Physics as (Higher) Algebra

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- work with Zwiebach 2017
- work with Samtleben, 2018, and Bonezzi, 2019
- discussions with Dennis Sullivan

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Overview

Underlying question of this talk:

Can a physical theory be entirely encoded in algebraic data?

- Part I: Classical field theory in terms of Lie-infinity (L_∞) algebras

- Part II: Duality covariant formulations of string-/M-theory in terms of Leibniz-Loday algebras and embedding tensor maps

- Part III: Duality hierarchy in terms of differential graded Lie algebras.

Part I: Classical field theory in terms of L_∞ algebras

Reminder: Lie algebras and Maurer-Cartan

Lie algebra: vector space $\mathfrak{g}$ equipped with bilinear antisymmetric bracket,

$$[x, y] = -[y, x]$$

satisfying Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

These data are sufficient for (topological) Chern-Simons gauge theory:

$$I = \int_{M_3} \left\langle A, \mathrm{d}A + \tfrac{1}{3}[A, A] \right\rangle$$

with gauge invariance

$$\delta_\xi A = \mathrm{d}\xi + [A, \xi]$$

where ξ and A are $\mathfrak{g}$-valued zero- and one-forms. Field equations of Maurer-Cartan form

$$F(A) \equiv \mathrm{d}A + \tfrac{1}{2}[A, A] = 0$$

Yang-Mills, Gravity, String Theory need more structure $\rightarrow$ L_∞ algebras

Strongly Homotopy Lie or L_∞ Algebras

An L_∞ algebras is a graded vector space [Zwiebach (1993), Lada & Stasheff (1993)]

$$X = \bigoplus_{n \in \mathbb{Z}} X_n ,$$

equipped with *multilinear and graded antisymmetric* brackets or maps

$$[x_1, \ldots, x_n] \equiv b_n(x_1, \ldots, x_n) \in X_{\sum_i |x_i| - 1} ,$$

satisfying the *generalized Jacobi identities*

$$D^2 = 0 , \qquad D \equiv \sum_{i=1}^{\infty} b_i ,$$

where D acts as *coderivation* on symmetric tensor algebra.

Explicit L_∞-relations

For $n = 1$ we learn that $b_1 \equiv \partial$ is nil-potent:

$$\partial^2 = 0$$

For $n = 2$ we learn that b_1 is a derivation of $b_2 \equiv [\cdot, \cdot]$:

$$b_1(b_2(x_1, x_2)) + b_2(b_1(x_1), x_2) + (-1)^{x_1} b_2(x_1, b_1(x_2)) = 0$$

For $n = 3$ we learn that $b_2 \equiv [\cdot, \cdot]$ satisfies Jacobi only 'up to homotopy'

$$\begin{aligned} 0 \;&=\; b_2(b_2(x_1, x_2), x_3) + \text{2 terms} \\ &+ b_1(b_3(x_1, x_2, x_3)) \\ &+ b_3(b_1(x_1), x_2, x_3) + \text{2 terms} \end{aligned}$$

For $n = 4$ we learn that $b_2 b_3 + b_3 b_2$ is zero 'up to homotopy', i.e., up to the the failure of b_1 to act as a derivation on b_4

plus infinitely more relations

General Field Theories

Dictionary L_∞ algebra $\longleftrightarrow$ field theory:

$$\cdots \to X_2 \xrightarrow{\partial} X_1 \xrightarrow{\partial} X_0 \xrightarrow{\partial} X_{-1} \to \cdots$$
$$\chi \qquad \xi \qquad \Psi \qquad \text{EOM}$$

Action for Ψ completely encoded in these data (plus inner product):

$$I = \frac{1}{2}\langle\Psi, \partial\Psi\rangle + \frac{1}{3!}\langle\Psi, \big[\Psi, \Psi\big]\rangle + \frac{1}{4!}\langle\Psi, \big[\Psi, \Psi, \Psi\big]\rangle + \cdots \ .$$

Field equations

$$0 = \partial\Psi + \tfrac{1}{2}\big[\Psi, \Psi\big] + \tfrac{1}{3!}\big[\Psi, \Psi, \Psi\big] + \tfrac{1}{4!}\big[\Psi, \Psi, \Psi, \Psi\big] + \cdots \ ,$$

Gauge symmetries

$$\delta_\Lambda\Psi = \partial\Lambda + \big[\Lambda, \Psi\big] + \tfrac{1}{2}\big[\Lambda, \Psi, \Psi\big] + \tfrac{1}{3!}\big[\Lambda, \Psi, \Psi, \Psi\big] + \cdots \ .$$

Note: graded bracket may be symmetric (as for fields: $[\Psi, \Psi]$)
or antisymmetric (as for gauge parameters: $[\Lambda_1, \Lambda_2]$).

Example 1: 0-dimensional scalar theory

Space of “fields”: $\mathbf{v} \in X_0 = \mathbb{R}^n$, action is function

$$I(\mathbf{v}) = \frac{1}{2} v^i A_{ij} v^j + \sum_{n=3}^{\infty} \frac{1}{n!} A_{i_1 \ldots i_n} v^{i_1} \ \cdots \ v^{i_n}$$

2-term L_∞ algebra:

$$X_0 \xrightarrow{\partial} X_{-1} = (\mathbb{R}^n)^*$$

Brackets such that for $\langle\ ,\ \rangle : X_0 \otimes X_1 \to \mathbb{R}$ given by $\langle \mathbf{v}, \mathbf{a} \rangle \equiv v^i a_i$

$$I(\mathbf{v}) = \sum_{n=2}^{\infty} \frac{1}{n!} \langle \mathbf{v}, [\mathbf{v}, \ \cdots \ , \mathbf{v}] \rangle$$

i.e., $\partial : X_0 \to X_{-1}$ and b_n non-trivial on X_0

$$(\partial \mathbf{v})_i = A_{ij} v^j \ , \quad [\mathbf{v}, \ \cdots \ , \mathbf{v}]_i = A_{i j_1 \ldots j_{n-1}} v^{j_1} \ldots v^{j_{n-1}} \ \in \ X_{-1}$$

L_∞ relations identically satisfied, e.g., $\partial([\mathbf{v}, \mathbf{w}]) \equiv 0 \equiv \pm[\partial \mathbf{v}, \mathbf{w}] \pm [\mathbf{v}, \partial \mathbf{w}]$
$\to$ no constraints on scalar theories

Example 2: Einstein gravity

Perturbation theory about flat space:

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} \;+\; h_{\mu\nu}$$

Vector space

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_0 & \longrightarrow & X_{-1} \\ \xi^\mu & & h_{\mu\nu} & & R_{\mu\nu} \end{array}$$

Non-vanishing brackets:

i) diffeomorphism Lie algebra for ξ,

$$b_2(\xi_1, \xi_2)^\mu \;\equiv\; \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu$$

ii) gauge transformations,

$$b_1(\xi)_{\mu\nu} \;=\; \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \;, \qquad b_2(\xi, h)_{\mu\nu} \;=\; L_\xi h_{\mu\nu}$$

iii) ℓ_n for arbitrary n involving only the field h (dynamics),

$$b_n(h, \ldots, h) \qquad \text{for} \qquad h \in X_0$$

iv) gauge parameter and field equations, $b_2(\xi, R) \;=\; L_\xi R$

Homotopy Transfer I: Quasi-Isomorphisms

Two chain complexes, one projection of the other, $\bar{X}_i \;=\; p(X_i)$,

$$\dots \;\longrightarrow X_{i+1} \xrightarrow{\partial} X_i \xrightarrow{\partial} X_{i-1} \longrightarrow \;\dots$$

$$\dots \;\longrightarrow \bar{X}_{i+1} \xrightarrow{\bar{\partial}} \bar{X}_i \xrightarrow{\bar{\partial}} \bar{X}_{i-1} \longrightarrow \;\dots$$

Question: Can algebraic structure on X be transported to $\bar{X}$?

Obviously not in general. If projection *chain homotopic* to identity, then algebraic structure on X yields ∞-version on $\bar{X}$.

If projection $p : X \to \bar{X}$ and inclusion $\iota : \bar{X} \to X$ obey

$$p \circ \iota \;=\; \mathrm{id}_{\bar{X}} \;, \qquad \iota \circ p \;=\; \mathrm{id}_X + \partial \circ h + h \circ \partial \;,$$

where $h = h_i : X_i \to X_{i+1}$, then homologies equivalent,

$$H_i \;\equiv\; \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i+1})} \;\equiv\; \Big\{ [x] \,|\, x \in X_i \,,\; \partial x = 0 \Big\} \;\cong\; \bar{H}_i$$

$\Rightarrow X$, $\bar{X}$ quasi-isomorphic $\Rightarrow$ Lie algebra on X transported to L_∞ on $\bar{X}$

Homotopy Transfer II: Resulting L_∞ algebra

Given Lie bracket $[\cdot,\cdot]$ on X, how do we define L_∞ algebra on $\bar{X}$?

For $\bar{x},\bar{y}\in\bar{X}$ set $x=\iota(\bar{x}), y=\iota(\bar{y})\in X$ and

$$b_2(\bar{x},\bar{y}) \;\equiv\; p([x,y])$$

Jacobi in general violated:

$$\text{Jac}(\bar{x}_1,\bar{x}_2,\bar{x}_3) \;\equiv\; b_2(b_2(\bar{x}_1,\bar{x}_2),\bar{x}_3) + \text{two terms} \;\neq\; 0$$

Can be compensated by defining 3-bracket

$$b_3(\bar{x}_1,\bar{x}_2,\bar{x}_3) \;=\; -p\Big(\big[h([x_1,x_2]),x_3\big] + \text{two terms}\Big)$$

$\rightarrow$ works to all orders $\;\Rightarrow\;$ L_∞ algebra on $\bar{X}$

$\rightarrow$ splitting $v^i=(v^a,v^\alpha)$ in 0D scalar theory, “integrating out” v^α

$\leftrightarrow$ projecting to v^a, homotopy transfer for $h^{ab}=-(A^{-1})^{ab}$

[Arvanitakis, O.H., Hull & Lekeu, to appear]

L_∞ in Field Theory & Homotopy Transfer

- "bootstrap" program for NC gauge theories

  [Blumenhagen, Brunner, Kupriyanov & Lüst (2018)]

- Seiberg-Witten map between commutative and non-commutative (NC) gauge theories is quasi-isomorphism

  [Blumenhagen, Brinkmann, Kupriyanov & Traube (2018)]

- A_∞ algebras as deformations of higher-spin algebras

  [Sharapov & Skvortsov (2018)]

- Recursion relations for amplitudes (e.g. Parke-Taylor formula) obtained through quasi-isomorphism

  [Macrelli, Saemann & Wolf (2019), Arvanitakis (2019)]

- General relativity as Maurer-Cartan equation for Lie algebra

  [Reiterer & Trubowitz, 2018]

Part II: Duality covariant formulation of string-/M-theory

Leibniz-Loday Algebras

Gauge algebra of DFT and ExFT is governed by Leibniz-Loday algebra, a vector space X_0 with bilinear map $\circ$ obeying

$$V \circ (W \circ U) \;=\; (V \circ W) \circ U + W \circ (V \circ U)$$

Minimal structure needed to have consistent infinitesimal variations

$$\delta_V W \;\equiv\; \mathcal{L}_V W \;\equiv\; V \circ W$$

Closure implied by Leibniz relation: $[\mathcal{L}_V, \mathcal{L}_W]U \;=\; \mathcal{L}_{V\circ W}U$.

Symmetric part

$$\{V, W\} \equiv \tfrac{1}{2}(V \circ W + W \circ V) \;\equiv\; \tfrac{1}{2}\mathfrak{D}(V \bullet W)$$

Differential $\mathfrak{D}$ and $\bullet : X_0 \otimes X_0 \to X_1$. If trivial Leibniz reduces to Lie.

In general, higher algebra on chain complex (in particular L_∞ algebra)

$$\cdots \to X_1 \;\xrightarrow{\mathfrak{D}}\; X_0$$

[O.H. & Samtleben (2018), Kotov & Strobl (2018), O.H. & Bonezzi (2019)]

Embedding Tensor

Derive Leibniz algebra from *Lie* algebra $\mathfrak{g}$ and embedding tensor map. Given Lie brackets $[\cdot,\cdot]$ we have *adjoint* and *coadjoint* reps

$$\delta_\zeta a \equiv \mathsf{ad}_\zeta a \equiv [\zeta, a]\,, \qquad \delta_\zeta \mathcal{A} \equiv \mathsf{ad}^*_\zeta \mathcal{A}$$

leaving pairing $\mathcal{A}(a) \in \mathbb{R}$ invariant. Given representation R, $\delta_\zeta V = \rho_\zeta V$, embedding tensor is map

$$\vartheta \ : \ R \ \rightarrow \ \mathfrak{g}$$

Transport Lie algebra to higher algebra on R:

$$V \circ W \equiv \rho_{\vartheta(V)} W$$

defines Leibniz algebra provided *quadratic constraint* obeyed

$$\vartheta(V \circ W) \ = \ [\vartheta(V), \vartheta(W)]$$

Equivalently, embedding tensor $\Theta \,:\, R \otimes \mathfrak{g}^* \rightarrow \mathbb{R}$ defined by pairing

$$\Theta(V, \mathcal{A}) \equiv -\mathcal{A}(\vartheta(V))$$

is Leibniz invariant.

Embedding Tensor for Double/Exceptional Field Theory

$\mathfrak{g}_0$ Lie algebra of U-duality group (e.g. $\mathrm{E}_{7(7)}$), $\quad [t_\alpha, t_\beta] \;=\; f_{\alpha\beta}{}^\gamma t_\gamma$

Pick representation R_0 with matrices $(t_\alpha)_M{}^N$, $M, N = 1, \ldots, \mathrm{dim}(R_0)$.

Infinite-dim. Lie algebra $\mathfrak{g}$ as functions $\zeta \equiv (\lambda^M, \sigma^\alpha)$ of Y^M for $\mathbb{R}^{\mathrm{dim}(R_0)}$,

$$\left[\zeta_1, \zeta_2\right] \equiv \left(2\,\lambda_{[1}{}^N \partial_N \lambda_{2]}{}^M, \;\; 2\,\lambda_{[1}{}^N \partial_N \sigma_{2]}{}^\alpha + f_{\beta\gamma}{}^\alpha \sigma_1{}^\beta \sigma_2{}^\gamma\right)$$

Representation R of $\mathfrak{g}$ given by functions $V^M(Y)$:

$$\rho_\zeta V^M \equiv \lambda^N \partial_N V^M - \sigma^\alpha (t_\alpha)_N{}^M V^N$$

Define embedding tensor map

$$\vartheta(V) \;=\; \left(V^M\,,\; -\kappa (t^\alpha)_M{}^N \partial_N V^M\right) \;\in\; \mathfrak{g}$$

Derived Leibniz structure defines *generalized Lie derivative*

$$\mathcal{L}_\Lambda V^M \equiv \rho_{\vartheta(\Lambda)} V^M = \Lambda^N \partial_N V^M + \;\kappa\,(t^\alpha)_N{}^M (t_\alpha)_L{}^K\, \partial_K \Lambda^L\, V^N$$

closure/quadratic constraint $\Rightarrow$ strong section constraint on ∂_M.

$O(d,d)$ or Double Field Theory

$\mathfrak{g}_0 = \mathfrak{o}(d,d)$ with representation matrices and invariant metric

$$(t^{IJ})_M{}^N \;=\; 2\,\delta^{[I}{}_M\,\eta^{J]N}\;, \qquad \eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

∞-dimensional Lie algebra $\mathfrak{g}$ consists of functions $\zeta = (\lambda^M, \sigma^{IJ})$,
with embedding tensor map $\vartheta : R \to \mathfrak{g}$

$$\vartheta(V) \;=\; \big(V^M\,,\; -\kappa(t^{IJ})_M{}^N \partial_N V^M\big) \;=\; \big(V^M\,,\; 2\,\partial^{[I}V^{J]}\big)$$

Derived Leibniz algebra

$$(V \circ W)^M \;=\; V^N \partial_N W^M + \partial^M V_N W^N - \partial_N V^M W^N$$

The symmetric part obeys $\{V,W\} = \frac{1}{2}\mathfrak{D}(V \bullet W)$ for

$$\mathfrak{D} \;:\; X_1 \to X_0\;, \qquad (\mathfrak{D}f)^M \;\equiv\; \partial^M f$$

$$\bullet \;:\; X_0 \otimes X_0 \to X_1\;, \qquad V \bullet W \;\equiv\; \eta_{MN} V^M W^N$$

X_0: $O(d,d)$ vectors, X_1: $O(d,d)$ scalars $\to$ Courant algebroid

$E_{7(7)}$ Exceptional Field Theory

Generators t_α, $\alpha = 1, \ldots, 133$, $\quad \mathrm{dim} R_0 = 56$, $\quad M, N = 1, \ldots, 56$,
symplectic embedding $E_{7(7)} \subset \mathrm{Sp}(56) \;\Rightarrow\;$ invariant form Ω_{MN}
Embedding tensor yields Leibniz algebra

$$(V \circ W)^M \;= V^N \partial_N W^M - W^N \partial_N V^M - \tfrac{1}{2} \partial^M V_N\, W^N$$
$$- \,12\, (t_\alpha)^{MN} (t^\alpha)_{KL}\, \partial_N V^K W^L$$

Symmetric part obeys $\{V, W\} = \frac{1}{2} \mathfrak{D}(V \bullet W)$ for $\bullet : X_0 \otimes X_0 \to X_1 \cong \mathfrak{g}^*$,
$\mathcal{A} = (A_\alpha, B_M) \in \mathfrak{g}^*$. The bullet operation is defined by

$$V \bullet W \;\equiv\; \Big((t_\alpha)_{KL} V^K W^L,\; \tfrac{1}{2} \big(V_N\, \partial_M W^N + W_N\, \partial_M V^N \big) \Big) \;\in\; \mathfrak{g}^*$$

and the differential $\mathfrak{D} : \mathfrak{g}^* \to R$ by

$$\big(\mathfrak{D} \mathcal{A} \big)^M \;\equiv\; -12 \Big((t^\alpha)^{MN} \partial_N A_\alpha \;-\; \frac{1}{12} \Omega^{MN} B_N \Big) \in R$$

Quadratic constraints require

$$(t_\alpha)^{MN}\, \partial_M f\, \partial_N g \;=\; 0\,, \quad \Omega^{MN}\, \partial_M f\, \partial_N g \;=\; 0$$

$\mathsf{E}_{8(8)}$ Exceptional Field Theory

$\mathfrak{e}_{8(8)}$ representation given by (co)adjoint

$\to \infty$-dimensional Lie algebra given by pairs $\zeta = (\lambda^M, \sigma_M)$, $M = 1, \ldots, 248$

$\to$ co-adjoint vectors are pairs $\mathcal{A} = (A^M, B_M)$

Embedding tensor is map $\vartheta : \mathfrak{g}^* \to \mathfrak{g}$ for $\mathfrak{g}^* \ni \Upsilon = (\Lambda^M, \Sigma_M)$ given by

$$\vartheta(\Upsilon) = \big(\Lambda^M,\ f_M{}^N{}_K \partial_N \Lambda^K + \Sigma_M\big) \equiv \big(\Lambda^M, R_M(\Lambda, \Sigma)\big)$$

Leibniz algebra defined on $\mathfrak{g}^*$

$$\Upsilon_1 \circ \Upsilon_2 \ \equiv\ \Big(\mathcal{L}^{[1]}_{\Upsilon_1} \Lambda_2{}^M\ ,\ \mathcal{L}^{[0]}_{\Upsilon_1} \Sigma_{2M}\ +\ \Lambda_2{}^N \partial_M R_N(\Upsilon_1)\Big)$$

$\to$ tensor hierarchy of ExFT necessarily employs ‘doubled’ vectors

$$\mathcal{A}_\mu \ \equiv\ \big(A_\mu{}^M,\ B_{\mu M}\big)$$

Bilinear form

$$\Theta(\mathcal{A}_1, \mathcal{A}_2) = -\int \mathrm{d}Y \big(2 A_{(1}{}^M B_{2)M} - f^M{}_{NK} A_1{}^N \partial_M A_2{}^K\big)$$

gives invariant 3D Chern-Simons theory.

Part III: Duality hierarchy & differential graded Lie algebras

Derived from differential graded Lie algebra

Upon suspension (overall shift of degree) and addition of vector spaces the graded symmetric $\bullet$ can be interpreted as dgLa (only b_1 and b_2) on

$$\cdots \longrightarrow \; X_2 \xrightarrow{\mathfrak{D}} X_1 \xrightarrow{\mathfrak{D}=\vartheta} X_0 = \mathfrak{g} \xrightarrow{\mathfrak{D}} X_{-1} \longrightarrow \; \cdots$$

Leibniz algebra on X_1 then "derived" from dgLa bracket:

[Lavau & Palmkvist (2019), O.H. & Bonezzi (2019)]

$$x \circ y \;\equiv\; -\mathfrak{D}x \bullet y \;=\; \pm[\mathfrak{D}x, y]$$

Leibniz relations follow from dgLa axioms.

Entire chain complex forms representation space of $\mathfrak{g}$.

Gauge theory or tensor hierarchy

Given dgLa X take dgLa $Z \equiv X \otimes \Omega(M)$ of forms on M valued in X

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[0]} & \xrightarrow{\mathfrak{D}} & Z^2_{[0]}(\chi_0) & \xrightarrow{\mathfrak{D}} & Z^1_{[0]}(\lambda_0) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[1]}(\chi_1) & \xrightarrow{\mathfrak{D}} & Z^2_{[1]}(\lambda_1) & \xrightarrow{\mathfrak{D}} & Z^1_{[1]}(A_1) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[2]}(\lambda_2) & \xrightarrow{\mathfrak{D}} & Z^2_{[2]}(A_2) & \xrightarrow{\mathfrak{D}} & Z^1_{[2]}(F_2) \\
\downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d} \\
\cdots & \xrightarrow{\mathfrak{D}} & Z^3_{[3]}(A_3) & \xrightarrow{\mathfrak{D}} & Z^2_{[3]}(F_3) & \xrightarrow{\mathfrak{D}} & Z^1_{[3]}(\mathrm{d}F_2)
\end{array}
$$

dgLa structure w.r.t. diagonal grading $|Z^k_{[p]}| = k - p$ and differential

$$\partial \equiv d + \mathfrak{D}$$

Combing forms into formal sums, remembering only diagonal degree,

$$\cdots \longrightarrow Z_1 \xrightarrow{\partial} Z_0 \xrightarrow{\partial} Z_{-1} \longrightarrow \cdots ,$$
$$\Lambda \qquad\qquad \mathcal{A} \qquad\qquad \mathcal{F}$$

where Λ gauge parameters, $\mathcal{A}$ gauge fields, and $\mathcal{F}$ field strengths.

Maurer-Cartan forms

$$\Omega \equiv e^{-\mathcal{A}}\,\partial\, e^{\mathcal{A}}$$

identically satisfy Maurer-Cartan equations

[Greitz, Howe & Palmkvist (2014)]

$$\partial\Omega + \tfrac{1}{2}[\Omega, \Omega] = 0$$

In terms of $\Omega = \mathcal{F} + \mathfrak{D}A_1$ yields *Bianchi identity* of tensor hierarchy

$$DF_p + \frac{1}{2}\sum_{k=2}^{p-1}[F_k, F_{p+1-k}] + \mathfrak{D}F_{p+1} = 0$$

Scalars and generalization of G/H

Include zero-forms (scalars) $\phi \in \mathfrak{g} = X_0$:

$$\Omega \equiv e^{-\phi} e^{-\mathcal{A}} \partial \left(e^{\mathcal{A}} e^{\phi} \right)$$

satisfies Maurer-Cartan $\Leftrightarrow$ $\partial_\Omega = \partial + \Omega$ squares to zero: $\partial_\Omega^2 = 0$

in components:

$$\partial_\Omega = D_Q + P + T + \sum_{p=2}^{\infty} \mathcal{V}^{-1} F_p \mathcal{V} \, ,$$

"coset representative" $\mathcal{V} \equiv e^{\phi} \in G$, $\mathcal{V}^{-1} D \mathcal{V} = P + Q$, $Q \in \mathfrak{h} \subset \mathfrak{g}$

and embedding tensor $\Theta \in X_{-1}$ and T-tensor

$$\mathcal{V}^{-1} \mathfrak{D} \mathcal{V} = [\Theta, \phi] + \cdots \qquad T \equiv \mathcal{V}^{-1} \Theta \mathcal{V}$$

Duality Hierarchy & Dynamics

Duality relations: 1st order formulation of dynamics, e.g., in $D = 2$:
[c.f. Inverso's talk on $E_{9(9)}$ theory]

$$\partial^\mu \phi = \varepsilon^{\mu\nu} \partial_\nu \varphi \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi = \varepsilon^{\mu\nu} \partial_\mu \partial_\nu \varphi \equiv 0$$

Goal: complete gauged supergravity/ExFT as tower of duality relations
[Bergshoeff, Hartong, O. H., Huebscher & Ortin (2009)]

In n external dimensions following structures needed:

- G covariant isomorphisms $I_p : X_p^* \to X_{n-p-2}$
- H invariant metric $\Delta_p : X_p \to X_p^*$

$\to$ 'generalized metric' $\mathcal{M}_1 \equiv \mathcal{V} \Delta_1 \mathcal{V}^\mathsf{T}$ extended to map $\mathcal{M}$ on entire X

$$\text{Duality relations:} \qquad \mathcal{F} = \star I \mathcal{M} \mathcal{F}$$

Integrability conditions imply non-linear equations, including scalars with

$$V = \tfrac{1}{2}(T, \Delta_{-1} T)$$

Summary & Outlook

- New (non-Wilsonian) consistent subsectors of string theory via homotopy transfer [A. Sen (2016)]
- Relaxation of section constraint via embedding tensor ?
- Universal dgLa unifying all $E_{n(n)}$, $n = 2, \ldots, 9$?