Exact Holography from Integrability

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ETH Zürich

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Plan

1. Review of the AdS/CFT setup
2. Integrability for the planar two-point function
3. Tackling three-point functions
4. Non-planar and higher-point functions
5. AdS_3/CFT_2 as a laboratory
6. Conclusions and outlook

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- $\mathrm{AdS}_4 \times \mathrm{CP}^3$ superstrings / ABJM theory.
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In general, **many parameters**. Simplest case is $\mathcal{N} = 4$ SYM:

$$(g_{YM},\, N) \quad \to \quad (\lambda = g_{YM}^2 N,\, \frac{1}{N})$$

The correspondence is natural when $N \to \infty$, but **λ can be arbitrary**.

Large-N Yang-Mills theory

If $\psi^\alpha(x) \equiv [\psi^\alpha(x)]^i{}_j$ are "gluons", consider ['t Hooft]

$$\mathcal{O}(x) = C_{\alpha_1 \ldots \alpha_n} \mathrm{Tr}\left[\psi^{\alpha_1} \cdots \psi^{\alpha_n}\right](x)\,.$$

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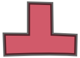
vertex (*e.g.*): $\psi_i{}^k$ $\psi^i{}_j$  $\psi_j{}^k$

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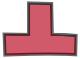
propagator: $\psi^{i}{}_{j}$  $\psi_{j}{}^{i}$ vertex (*e.g.*): $\psi^{i}{}_{j}$  $\psi_{j}{}^{k}$, $\psi_{i}{}^{k}$

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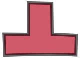
vertex (*e.g.*): $\psi^i{}_j$  $\psi_j{}^k$

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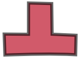
propagator: $\psi^i{}_j$  $\psi_j{}^i$ vertex (*e.g.*): $\psi^i{}_j$  $\psi_j{}^k$, $\psi_i{}^k$

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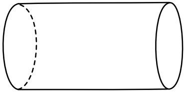
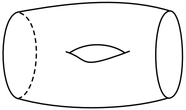
$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \rangle = \frac{\delta_{AB}}{|x_1 - x_2|^{2\Delta_A(\lambda, N)}} =$$  $+$  $+ \cdots$

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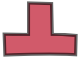
propagator: $\psi^i{}_j$  $\psi_j{}^i$ vertex (*e.g.*): $\psi^i{}_j$  $\psi_j{}^k$

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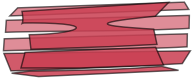
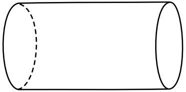
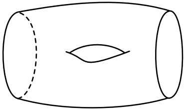


$O(1/N^2)$

$$\langle \mathcal{O}_A(x_1) \mathcal{O}_B(x_2) \rangle = \frac{\delta_{AB}}{|x_1 - x_2|^{2\Delta_A(\lambda, N)}} =$$  $+$  $+ \cdots$

The planar two-point funtion

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\rangle = \frac{\delta_{AB}}{|x_1 - x_2|^{2\Delta_A(\lambda)}} + O(1/N^2)\,.$$

In the gauge theory we can perturbatively compute, for a **generic operator** $\mathcal{O}_A$,

$$\Delta_A(\lambda) = \Delta_A^{(0)} + \lambda\Delta_A^{(1)} + \lambda^2\Delta_A^{(2)} + \dots$$

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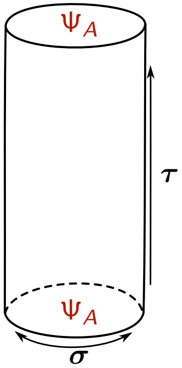

In string theory, we get the propagation of a **closed-string state** Ψ_A.

A perturbative computation from the string NLSM gives

$$\Delta_A(\lambda) = \sqrt[4]{\lambda}\left(\Delta_A^{[0]} + \frac{1}{\sqrt{\lambda}}\Delta_A^{[1]} + \frac{1}{\lambda}\Delta_A^{[2]} + \dots\right) + O(1/N^2)$$

String-spectrum computation depends on $\frac{1}{\sqrt{\lambda}} \approx \frac{\ell^2_{\text{string}}}{R^2_{\text{AdS}}}$.

Three-point functions

Three-point functions are also **hugely constrained** in terms of $\mathbf{C}(\lambda)$:

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\mathcal{O}_C(x_3)\rangle = \frac{1}{N}\frac{\mathbf{C}_{ABC}(\lambda)}{|x_1-x_2|^{2\Delta_{AB}}|x_1-x_3|^{2\Delta_{AC}}|x_2-x_3|^{2\Delta_{BC}}} + O(1/N^3)$$

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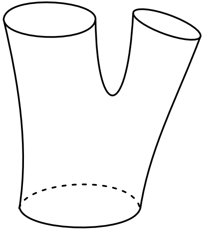
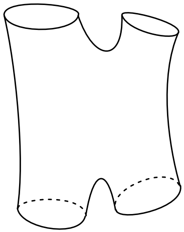


$$+\ O(1/N^3)$$

Four-point functions

Four-point functions instead depend on **conformal cross-ratios**

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\mathcal{O}_C(x_3)\mathcal{O}_D(x_4)\rangle = \frac{1}{N^2}\frac{\mathcal{F}_{ABCD}\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{14}x_{23}}{x_{13}x_{24}}; \lambda\right)}{\prod_{j<k}^{4} |x_j - x_k|^{2\Delta_{ABCD}[j,k]}} + O(1/N^4)$$

$$\langle \mathcal{O}_A(x_1)\mathcal{O}_B(x_2)\mathcal{O}_C(x_3)\mathcal{O}_D(x_4)\rangle =$$  $$+ \; O(1/N^4)$$

Plan

The string non-linear sigma model (NLSM)

We want to find the energy spectrum of strings, $E = \int_0^R \mathrm{d}\sigma P^0$, where $P_\mu = \dfrac{\delta S}{\delta \partial_\tau X^\mu}$.

$$S = \sqrt{\lambda} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^R \mathrm{d}\sigma \Big(\sqrt{|g|} g^{\mu\nu} G_{JK}(X) + \epsilon^{\mu\nu} B_{JK}(X) \Big) \partial_\mu X^J \partial_\nu X^K + \text{fermions}$$

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- For our backgrounds, the action is **classically integrable**.
- Has reparametrisation invariance (and κ-symmetry).
- Needs to be gauge-fixed before studying it at quantum level.
- Fix light-cone gauge so that $R = \text{R-charge}$, and the Hamiltonian is $H_{\text{w.s.}} \approx E$.

Factorised scattering for $R \to \infty$

At the quantum level, integrability means **scattering factorisation**. [Zamolodchikov^2]

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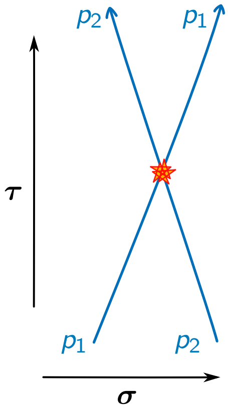

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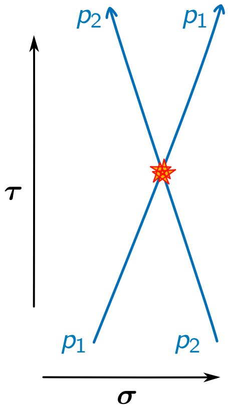

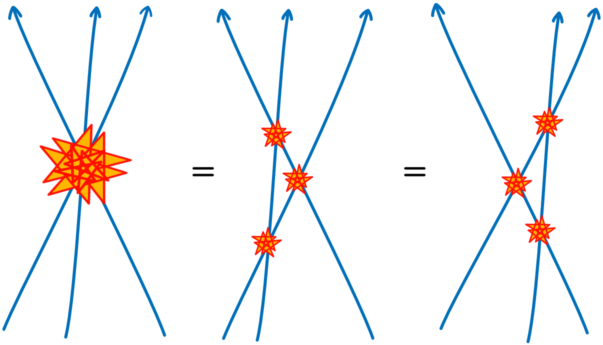

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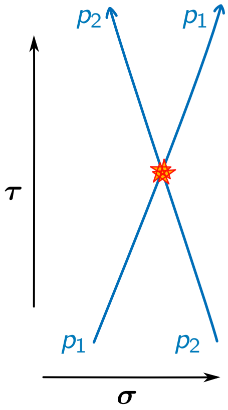

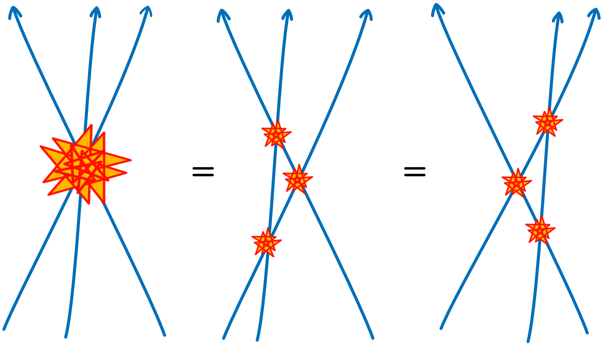

"Yang-Baxter equation"

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The Hamiltonian $\mathbf{H}$ can be computed at $\tau = -\infty$, where particles are well-separated.

$$\mathbf{H}\,|p_1, \dots p_n\rangle = \sum_{j=1}^{n} \omega(p_j)\,|p_1, \dots p_n\rangle$$

where $\omega(p)$ **follows from symmetry** too.

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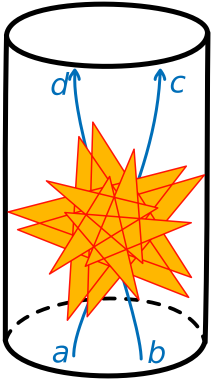
n particles:

$$e^{ip_k R} \prod_{j \neq k}^{n} S(p_k, p_j) = 1, \quad k = 1, \ldots n, \qquad H = \sum_{j=1}^{n} \omega(p_j)\,.$$

The (infamous) wrapping corrections

We assumed that the particles are "well-separated". This is an approximation when R is finite.

[Lüscher] [Ambjørn, Janik, Kristjansen]



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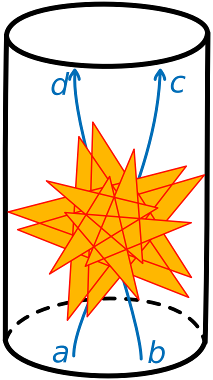

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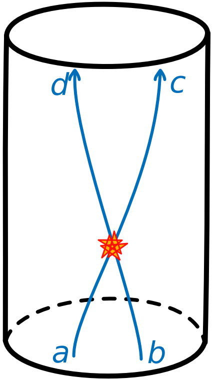

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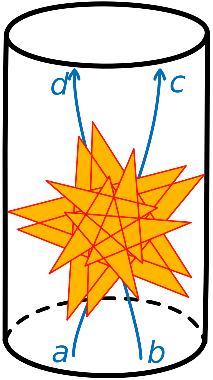

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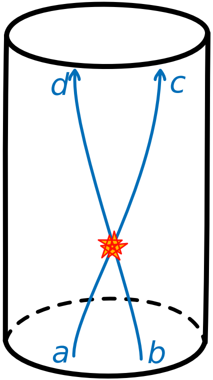
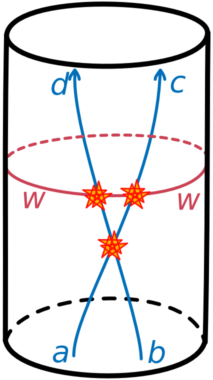


$+ \sum_{w} \int \mathrm{d}p_w$



$e^{-\omega(p_w)R} + \dots$

Again on wrapping

Perhaps a more familiar idea is that we "cut" the cylinder by inserting

$$\mathbf{1} = 1 + \sum_{w} |p_w\rangle\langle p_w| + \sum_{w,w'} |p_w, p_{w'}\rangle\langle p_w, p_{w'}| + \dots$$

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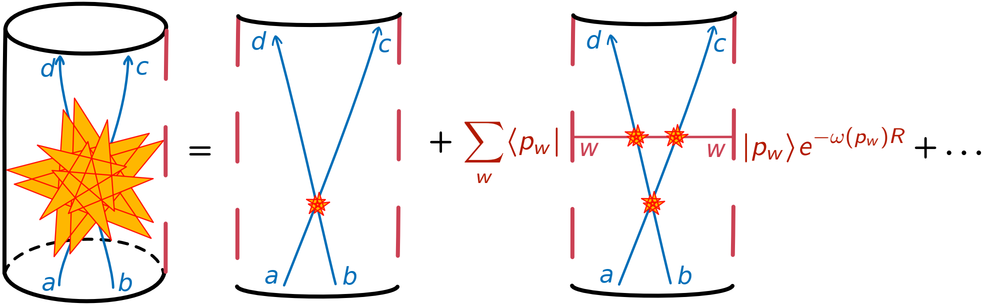


Finite-volume spectrum

The **finite-volume spectrum** comes from the **thermodynamic** of a "mirror theory" $(\sigma \leftrightarrow \tau)$.

[Yang^2] [Zamolodchikov] [Arutyunov, Frolov] [Gromov, Kazakov, Vieira] [Bombardelli, Fioravanti, Tateo] [...]

C. Marboe, D. Volin / Nuclear Physics B 899 (2015) 810–847 831

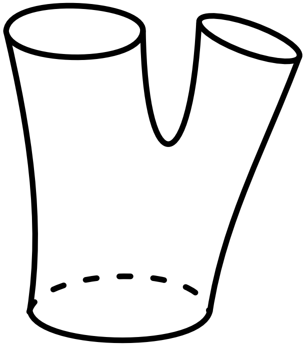
$(\lambda = 4\pi^2 g^2)$

$$
\begin{aligned}
\Delta = {} & 4 + 12g^2 - 48g^4 + 336g^6 + g^8(-2496 + 576\,\zeta_3 - 1440\,\zeta_5) \\
& + g^{10}(15\,168 + 6912\,\zeta_3 - 5184\,\zeta_3^2 - 8640\,\zeta_5 + 30\,240\,\zeta_7) \\
& + g^{12}(-7680 - 262\,656\,\zeta_3 - 20\,736\,\zeta_3^2 + 112\,320\,\zeta_5 + 155\,520\,\zeta_3\,\zeta_5 \\
& + 75\,600\,\zeta_7 - 489\,888\,\zeta_9) \\
& + g^{14}(-2\,135\,040 + 5\,230\,080\,\zeta_3 - 421\,632\,\zeta_3^2 + 124\,416\,\zeta_3^3 - 229\,248\,\zeta_5 \\
& + 411\,264\,\zeta_3\,\zeta_5 - 993\,600\,\zeta_5^2 - 1\,254\,960\,\zeta_7 - 1\,935\,360\,\zeta_3\,\zeta_7 - 835\,488\,\zeta_9 \\
& + 7\,318\,080\,\zeta_{11}) + \ldots
\end{aligned}
$$

Plan

Cutting up three-point functions

Three-point functions have strange topology. . . What is $R \to \infty$? [Basso, Komatsu, Vieira]



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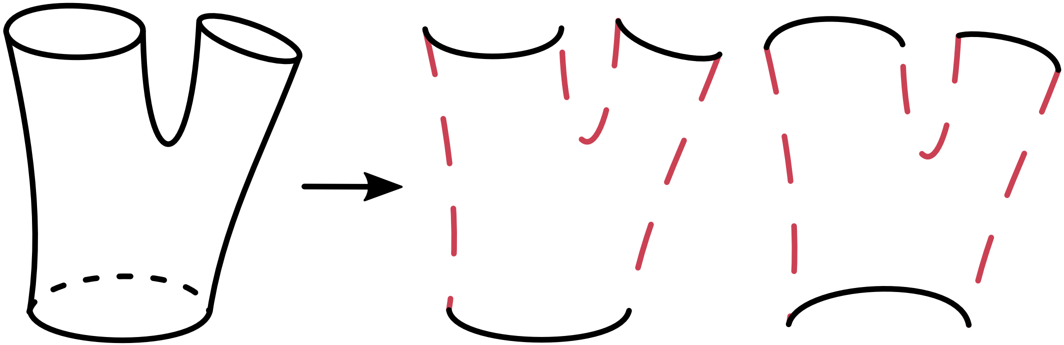
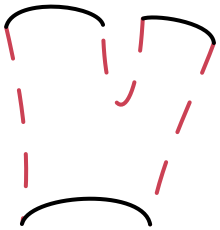


The “hexagon operator”

Each patch has six edges. QFT interpretation? [Basso, Komatsu, Vieira]



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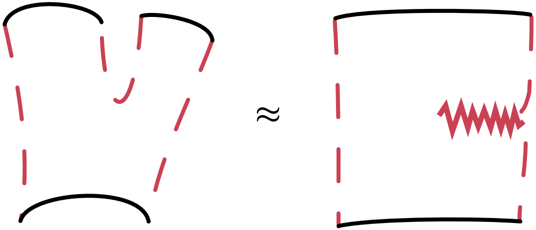

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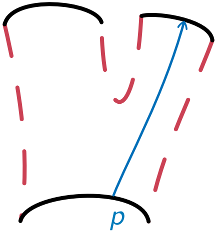
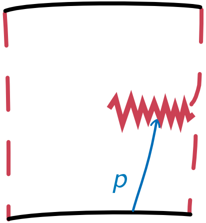
Each patch has six edges. QFT interpretation? [Basso, Komatsu, Vieira]



$\approx$



$\approx \quad \langle \mathbf{h} | p \rangle$

This is the computation of a **form factor** $\langle \mathbf{h} | p \rangle$ for the non-local **“hexagon” operator**.

The form-factor bootstrap

We want the form-factor $\langle \mathbf{h}|$.

- 1 particle: $\langle \mathbf{h}|p\rangle$ has 16 entries.
- 2 particle: $\langle \mathbf{h}|p_1, p_2\rangle$ has 256 entries.
- n particles: $\langle \mathbf{h}|p_1, \ldots p_n\rangle$ has 16^n entries.

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- n particles: $\langle \mathbf{h}|p_1, \ldots p_n\rangle$ has 16^n entries.

Symmetry plus compatibility with $\mathbf{S}(p_1, p_2)$ fixes (almost) uniquely

$$\langle \mathbf{h}|p\rangle \qquad \text{and} \qquad \langle \mathbf{h}|p_1, p_2\rangle$$

Factorisation gives arbitrary $\langle \mathbf{h}|p_1, \ldots p_n\rangle$.

"Asymptotic" three-point functions

Using our knowledge of $\langle \mathbf{h}|$, we may compute "aymptotic" correlators. For instance, take

$$\mathcal{O}_2, \quad \mathcal{O}_3 \quad \text{protected (BPS)}, \qquad \mathcal{O}_1 \quad \text{generic.}$$

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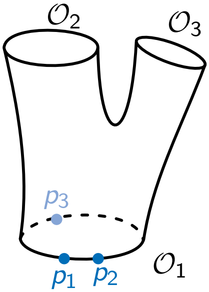
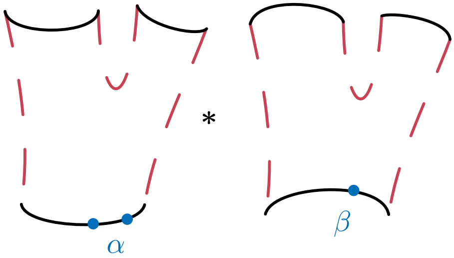


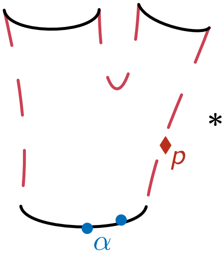
$$\approx \sum_{\{p_j\}=\alpha\cup\beta} \mathcal{W}(\alpha,\beta)$$



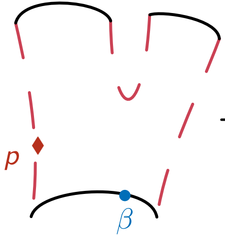
"Asymptotic" three-point functions

The previous recipe must be corrected due to **wrapping**. Simplest correction:

$$\sum_{\{p_j\}=\alpha\cup\beta} \mathcal{W}(\alpha,\beta) \sum_{\blacklozenge} \int \mathrm{d}p\, e^{-\omega(p)\,R_{13}}$$



$*$



$+ \cdots$

Now we must add corrections along **every cut**.

Some examples

This gives correct results, but breaks down eventually due to **wrapping**.

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s	$\left(\frac{C^{\bullet\circ\circ}}{C^{\circ\circ\circ}}\right)^2$ for twist $L=2$, bridge $\ell=1$ and spin s
2	$\frac{1}{6} - g^2 + (7+3\zeta_3)g^4 - (48+8\zeta_3+25\zeta_5)g^6 + \dots$
4	$\frac{1}{70} - \frac{205}{1764}g^2 + \left(\frac{76393}{74088} + \frac{5}{14}\zeta_3\right)g^4 - \left(\frac{242613655}{28005264} + \frac{1315}{1323}\zeta_3 + \frac{125}{42}\zeta_5\right)g^6 + \dots$
6	$\frac{1}{924} - \frac{553}{54450}g^2 + \left(\frac{880821373}{8624880000} + \frac{7}{220}\zeta_3\right)g^4 - \left(\frac{1364275757197}{1423105200000} + \frac{520093}{6534000}\zeta_3 + \frac{35}{132}\zeta_5\right)g^6 + \dots$
8	$\frac{1}{12870} - \frac{14380057}{18036018000}g^2 + \left(\frac{5944825782678337}{682443241880400000} + \frac{761}{300300}\zeta_3\right)g^4$
	$-\left(\frac{75807280363428746576 5957}{860738363254073304000000 0}\right)$
10	$\frac{1}{184756} - \frac{3313402433}{55983859495200}g^2 + \left(\frac{171050793565932326659}{2488046776199329363200000} + \frac{671}{3527160}\zeta_3\right)g^4$
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[Eden, AS] [Basso, Komatsu, Goncalves, Vieira]

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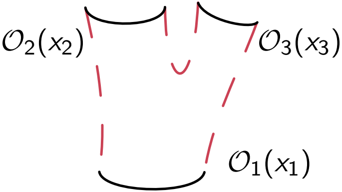
[Eden, AS] [Basso, Komatsu, Goncalves, Vieira]

No exact method exists here!

Plan

Four-point functions and position dependence

To tackle n-point functions, we need dependence on $x_1, \dots x_n$.



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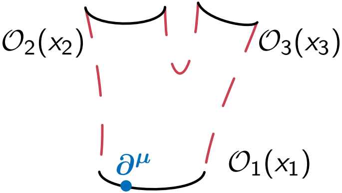

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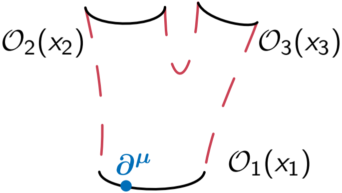

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The excited object must transform as a vector

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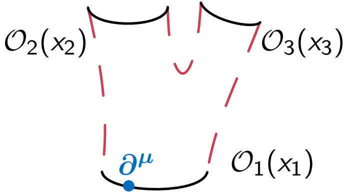

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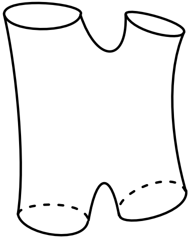
This allows to associate **weights** to **excitations** on the hexagon, [Eden, AS] [Fleury, Komatsu]

$$\bullet = \partial_\pm \to \frac{x_{23}^\pm}{x_{12}^\pm x_{13}^\pm} \, .$$

Four-point function tessellation

We need to "cut open" the four point function. Two options:

1. Cut into two three-point functions. [Basso, Coronado, Komatsu, Lam, Vieira, Zhong]
2. Cut straight into hexagons. [Eden, AS] [Fleury, Komatsu]



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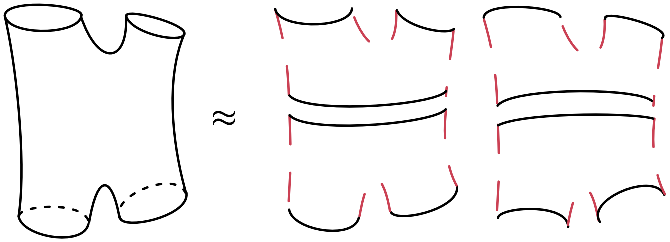

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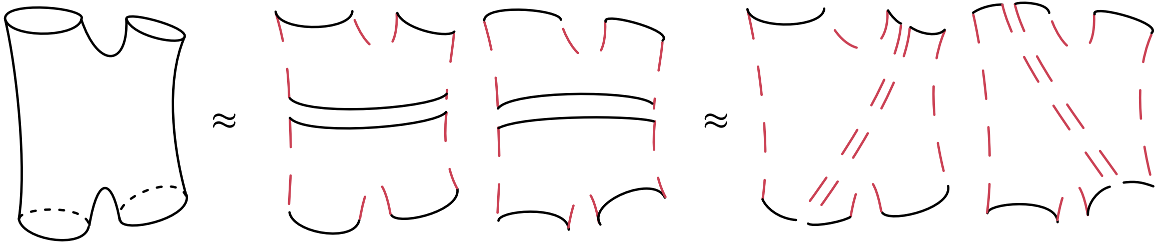

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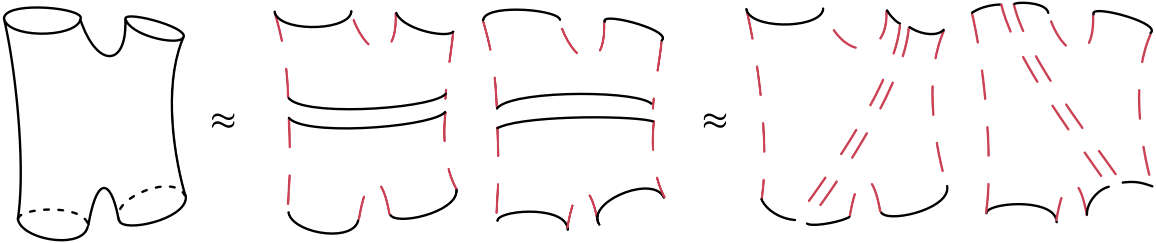
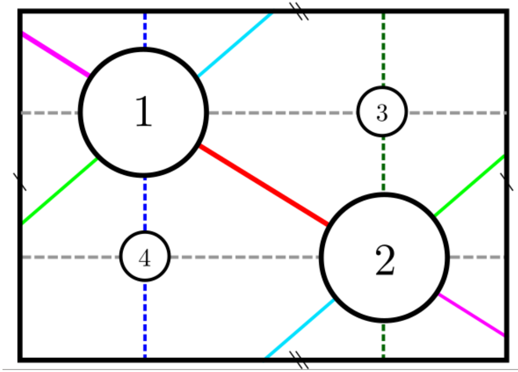


Caveat: more cuts = more wrapping!

Higher-genus and higher-point functions

More complicated tessellations can describe non-planar observables.

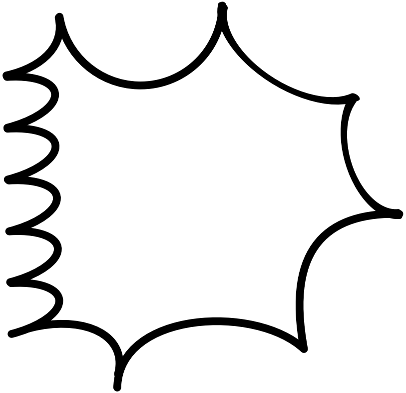
Example: **torus** four-point function. [Eden, Jiang, le Plat, AS] [Bargheer, Caetano, Fleury, Komatsu, Vieira]



Plan

The AdS_3/CFT_2 parameter space

This setup has half of the supersymmetry of $AdS_5 \times S^5$, and more parameters.



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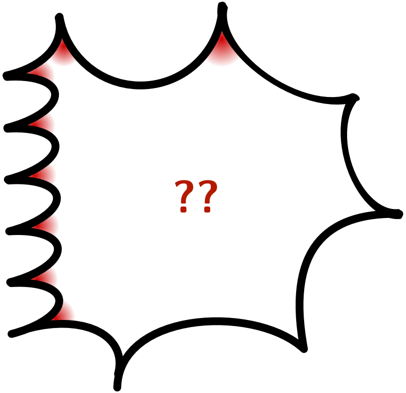

Can tune string tension and RR/NSNS fluxes.

Special points

- large tension (sugra)
- NSNS flux only (WZW model)

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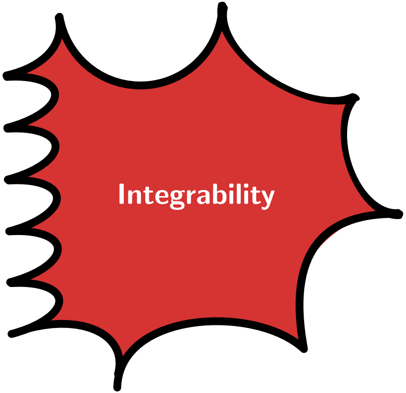

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Integrability holds in general.

[Babichenko, Stefanski, Zarembo] [Borsato, Ohlsson-Sax, AS] [. . .]

The pure-NSNS points

The spectrum for the level-k WZW models is simple [Maldacena, Ooguri]

$$E(n_1, \dots n_N) \approx \sqrt{R^2 + 2k(n_1 + \cdots + n_N)} - R$$

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In integrability, this is reflected in the **simple S-matrix** and **dispersion** [Hoare, Tseytlin] [Baggio, AS]

$$\mathbf{S}(p_1, p_2) = e^{i\Phi(p_1,p_2)}\,\mathbf{1}\,, \qquad \omega_\mu(p) = \left|\frac{k}{2\pi}p + \mu\right|.$$

$$\Phi(p_1, p_2) = \begin{cases} -\frac{k}{2\pi}p_1\,p_2 & p_1 \text{ left-mover, } p_2 \text{ right-mover} \\ +\frac{k}{2\pi}p_1\,p_2 & p_2 \text{ left-mover, } p_1 \text{ right-mover} \\ 0 & \text{else} \end{cases}$$

The Bethe ansatz (dirty derivation)

$$\Phi(p_1, p_2) = \pm \frac{k}{2\pi} p_1 p_2 \,, \qquad \omega_\mu(p) = \left| \frac{k}{2\pi} p + \mu \right|$$

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We write the usual Bethe equations [Baggio, AS] [Dei, AS]

$$e^{ip_j R} \prod_{k \neq j}^{N} S(p_k, p_j) = 1 \qquad \Rightarrow \qquad p_j R + \sum_{j}^{N} \Phi(p_j, p_k) = 2\pi n_j, \quad n_j \in \mathbb{Z}$$

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Introduce the notation: $p_j^+ \equiv$ "left", $p_j^- \equiv$ "right", and $P^\pm \equiv \sum p_j^\pm$

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Sum over j, use that $P^+ + P^- = 0$ and introduce $\mathcal{N}^+ = \mathcal{N}^-$

$$P^\pm \left(R + \frac{k}{2\pi} P^\pm \right) = 2\pi \mathcal{N}^\pm \,, \qquad H \approx E \approx \sqrt{R^2 + 2k(\mathcal{N}^+ + \mathcal{N}^-)} - R$$

The spectrum from integrability

The integrability construction reproduces the short-string spectrum. [Baggio, AS] [Dei, AS]

- TBA can be solved exactly (!!)
- Wrapping effects can be resummed explicitly.
- "Spectral flow" is automatically implemented.
- Can be done for $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ and $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$

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Ideal testing ground for higher-point functions?

Plan

Conclusions and outlook

- **Integrability** can be used to compute **non-protected observables**.
- The spectrum can be studied in detail using **thermodynamic** Bethe ansatz.

  [Ambjørn, Janik, Kristjansen] [Arutyunov, Frolov] [Gromov, Kazakov, Vieira] [Bombardelli, Fioravanti, Tateo]
- **Higher-point functions** and **non-planar observables** can also be studied.

  [Basso, Komatsu, Vieira][Eden, AS][Fleury, Komatsu] [Eden, le Plat, Jiang, AS][Bargheer, Coronado, Caetano, . . .]
- **Wrapping effects** need to be tamed.

  [Basso, Coronado, Komatsu, Lam, Vieira, Zhong] [de Leeuw, Eden, Jiang, le Plat, Müeller, AS]
- $\mathrm{AdS}_3/\mathrm{CFT}_2$ may be an ideal ground to do this.

  [Hoare, Tseytlin] [Lloyd, Ohlsson-Sax, Stefanski, AS] [Baggio, AS] [Dei, AS]

  (On top of being hugely interesting by itself.)

  [Giribet, Hull, Kleban, Porrati, Rabinovici] [Dei, Eberhardt, Gaberdiel, Gopakumar]