{v^{\hat{I}n}}_{\checkmark} + M'^{\hat{I}n}$$

deviation from point of enhancement $\ \delta(g+B)_{mn} \ , \delta A^I_m$

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$$+ \underbrace{D_{\mu} M_{am} D^{\mu} M^{am}}_{\bullet} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n} \quad \text{•quartic potential}$$

$$- \underbrace{A^{\alpha} \text{ acquire mass}^2 \sim vv^t}_{\bullet}$$

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Higgs mechanism

 $M^{\hat{I}n} = v^{\hat{I}n} + M'^{\hat{I}n}$ We get these actions from DFT !! deviation from
point of enhancement $\delta(q + B)_{mn}, \delta A_m^I$ We get these actions from DFT !! G.Aldazabal, MG, S. Iguri, M. Mayo, C. Nuñez 15
Y. Cagnacci, MG, S. Iguri, C. Nuñez 17

- In the bosonic theory
- In the heterotic theory

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Gives the point in moduli space where the enhancement occurs

 $E_{9-p} \times E_{9-q} \times SU(p+q)$ $E_{9-p} \times SU(9+p)$ $E_{9-p} \times SO(16+2p)$ $SU(18) \quad SO(34)$

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```

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Possible groups for heterotic on S¹: from the generalized Dynkin diagram of Γ^{17,1} erasing 2 nodes
 E_{9-p} × E_{9-q} × SU(p+q)

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```
E_{9-p} \times E_{9-q} \times SU(p+q)E_{9-p} \times SU(9+p)E_{9-p} \times SO(16+2p)SU(18) \quad SO(34)
```

- Heterotic on T²: found all the groups that appear in F-theory on K3 Gives strong support to duality!
- There does not seem to exist a (unique) generalized Dynkin diagram

- Effective action:
 - O(p,q)-covariance
 - p=q=n for bosonic string n=dimension of G (rank d simply-laced group)
 p=n, q=k for heterotic string n=dimension of G (rank d+16 simply-laced group)
- A^I, A^I n+n non-abelian vectors, M^{IJ} n^2 scalars in adj x adj for bosonic
- A^I, A^m n non-abelian+d abelian vectors, M^{Im} n x d scalars for heterotic
- M³ potential in the bosonic theory
- \bullet M^4 potential in the heterotic theory
- Higgs mechanism describing symmetry breaking

DFT O(N, \overline{N}) action $S = \int dX \left(-\partial_{MN} \mathcal{H}^{MN} + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} \right)$ $M=1,...,N+\overline{N}$

Equivalent to

 $S = \int dX \,\mathbb{R}$

generalized Ricci scalar

Coimbra, Strickland-Constable, Waldram 09

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Generalized Scherk-Schwarz reduction of DFT action

$$\mathcal{H}^{MN} = \delta^{AB} E_A{}^M E_B{}^N \qquad \qquad E_A(x, y) = U_A{}^{A'}(x) E'_{A'}(y)$$

$$O(N,\overline{N}) \longrightarrow O(D,D) \times O(n-D,\overline{n}-D)$$

external

internal

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$$\mathcal{H}^{MN} = \delta^{AB} E_A{}^M E_B{}^N \qquad E_A(x, y, \tilde{y}) = U_A{}^{A'}(x) E'_{A'}(y, \tilde{y}) \qquad \overline{\mathsf{N-D=n}}$$
$$\overset{\mathsf{N-D=n}}{\underset{\mathsf{D}}{\overset{\mathsf{N-D=n}}{\underset{\mathsf{D}}{\overset{\mathsf{O}}{\underset{\mathsf{n+n}}}}}} \partial_M = (\partial_\mu, \partial_m, \partial_m, \partial_m)$$
$$\overset{\mathsf{N-D=n}}{\underset{\mathsf{D}}{\overset{\mathsf{O}}{\underset{\mathsf{n+n}}}} \partial_M = I$$

$E_A(x, y, \tilde{y}) = U_A^{A'}(x) E'_{A'}(y, \tilde{y})$

 $E_A(x, y, \tilde{y}) = U_A^{A'}(x) E'_{A'}(y, \tilde{y}) \qquad \qquad \mathcal{H} = E^t E$

 $E_A(x, y, \tilde{y}) = U_A^{A'}(x) E'_{A'}(y, \tilde{y}) \qquad \qquad \mathcal{H} = E^t E = E'^t U^T U E' \equiv E'^t \mathcal{M} E'$

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Aldazabal, Baron, Marques, Nuñez II Geissbuhler II

$$H = dB + F^I \wedge A_I$$

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Aldazabal, Baron, Marques, Nuñez I I Geissbuhler I I

$$H = dB + F^{I} \wedge A_{I}$$
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Aldazabal, Baron, Marques, Nuñez II Geissbuhler II

$$H = dB + F^{I} \wedge A_{I}$$

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Claim: this action reproduces the string theory action for compactifications of bosonic and heterotic on T^d close to enhancement point

Let's look at internal piece only : O(d,d(+|6))

Let's look at internal piece only : O(d,d(+16))

Frame $E_A{}^M = E$ M=1,...,d+d for bosonic M=1,...,d+d+16 for heterotic

Let's look at internal piece only : O(d,d(+|6))

Frame $E_A{}^M = E$ M=1,...,d+d for bosonic M=1,...,d+d+16 for heterotic

 $E = E_0 + \delta E$ iconstant (g_0, B_0, A_0) || $e_0^t e_0$

Let's look at internal piece only : O(d,d(+|6))

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 $\mathcal{H} = E^t E = E^{\prime t} U^T U E^\prime \equiv E^{\prime t} \mathcal{M} E^\prime$

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 $\mathcal{H} = E^{t}E = E^{\prime t}U^{T}UE^{\prime} \equiv E^{\prime t}\mathcal{M}E^{\prime}$ we get $\mathcal{M} = \begin{pmatrix} I_{d} + \frac{1}{2}M^{t}M & M^{t} \\ M & I_{d(+16)} + \frac{1}{2}MM^{t} \end{pmatrix}$ where $M = \begin{pmatrix} \hat{e}_{0}(\delta G - \delta B^{\prime})\hat{e}_{0}^{t} \\ \delta A \hat{e}_{0}^{t} \end{pmatrix}$

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$$E = E_0 + \delta E = \underbrace{(1 + \delta E E_0^{-1})E_0}_{\text{indep of y}} \qquad E(x, y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I_{JK}E'_K$$

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$$Constant \qquad U(x) \quad E'(y) \qquad so far \\ indep of y \qquad indep of y$$

$$e_0^t e_0$$

Plug that in

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_{\mu}\mathcal{M})_{IJ} (D^{\mu}\mathcal{M})^{IJ}$$

$$- \frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3 \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2 \eta^{IL} \eta^{JM} \eta^{KN} \right)$$

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$$E = E_0 + \delta E = \underbrace{(1 + \delta E E_0^{-1})E_0}_{\text{for equation of }} \qquad E(x, y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I_{JK}E'_K$$

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$$E(x, y) = U(x)E'(y) \qquad [E'_J, E'_K]_C = f^I_{JK}E'_K$$

 $\mathcal{H} = E^{t}E = E^{\prime t}U^{T}UE^{\prime} \equiv E^{\prime t}\mathcal{M}E^{\prime}$ we get $\mathcal{M} = \begin{pmatrix} I_{d} + \frac{1}{2}M^{t}M & M^{t} \\ M & I_{d(+16)} + \frac{1}{2}MM^{t} \end{pmatrix}$ where $M = \begin{pmatrix} \hat{e}_{0}(\delta G - \delta B^{\prime})\hat{e}_{0}^{t} \\ \delta A \hat{e}_{0}^{t} \end{pmatrix}$

Plug that in

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_{\mu}\mathcal{M})_{IJ} (D^{\mu}\mathcal{M})^{IJ}$$
No enhancement of symmetry

$$-\frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3 \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2 \eta^{IL} \eta^{JM} \eta^{KN} \right)$$
no double field theory

Action close to a special point in moduli space (enhancement)

 $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$

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 $\mathcal{M}_D \times T^d$

Frame $E_A{}^M = E$ M=1,..., \overline{n} +n

Frame $E_A{}^M = E$ M=1,..., \overline{n} +n • for bosonic \overline{n} =n= dim of group G_{bos} of rank d

- for bosonic n=n= dim of group G_{bos} of rank d
- Frame $E_A{}^M = E$ M=1,..., \overline{n} +n • for beterotic \overline{n} =d, n:
 - for heterotic n=d, n=dim of group G_{het} of rank d+16

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 y_R^m, y_L^m "Double coordinates": only the Cartan directions: d+d(+16)

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 $\mathcal{H} = E^t E = E'^t(y) \mathcal{M}(x) E'(y)$

- for bosonic $\overline{n}=n=$ dim of group G_{bos} of rank d
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 $\mathcal{H} = E^{t}E = E^{\prime t}(y)\mathcal{M}(x)E^{\prime}(y)$ $U^{T}U$

• for bosonic n=n= dim of group G_{bos} of rank d

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 y_R^m , y_L^m "Double coordinates": only the Cartan directions: d+d(+16)

Frame $E_A{}^M = E$ M=1,...,**n**+n

E(x,y) = U(x)E'(y) $\mathcal{H} = E^{t}E = E'^{t}(y)\mathcal{M}(x)E'(y)$ $\bigcup U^{T}U$ $\mathcal{M} = \begin{pmatrix} I_{n} + \frac{1}{2}M^{t}M & M^{t}\\ M & I_{\overline{n}} + \frac{1}{2}MM^{t} \end{pmatrix}$ $\widehat{\mathcal{M}}^{aa'}$

• for bosonic
$$\overline{n}=n=$$
 dim of group G_{bos} of rank d

• for heterotic n=d, n=dim of group G_{het} of rank d+16

 $\mathcal{M}_{R}^{m}, \mathcal{Y}_{L}^{m}$ "Double coordinates": only the Cartan directions: d+d(+16) E(x,y) = U(x)E'(y) $\mathcal{H} = E^{t}E = E''(y)\mathcal{M}(x)E'(y)$ $\int U^{T}U$ $[E'_{J}, E'_{K}]_{C} = f^{I}{}_{JK}E'_{K}$ $\mathcal{M} = \begin{pmatrix} I_{n} + \frac{1}{2}M^{t}M & M^{t} \\ M & I_{\overline{n}} + \frac{1}{2}MM^{t} \end{pmatrix}$

Frame $E_A{}^M = E$ M=1,..., \overline{n} +n

Frame
$$E_A{}^M = E$$
 $M=1,...,n+n$
• for bosonic $n=n=$ dim of group G_{bos} of rank d
• for heterotic $n=d$, $n=$ dim of group G_{het} of rank d+16
 $g_{R}^{(m)}, g_{L}^{(d)+16}$
"Double coordinates": only the Cartan directions: $d+d(+16)$
 $E(x,y) = U(x)E'(y)$
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 $\mathcal{U}^T U$
 $U^T U$
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 $\hat{f}_{aa'}$
 E' such that
• for bosonic $f_{ab}{}^c = f_{ab}{}^c$ = str const of G_{bos}

Frame $E_A{}^M = E$ M=1,..., \overline{n} +n

• for heterotic n=d, n=dim of group G_{het} of rank d+16

 $\begin{aligned} & \int_{R}^{d} \int_{Y_{L}^{m}}^{d(+16)} & \text{``Double coordinates'': only the Cartan directions: d+d(+16)} \\ & E(x,y) = U(x)E'(y) \\ & \mathcal{H} = E^{t}E = E^{\prime t}(y)\mathcal{M}(x)E'(y) \\ & \downarrow U^{T}U \\ & \downarrow U^{T}U \\ & \downarrow U^{T}U \\ & \mathcal{M} = \begin{pmatrix} I_{n} + \frac{1}{2}M^{t}M & M^{t} \\ M & I_{\bar{n}} + \frac{1}{2}MM^{t} \end{pmatrix} \\ & \stackrel{\uparrow}{\underset{M^{aa'}}{\overset{\bullet}}} & \text{ F' such that} \\ & \bullet \text{ for bosonic } f_{ab}{}^{c} = f_{ab}{}^{c} = \text{ str const of } G_{\text{bos}} \end{aligned}$

• for heterotic $f_{ab}{}^{c} = 0$, $f_{ab}{}^{c} =$ str const of G_{het}

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^tM & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_{\mu}\mathcal{M})_{IJ} (D^{\mu}\mathcal{M})^{IJ} - \frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3 \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2 \eta^{IL} \eta^{JM} \eta^{KN} \right)$$

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And we get

$$\mathcal{M} = \begin{pmatrix} \mathbf{I}_n + \frac{1}{2}M^t M & M^t \\ M & \mathbf{I}_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

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And we get

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{a}_{\mu\nu} \bar{F}^{a\mu\nu}_{a} + \frac{1}{4} M_{aa'} F^{a}_{\mu\nu} \bar{F}^{a'\mu\nu} + D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}$$
bosonic

$$\mathcal{M} = \begin{pmatrix} \mathbf{I}_n + \frac{1}{2}M^t M & M^t \\ M & \mathbf{I}_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

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$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{m}_{\mu\nu} \bar{F}^{\mu\nu}_{m} + \frac{1}{4} M_{am} F^{a}_{\mu\nu} \bar{F}^{m\mu\nu} + D_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n}$$
 heterotic

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$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_{\mu}\mathcal{M})_{IJ} (D^{\mu}\mathcal{M})^{IJ} - \frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3 \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2 \eta^{IL} \eta^{JM} \eta^{KN} \right)$$

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 heterotic

Precisely the string theory actions !

$$\mathcal{M} = \begin{pmatrix} \mathbf{I}_n + \frac{1}{2}M^tM & M^t \\ M & \mathbf{I}_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

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$$+ D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \underbrace{\frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}}_{\bullet \text{cubic potential, unbounded from below}}$$

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{m}_{\mu\nu} \bar{F}^{\mu\nu}_{m} + \frac{1}{4} M_{am} F^{a}_{\mu\nu} \bar{F}^{m\mu\nu}_{m}$$

$$+ D_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n}$$

$$\text{heterotic}$$

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Precisely the string theory actions !

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$$+ D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \underbrace{\frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}}_{\bullet \text{cubic potential, unbounded from below}}$$

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$$+ D_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n}$$

$$+ D_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n}$$

$$+ Q_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n}$$

Precisely the string theory actions !

Higgs mechanism $M^{\hat{I}n} = v^{\hat{I}n} + M'^{\hat{I}n}$

$$\mathcal{M} = \begin{pmatrix} I_n + \frac{1}{2}M^tM & M^t \\ M & I_{\bar{n}} + \frac{1}{2}MM^t \end{pmatrix}$$

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} \mathcal{M}_{IJ} F^{I\mu\nu} F^{J}_{\mu\nu} + (D_{\mu}\mathcal{M})_{IJ} (D^{\mu}\mathcal{M})^{IJ} - \frac{1}{12} f_{IJK} f_{LMN} \left(\mathcal{M}^{IL} \mathcal{M}^{JM} \mathcal{M}^{KN} - 3 \mathcal{M}^{IL} \eta^{JM} \eta^{KN} + 2 \eta^{IL} \eta^{JM} \eta^{KN} \right)$$

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$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{a}_{\mu\nu} \bar{F}^{\mu\nu}_{a} + \frac{1}{4} M_{aa'} F^{a}_{\mu\nu} \bar{F}^{a'\mu\nu}$$

$$+ D_{\mu} M_{aa'} D^{\mu} M^{aa'} - \underbrace{\frac{1}{12} f_{abc} \bar{f}_{a'b'c'} M^{aa'} M^{bb'} M^{cc'}}_{\bullet \text{cubic potential, unbounded from below}} \bullet \text{cubic potential, unbounded from below} \bullet \text{tachyonic masses}$$

$$\mathcal{L} = R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \frac{1}{4} \bar{F}^{m}_{\mu\nu} \bar{F}^{\mu\nu}_{m} + \frac{1}{4} M_{am} F^{a}_{\mu\nu} \bar{F}^{m\mu\nu} + \frac{1}{4} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n} \bullet \text{heterotic} \\ + D_{\mu} M_{am} D^{\mu} M^{am} - \frac{1}{12} f_{abc} f_{a'b'c} M^{am} M^{a'm} M^{bn} M^{b'n} \bullet \text{squartic potential} \\ \bullet \text{positive masses} \quad m^{2}_{\alpha} = |\alpha \cdot v|^{2}$$

Precisely the string theory actions !

Higgs mechanism $M^{\hat{I}n} = v^{\hat{I}n} + M'^{\hat{I}n}$

What about E'?

What about E'? $[E'_J, E'_K]_C = f^I_{JK}E'_K$

What about E?
$$[E'_J, E'_K]_C = f^I_{JK}E'_K$$

left and right split discuss the left part

What about E'?
$$[E'_J, E'_K]_C = f^I_{JK}E'_K$$

left and right split discuss the left part

$$[V_1, V_2]_C = \frac{1}{2} (\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1)$$
 C-bracket

 $(\mathcal{L}_{V_1}V_2)^I = V_1^J \partial_J V_2^I + \ (\partial^I V_{1J} - \partial_J V_1^I) V_2^J \quad \text{generalized Lie derivative}$
What about E'?
$$[E'_J, E'_K]_C = f^I_{JK}E'_K$$

$$[V_1, V_2]_C = \frac{1}{2} (\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1) \quad \text{C-bracket}$$
$$(\mathcal{L}_{V_1} V_2)^I = V_1^{[J]} \partial_J V_2^I + (\partial^I V_{1J} - \partial_J V_1^I) V_2^J \quad \text{and} \quad \mathbf{n} = d(+16) + 2p$$

generalized Lie derivative

Cartans Roots
$$lpha^{\pm}$$

What about E? $[E'_J, E'_K]_C = f^I_{JK}E'_K$

left and right split discuss the left part

$$[V_1, V_2]_C = \frac{1}{2} (\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1) \quad \text{C-bracket}$$
$$(\mathcal{L}_{V_1} V_2)^I = V_1 \overset{J}{\longrightarrow} \partial_J V_2^I + (\partial^I V_{1J} - \partial_J V_1^I) V_2^J$$
$$\overset{h}{\longrightarrow} n = d(+16) + 2p$$

generalized Lie derivative

Cartans Roots α^{\pm}

What about E'?
$$[E'_J, E'_K]_C = f^I_{JK}E'_K$$

$$\begin{split} [V_1, V_2]_C &= \frac{1}{2} (\mathcal{L}_{V_1} V_2 - \mathcal{L}_{V_2} V_1) \quad \text{C-bracket} \\ (\mathcal{L}_{V_1} V_2)^I &= V_1^{[J]} \partial_J V_2^I + (\partial^I V_{1J} - \partial_J V_1^I) V_2^J \quad \text{gene} \\ & \ddots \\ & n = d(+16) + 2p \\ & \text{Cartans} \quad \text{Roots} \quad \alpha^{\pm} \end{split}$$

generalized Lie derivative

The following E' does the job for SU(2) algebra

$$E' = \begin{pmatrix} e^{\sqrt{2}iy^L} & 0 & 0 \\ 0 & e^{-\sqrt{2}iy^L} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Straightforward generalization to SU(2)^d





 $SU(2)^2 imes SU(2)^2$ SU(3) imes SU(3) 3 positive roots : 2 simple, 1 non-simple



 $SU(2)^2 imes SU(2)^2$ SU(3) imes SU(3) 3 positive roots : 2 simple, 1 non-simple for simple roots $[E'_{lpha}, E'_{-lpha}] = lpha \cdot H$

for simple roots same construction

$$[E'_{\alpha}, H_m] = \alpha^m$$

 $E_{\alpha} \sim e^{i\alpha \cdot y_L}$



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$$\begin{bmatrix} \mathbf{n} \\ \alpha \end{bmatrix} \begin{bmatrix} E'_{\alpha}, H_m \end{bmatrix} = \alpha^m$$

for non simple roots no coordinate associated

still
$$[E'_{\alpha_1+\alpha_2},H_m]=(\alpha_1+\alpha_2)^m\,\checkmark$$



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Deformed generalized Lie derivative

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$$\begin{bmatrix} E'_{\alpha}, E'_{-\alpha} \end{bmatrix} = \alpha \cdot H$$

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 $SU(2)^2 \times SU(2)^2$ $SU(3) \times SU(3)$ 3 positive roots : 2 simple, 1 non-simple $[E'_{\alpha}, E'_{-\alpha}] = \alpha \cdot H$ $[E'_{\alpha}, H_m] = \alpha^m \qquad \checkmark$

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Deformed generalized Lie derivative

$$\tilde{\mathcal{L}}_{E_I} E_J = \mathcal{L}_{E_I} E_J + \Omega_{IJ}{}^K E_K$$

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Cocycle tensor

$$\Omega_{IJK} = \begin{cases} (-1)^{\alpha*\beta} \,\delta_{\alpha+\beta+\gamma} \\ -(-1)^{\alpha*\beta} \,\delta_{\alpha+\beta+\gamma} \end{cases}$$

if two roots are positive if two roots are negative

 $[E'_{\alpha}, E'_{-\alpha}] = \alpha \cdot H$ $[E'_{\alpha}, H_m] = \alpha^m$ $E_{\alpha} \sim e^{i\alpha \cdot y_L} \equiv e^{iy^{\alpha}}$

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 $[E'_{\alpha}, H_m] = \alpha^m$

does not arise from any obvious extension of the previous construction

This reproduces

$$\left[E_J', E_K'\right]_{\tilde{C}} = f^I{}_{JK}E_K' \qquad \text{for }$$

or any group

Heterotic SO(32) and E8 x E8 have exactly the same cocycles !

Supports idea that heterotic SO(32) and $E8 \times E8$ different vacua of same theory

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- For T^d, is there a vielbein depending on Cartan coordinates only that satisfies algebra under ordinary bracket?

$-(k+{ ilde k})$ 19	$\sum (A_n - \frac{1}{2})^2 \ge 2(1 - R^2)$						$\sum A_n^2 \ge 2(1-R^2)$						18 $-\lambda_{sp}$ +	$k-\tilde{k}$
k- heta 17	$A_1 \ge -A_2$						$A_{16} \ge 1 - A_{15}$.5	16	
1	2 3	4	5	6	7	8	9	10	11	12	13	14	15	
$A_1 \ge A_2$	$\geq A_3 \geq A_3$	$A_4 \ge A_4$	$A_5 \ge A_5$	$A_6 \ge A_6$	$A_7 \ge A_7$	$A_8 \ge A_8$	$_9 \ge A_1$	$_0 \ge A_1$	$A_{11} \ge A$	$_{12} \ge A$	$l_{13} \ge 1$	$A_{14} \ge$	$A_{15} \ge A_{16}$	

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