$\mathcal{N} = 1$ Minkowski backgrounds: Involutivity, moment maps and moduli

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1910.04795 & 1912.????? with Anthony Ashmore, David Tennyson, & Dan Waldram

General motivation

- $\exists$ "good" understanding of sugra/strings on "nice" geometries

(Special holonomy manifolds e.g. CY3 [Candelas, Horowitz, Strominger & Witten '85])

- Want: to understand general SUSY backgrounds (with fluxes)
  - for phenomenology (moduli stabilisation, SUSY breaking)
  - for AdS/CFT (CFT data $\leftrightarrow$ geometry)
  - for theoretical / mathematical understanding
- Much work on this! [Too many people to list!]
- Generalised geometry provides a useful toolkit for fluxes!

Generalised geometry and supergravity

- Geometry of extended tangent bundle $E \simeq TM \oplus (\text{"forms"})$
- Combines : $\begin{cases} \text{diffeo} + \text{gauge symmetry} \to \text{Generalised diffeos} \\ \text{metric} + \text{gauge fields} \to \text{Generalised metric} \end{cases}$
- Enhanced sym $\begin{cases} GL(d,\mathbb{R}) \to O(d,d) \times \mathbb{R}^+,\ E_{d(d)} \times \mathbb{R}^+, \text{etc.} \\ SO(d) \to SO(d) \times SO(d),\ H_d, \text{etc.} \end{cases}$
- Also combines : various G-structures on TM for $\mathcal{N}$ SUSYs

  $\to$ Unique generalised structure group $\mathcal{G}_{\mathcal{N}}$ for each $\mathcal{N}$
- Notion of integrability [Coimbra, CS-C & Waldram '14; Coimbra & CS-C '16]

$\mathcal{N}$-SUSY Minkowski background $\Leftrightarrow$ Integrable $\mathcal{G}_{\mathcal{N}}$ structure

Outline of talk

- Generalised geometry and $\mathcal{N} = 1$ SUSY backgrounds
- Complex and $SL(3, \mathbb{C})$ structures

  $\rightarrow$ Integrability, moment maps and moduli
- J and ψ structures for $\mathcal{N} = 1$

  $\rightarrow$ Integrability and moment maps
- Examples: G_2 and GMPT
- Moduli
- Superpotentials, F-terms and D-terms
- Extremisation of $\mathcal{K}$ and Hitchin's variational principle
- Summary and comments

$\mathcal{N} = 1$ backgrounds of $D = 11$ supergravity

Warped metric ansatz

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta(y)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}\mathrm{d}y^m\mathrm{d}y^n$$

11d $=$ 4d $\otimes$ 7d spinors

One complex spinor $\zeta \to$ uplift of 4d Killing spinors η to 11d

$$\varepsilon_{(\eta)} = \eta_+ \otimes \zeta + \eta_- \otimes \zeta^*$$

7d Killing spinor equations

11d SUSY: $\delta_\varepsilon \psi_M = 0$ $\quad (F_4 = \mathrm{d}A_3,\ \tilde{F}_7 = \mathrm{d}\tilde{A}_6 - \frac{1}{2}A_3 \wedge F_4)$

$$\Rightarrow \nabla_m\zeta + \not{F}_m\zeta + \tilde{\not{F}}_m\zeta = 0$$
$$\not{\nabla}\zeta + \not{F}\zeta + \tilde{\not{F}}\zeta + \not{\partial}\Delta\zeta = 0$$

$E_{7(7)} \times \mathbb{R}^+$ Generalised Geometry [Hull '07; Pacheco & Waldram '08; CSW '11]

$$E \simeq TM_7 \oplus \Lambda^2 T^* M_7 \oplus \Lambda^5 T^* M_7 \oplus (T^* M_7 \otimes \Lambda^7 T^* M_7)$$

- Dorfman derivative: $L_V = \partial_V - (\partial \times_{\text{ad}} V) \cdot$
- Generalised torsion $T(V) \in \text{ad}(E_{7(7)} \times \mathbb{R}^+)$

$$T(V)\cdot = L_V^{(\partial \to D)} - L_V$$

- Generalised metric $\leftrightarrow$ bosonic physical d.o.f.

$$G \sim \{g_{mn}, A_{mnp}, \tilde{A}_{m_1 \dots m_6}, \Delta\} \in \frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)}$$

Generalised Intrinsic Torsion & SUSY backgrounds [CSW '11]

- Generalised Levi-Civita: $\exists$ Torsion-free $SU(8)$ connections

$$DG = 0 \qquad T(D) = 0$$

- Killing spinor ϵ defines (global) $SU(7)$ structure on E
- Killing spinor equations: Operators $D \times_{\text{SUSY}} \epsilon = 0$

Key result: [Coimbra, CS-C & Waldram '14]

Killing spinor eqns $(D \times_{\text{SUSY}} \epsilon) \equiv SU(7)$ Intrinsic torsion

$\mathcal{N} = 1$ Minkowski background $\equiv$ torsion-free $SU(7)$ structure

(Almost) Complex structures

Almost $\mathbb{C}$-structure $\quad (GL(3,\mathbb{C}) \subset GL(6,\mathbb{R}))$

- $I \in \mathrm{End}(TM)$ s.t. $I^2 = -1 \rightarrow$ generates $U(1) \subset GL(6,\mathbb{R})$
- $TM_{\mathbb{C}} \rightarrow T^{1,0} \oplus T^{0,1}$ $\boxed{\simeq L_{+1} \oplus L_{-1}}$ $\quad I \leftrightarrow$ choice of L_{-1}
- $L_{\pm 1}$ has eigenvalue $\pm \mathrm{i}$ under $U(1)$ generator I.

Integrability

- Complex manifold $\Leftrightarrow$ Nijenhuis tensor vanishes

  $\Leftrightarrow GL(3,\mathbb{C})$ intrinsic torsion vanishes

  $\Leftrightarrow$ $\boxed{\text{Involutive } [L_{-1}, L_{-1}] \subset L_{-1}}$

$SL(3,\mathbb{C})$ structures

Holomorphic volume form

- Line bundle $\mathcal{U}_I \sim \Lambda^3(L_{+1})^* \subset \Lambda^3 T^*_{\mathbb{C}}$
- If trivial (as bundle) $\exists$ section Ω

  $\rightarrow$ Structure group reduced to $SL(3,\mathbb{C}) \subset GL(3,\mathbb{C})$
- (Stable) 3-Form Ω also defines I [Hitchin '00]
- Rescaling $\Omega \rightarrow f\Omega$ gives same I where $f : M \rightarrow \mathbb{C}^*$

Integrability

- I is integrable $\Rightarrow \mathrm{d}\Omega = A \wedge \Omega$
- Additional condition for $SL(3,\mathbb{C})$ integrable

$$\boxed{\mathrm{d}\Omega = 0} \qquad (\mathcal{U}_I \text{ holomorphically trivial})$$

Kähler geometry of $SL(3,\mathbb{C})$ structures

Space of $SL(3,\mathbb{C})$ structures

(Pseudo-)Kähler coset: $Q = \dfrac{GL(6,\mathbb{R})}{SL(3,\mathbb{C})}$ Bundle: $Q \to \mathcal{Q} \to M$

Space of $SL(3,\mathbb{C})$ structures $\mathcal{Z} = \{\text{sections of } \mathcal{Q}\}$

$\mathcal{Z}$ inherits Kähler potential $\mathcal{K} = \mathrm{i}\int_M \Omega\wedge\bar{\Omega}$ (Hitchin functional)

Think of Ω as complex coordinate on $\mathcal{Z}$

... with integrable $GL(3,\mathbb{C})$

- Set $\hat{\mathcal{Z}} = \{\Omega \in \mathcal{Z} \mid I_\Omega \text{ is integrable}\}$
- $\hat{\mathcal{Z}}$ inherits Kähler potential from $\mathcal{Z}$

Diffeomorphisms and moment map

Diffeomorphisms

Diffeo group $\mathrm{Diff}(M)$ acts on $\hat{\mathcal{Z}}$

$$v \in \Gamma(TM) \mapsto \rho_v \in \Gamma(T\hat{\mathcal{Z}}) \quad \text{s.t.} \quad i_{\rho_v}\delta\Omega = \mathcal{L}_v\Omega$$

Moment map

Kähler form on $\hat{\mathcal{Z}}$ is (where $\delta = \partial' + \bar{\partial}'$)

$$\varpi = \mathrm{i}\partial'\bar{\partial}'\mathcal{K}$$

Find (equivariant) moment map $\mu : \hat{\mathcal{Z}} \to \mathfrak{diff}^*$

$$i_{\rho_v}\varpi = -\delta\mu(v) \quad \text{where} \quad \mu(v) = \int_M (\mathcal{L}_v\Omega) \wedge \bar{\Omega}$$

Moment map as intrinsic torsion

$$\mu(v) = \int (\mathcal{L}_v \Omega) \wedge \bar{\Omega} = \int (i_v \mathrm{d}\Omega) \wedge \bar{\Omega} - (i_v \Omega) \wedge \mathrm{d}\bar{\Omega} = 0 \quad \forall v$$

becomes the $SL(3, \mathbb{C})$ intrinsic torsion condition $\mathrm{d}\Omega = 0$

Moduli of $SL(3,\mathbb{C})$ structures

Moduli space

Symplectic (Kähler) quotient $\leftrightarrow$ complex quotient

$$\mathcal{M}_\Omega \simeq \hat{\mathcal{Z}} /\!\!/ \mathrm{Diff} \simeq \mu^{-1}(0) / \mathrm{Diff} \simeq \hat{\mathcal{Z}} / \mathrm{Diff}_{\mathbb{C}}$$

(nb. $\mathrm{Diff}_{\mathbb{C}}$ action only really defined on $\hat{\mathcal{Z}}$)

Relatively easy to formulate $\hat{\mathcal{Z}}$ and complexified quotient!

Moduli of $SL(3,\mathbb{C})$ structures

Deforming I

Define new $T^{0,1}$ graph of map $\mu : T^{0,1} \to T^{1,0}$

Decompose $\mathfrak{gl}(6,\mathbb{R}) \to \mathfrak{gl}(3,\mathbb{C}) \oplus \Omega^{0,1}(T^{1,0}) \oplus \Omega^{1,0}(T^{0,1})$

So $\mu|_{\text{point}}$ is a complex coordinate on coset $\frac{GL(6,\mathbb{R})}{GL(3,\mathbb{C})} \sim \frac{GL(6,\mathbb{C})}{GL(3,\mathbb{C}) \ltimes P_{\mathbb{C}}}$

$(\mu|_{\text{point}})^2 = 0$

$\to$ graph is just the (finite) action of $GL(6,\mathbb{R})$ on $T^{0,1}$.

Involutivity and Maurer-Cartan

Fixing $[(1+\mu)T^{0,1},(1+\mu)T^{0,1}] \subset (1+\mu)T^{0,1}$ we find

$$\bar{\partial}\mu + \frac{1}{2}[\mu,\mu] = 0$$

Moduli of $SL(3, \mathbb{C})$ structures

Quotient by Diff

Infinitesimally Diff acts by $\mu = \bar{\partial}\epsilon$ for $\epsilon \in T^{1,0}$

So we have that moduli of I are $H^{0,1}(T^{1,0})$

Moduli of Ω

For each I s.t. $\mathcal{U}_I$ trivial have unique Ω s.t. $\mathrm{d}\Omega = 0$ (up to $\mathbb{C}^*$)

$$\text{Moduli of } \Omega = \mathbb{C} \oplus H^{0,1}(T^{1,0}) \simeq H^{3,0} \oplus H^{2,1}$$

$\mathcal{N} = 1$ structures revisited : Bosonic description

$U(1)$ generator J ("Exceptional (almost) complex structure")

Let $J \in \mathbf{133}_0$ be the generator of the $U(1)$ commuting with $SU(7)$

$E_{\mathbb{C}}$ decomposes into eigenbundles

$$E \simeq L_{+3} \oplus L_{+1} \oplus L_{-1} \oplus L_{-3}$$

Involutivity

The $U(7) \times \mathbb{R}^+$ structure defined by J has vanishing torsion if

$$[L_{-3}, L_{-3}] \subset L_{-3}$$

$\mathcal{N} = 1$ structures revisited : Bosonic description

$SU(7)$ structure ψ [Pacheco & Waldram '07]

If line bundle $\mathcal{U}_J = \Lambda^7(L_{+3})$ trivial then have:

$\psi \in \mathbf{912}_{+3}$ defines $SU(7)$ structure

ψ defines a vol, J and generalised metric G (physical fields)

$$\mathrm{vol}_\psi \sim [\mathrm{i}s(\psi, \bar{\psi})]^{1/3} \qquad J \sim (\psi \times_{\mathbf{133}} \bar{\psi})\,\mathrm{vol}^{-3} \qquad G \sim (\psi \times_{\mathbf{1463}} \bar{\psi})\,\mathrm{vol}^{-4}$$

Local $\mathbb{C}^*$ action

$\psi' = f\psi$ gives the same J for $f : M \to \mathbb{C}^*$

Kähler geometry of $SU(7)$ structures

Space of $SU(7)$ structures

(Pseudo-)Kähler coset: $Q = \frac{E_{7(7)} \times \mathbb{R}^+}{SU(7)}$ Bundle: $Q \to \mathcal{Q} \to M$

Space of $SU(7)$ structures $\mathcal{Z} = \{\text{sections of } \mathcal{Q}\}$

$\mathcal{Z}$ inherits Kähler potential $\mathcal{K} = \int_M \text{vol}_\psi$ (Hitchin functional)

Think of ψ as complex coordinate on $\mathcal{Z}$

... with integrable $U(7) \times \mathbb{R}^+$

- Set $\hat{\mathcal{Z}} = \{\psi \in \mathcal{Z} \mid J \text{ is integrable}\}$
- $\hat{\mathcal{Z}}$ inherits Kähler potential from $\mathcal{Z}$

Moment map for GDiff

Diffeomorphisms

Generalised diffeo group $\mathrm{GDiff}(M)$ acts on $\hat{\mathcal{Z}}$

$$V \in \Gamma(E) \mapsto \rho_V \in \Gamma(T\hat{\mathcal{Z}}) \quad \text{s.t.} \quad i_{\rho_V}\delta\psi = L_V\psi$$

Moment map

Kähler form on $\hat{\mathcal{Z}}$ is (where $\delta = \partial' + \bar{\partial}'$)

$$\varpi = \mathrm{i}\partial'\bar{\partial}'\mathcal{K}$$

Find (equivariant) moment map $\mu : \hat{\mathcal{Z}} \to \mathfrak{gdiff}^*$

$$i_{\rho_V}\varpi = -\delta\mu(V) \quad \text{where} \quad \mu(V) = \int \tfrac{1}{3}\,\mathrm{vol}_\psi^{-2/3}\, s((L_V\psi), \bar{\psi})$$

Moment map and intrinsic torsion

Let D be (torsionful) $SU(7)$ connection, so $D\psi = 0$

$$\begin{aligned}\mu(V) &= \int \tfrac{1}{3}\,\mathrm{vol}_\psi^{-2/3}\, s((L^D_V\psi - T(V)\cdot\psi), \bar{\psi}) \\ &= \int \tfrac{1}{3}\,\mathrm{vol}_\psi^{-2/3}\, s(-T(V)\cdot\psi, \bar{\psi})\end{aligned}$$

$V \in \mathbf{7} + \mathbf{21} + \bar{\mathbf{7}} + \bar{\mathbf{21}}$ so must be those parts of T^{int}

Conclusion:

$$\boxed{T^{\text{int}}_{SU(7)} = 0 \quad \Leftrightarrow \quad L_{+3} \text{ involutive and } \mu(V) = 0}$$

A nice feature of $\mathcal{N} = 1$

$$\mathcal{M}_\psi = \hat{\mathcal{Z}} /\!\!/ \mathrm{GDiff} = \mu^{-1}(0)/\mathrm{GDiff} = \hat{\mathcal{Z}}/\mathrm{GDiff}_{\mathbb{C}}$$

Complexified quotient should implement the moment map!

(Use $E_{\mathbb{C}} \to$ generates $\mathrm{GDiff}_{\mathbb{C}}$)

So just need: L_{+3} Involutive and quotient by $\mathrm{GDiff}_{\mathbb{C}}$

Physical moduli

- Moduli space $\leftrightarrow$ space of chiral superfields in 4d
- Not all moduli physical ! Only those in $\frac{E_{7(7)} \times \mathbb{R}^+}{SU(8)/\mathbb{Z}_2}$
- Moduli moving $SU(7)$ inside $SU(8)$ changes Killing spinor ζ

  $\rightarrow$ absent for true $\mathcal{N} = 1$ backgrounds
- Constant $\mathbb{R}^+$ changes warp factor Δ and $U(1)$ rotating ζ

  $\rightarrow$ $\mathbb{C}^*_{\text{global}}$ which can be absorbed into 4d redefinitions
- Physical moduli space

$$\mathcal{M}_{\text{phys}} = \mathcal{M}_\psi / \mathbb{C}^*_{\text{global}}$$

- $\mathbb{C}^*_{\text{global}} \rightarrow$ Kähler transformations of $\mathcal{K}_{\text{phys}} = -3 \log \mathcal{K}$

Example 1: G_2 manifolds

J structure

$$J = \varphi^\sharp - \varphi$$

Involutivity

$$L_{+3} = \mathrm{e}^{\mathrm{i}\varphi} T_{\mathbb{C}}$$

$$L_{\mathrm{e}^{\mathrm{i}\varphi} v}(\mathrm{e}^{\mathrm{i}\varphi} w) = \mathrm{e}^{\mathrm{i}\varphi}\left(L_v^{\mathrm{i}\mathrm{d}\varphi + \frac{1}{2}\varphi\wedge\mathrm{d}\varphi} w\right) = \mathrm{e}^{\mathrm{i}\varphi}([v,w] + i_w i_v(\mathrm{i}\mathrm{d}\varphi + \tfrac{1}{2}\varphi\wedge\mathrm{d}\varphi))$$

$$\therefore \text{Involutive } \Leftrightarrow \mathrm{d}\varphi = 0$$

Example 1: G_2 manifolds

ψ structure

$$\psi \in \mathbf{912}_{+3} \simeq \mathbb{R} \oplus \Lambda^3 T^* \oplus (T^* \otimes \Lambda^5 T^*) \oplus \dots$$

$$\text{Set} \qquad \psi = \mathrm{e}^{\mathrm{i}\varphi} \cdot 1 = 1 + \varphi + j\varphi \wedge \varphi + \dots$$

Kähler potential

$$\mathcal{K} \sim \int_M \mathrm{vol}_\psi \sim \int_M \varphi \wedge *\varphi$$

Moment map

$$\mu(\mathrm{e}^{\mathrm{i}\varphi} \cdot V) \sim \int_M \mathrm{d}\omega \wedge *\varphi$$

Generalisations

Including fluxes

$$L_{+3} = \mathrm{e}^{A_3 + \tilde{A}_6} \mathrm{e}^{\mathrm{i}\varphi} T_{\mathbb{C}} \qquad \psi = \mathrm{e}^{A_3 + \tilde{A}_6} \mathrm{e}^{\mathrm{i}\varphi} \cdot 1$$

Involutive $\Leftrightarrow \mathrm{d}\varphi = F_4 = \tilde{F}_7 = 0$ $\qquad$ $\mathcal{K}$ and $\mu(V)$ as for zero-flux

Type-0

Require $\dim \pi(L_{+3}) = 7$

$$L_{+3} = \mathrm{e}^{\alpha+\beta} T_{\mathbb{C}} \qquad \psi = \mathrm{e}^{\alpha+\beta} \cdot 1$$

Similar treatment. E.g. Involutive for $\mathrm{d}\alpha = \mathrm{d}\beta = 0$

Example 2: GMPT solutions [Grana, Minasian, Petrini & Tomasiello '04]

$\mathcal{N}=1$ solutions of Type IIA/B

Have $SU(3)\times SU(3)\subset O(6,6)\subset E_{7(7)}$ structure (Φ^+,Φ^-) s.t.

$$\mathrm{d}\Phi_\pm = 0 \qquad F=-8\mathrm{d}^{\mathcal{J}_\pm}(\mathrm{e}^{-3\Delta}\mathrm{Im}\Phi_\mp)$$

$$\mathrm{d}(\mathrm{e}^{-\Delta}\mathrm{Re}\Phi_\mp)=0$$

[Tomasiello '07]

Involutivity

$$L_{+3}=\mathrm{e}^{C+8\mathrm{i}\mathrm{e}^{-3\Delta}\mathrm{Im}\Phi_\mp}(L_{+1}^{\mathcal{J}_\pm}\oplus\mathcal{U}_{\mathcal{J}_\pm})$$

Involutive $\Leftrightarrow$ $\mathrm{d}\Phi_\pm = \not{A}\Phi_\pm$ and $F=-8\mathrm{d}^{\mathcal{J}_\pm}(\mathrm{e}^{-3\Delta}\mathrm{Im}\Phi_\mp)$

Moment map condition

$$A=0 \quad \text{and} \quad \mathrm{d}(\mathrm{e}^{-\Delta}\mathrm{Re}\Phi_\mp)=0$$

Deformations of J

Decomposition of the adjoint

The adjoint of $E_{7(7)}$ decomposes under $SU(7) \times U(1)$ as

$$\mathbf{133} = \mathbf{1_0} + \mathbf{48_0} + (\mathbf{35_{+2}} + \mathbf{\bar{7}_{+4}}) + (\mathbf{\bar{35}_{-2}} + \mathbf{7_{-4}})$$

Deform $L'_{+3} = (1 + A)L_{+3}$ $\qquad A \in \mathcal{Q} \sim \mathbf{\bar{35}_{-2}} + \mathbf{7_{-4}}$

Involutivity and cohomology

Imposing $[L_{+3}, L_{+3}] \subset L_{+3}$ gives differential condition

$$\mathrm{d}_2 A = 0 \qquad \mathrm{d}_2 : \mathcal{Q} \to W^{\text{int}}_{U(7)\times\mathbb{R}^+}$$

Trivial deformations are $L'_{+3} = (1 + L_V)L_{+3}$ $\qquad V \in E_{\mathbb{C}}$

$$\text{I.e. } A = \mathrm{d}_1 V \qquad \mathrm{d}_1 : E_{\mathbb{C}} \to \mathcal{Q}$$

Cohomology of $E_{\mathbb{C}} \xrightarrow{\mathrm{d}_1} \mathcal{Q} \xrightarrow{\mathrm{d}_2} W^{\text{int}}_{U(7)\times\mathbb{R}^+}$ $\qquad (\mathrm{d}_2 \cdot \mathrm{d}_1 = 0)$

The neat thing about $\mathcal{N} = 1$

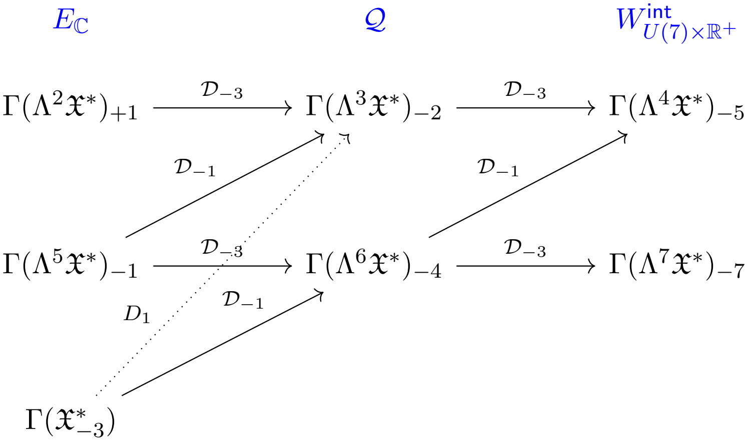
Complexified quotient should implement the moment map!

(Used $E_{\mathbb{C}} \to$ generates $\mathrm{GDiff}_{\mathbb{C}}$)

So just need to compute this cohomology to get moduli space of ψ

General problem seems hard!

Decomposing $\mathfrak{X} \sim (\mathbf{7})_{SU(7)}$ and $\mathcal{D} = \mathcal{D}_{+3} + \mathcal{D}_{+1} + \mathcal{D}_{-1} + \mathcal{D}_{-3}$



G_2 case easier

Neat parameterisations:

Just twist $GL(7,\mathbb{R})$ bundles by $\mathrm{e}^{\mathrm{i}\phi}$

$$L_{+3} = \mathrm{e}^{\mathrm{i}\varphi} T_{\mathbb{C}} \simeq T_{\mathbb{C}} \qquad E_{\mathbb{C}}/L_{+3} \simeq \Lambda^2 T^*_{\mathbb{C}} \oplus \Lambda^5 T^*_{\mathbb{C}} \oplus (T^* \otimes \Lambda^7 T^*)_{\mathbb{C}}$$

$$\mathcal{Q} \simeq \Lambda^3 T^*_{\mathbb{C}} \oplus \Lambda^6 T^*_{\mathbb{C}} \qquad W^{\mathsf{int}} \simeq \Lambda^4 T^*_{\mathbb{C}} \oplus \Lambda^7 T^*_{\mathbb{C}}$$

G_2 moduli

Infinitesimal deformations

$$L'_{+3} = (1 + \alpha + \beta) L_{+3} \qquad \alpha + \beta \in \Lambda^3 T^* + \Lambda^6 T^*$$

$$\text{Involutive} \quad \Leftrightarrow \quad \mathrm{d}\alpha = \mathrm{d}\beta = 0$$

Trivial deformations

$$L'_{+3} = (1 + L_V) L_{+3} = (1 + \mathrm{d}\tilde{\omega} + \mathrm{d}\tilde{\sigma}) L_{+3} \qquad \tilde{\omega} + \tilde{\sigma} \in \Lambda^2 T^*_{\mathbb{C}} \oplus \Lambda^5 T^*_{\mathbb{C}}$$

Cohomology: (usual de Rham groups)

$$H^3(M, \mathbb{C}) \oplus H^6(M, \mathbb{C}) \qquad (G_2 \Rightarrow H^6(M, \mathbb{C}) = 0.)$$

This also works for general type-0 case $L_{+3} = \mathrm{e}^{\alpha + \beta} T_{\mathbb{C}}$

GMPT moduli

Again $\exists$ nice (non-J-eigenspace) parameterisation of deformations

Assuming an analogue of the $\partial\bar{\partial}$-lemma, can show that moduli are

$$H^2_{\mathrm{d}_L}(M) \oplus H^0_{\bar{\partial}}(M) \oplus H^{-2}_{\bar{\partial}}(M) \oplus H^6(M, \mathbb{C})$$

Doesn't include deformations of sources/orientifolds.

The superpotential

Inspiration: AdS

$\mathcal{N} = 1$ AdS vacuum $\Leftrightarrow$ constant $SU(7)$ singlet intrinsic torsion Λ

[Coimbra & CS-C '15]

As $\mathcal{W} \sim \Lambda$ for AdS vacuum, guess:

Superpotential functional $\mathcal{W} \sim \int_M \Big(SU(7)$ singlet intrinsic torsion$\Big)$

Explicitly:

$$\mathcal{W} \sim \int_M \mathrm{tr}(J \cdot (D \times_{\mathrm{ad}} \psi))$$

(Note: This is well defined even though $D \times_{\mathrm{ad}} \psi$ is not!)

The superpotential

Is this holomorphic in ψ? (Naively not as J depends on $\bar{\psi}$)

$$\mathcal{W} \sim \int_M \frac{s(\bar{\psi}, (D \times_{\mathrm{ad}} \psi) \cdot \psi)}{\mathrm{i}s(\bar{\psi}, \psi)}$$

As $(D \times_{\mathrm{ad}} \psi) \cdot \psi \propto \psi$, [Pacheco & Waldram '07]

only scaling transformation $\bar{\psi} \to \bar{\lambda}\bar{\psi}$ appears.

$\mathcal{W}$ is clearly invariant under this scaling!

Volume form to integrate only if weight $+3$ i.e. $\psi \in \mathbf{912}_{+3}$

Examples

G_2 manifold

$$\mathcal{W} \sim \int_M \left(\tilde{F} + \tfrac{1}{2}\varphi \wedge \mathrm{d}\varphi - \mathrm{i}F \wedge \varphi \right)$$

[Acharya & Spence '00; Beasley & Witten '02; House & Micu '05; Lambert '05]

GMPT

$$\mathcal{W} \sim \int_M \left(\Phi_-, F + 8\mathrm{i}\mathrm{d}(\mathrm{e}^{-3\Delta}\mathrm{Im}\Phi_+) \right)$$

[Grana, Louis & Waldram; Benamachiche & Grimm; Koerber & Martucci]

F-terms vs involutivity

F-terms

$\delta\psi$ depends on parameters in $\mathbf{1}, \mathbf{7}$ and $\mathbf{\bar{35}}$

$$\therefore \frac{\delta\mathcal{W}}{\delta\psi} = 0 \quad \text{constrains } \bar{\mathbf{1}}, \bar{\mathbf{7}} \text{ and } \mathbf{35} \quad (\text{ and } \Rightarrow \mathcal{W} = 0)$$

Involutivity was only $\bar{\mathbf{1}}$ and $\mathbf{35}$ so not all the F-terms

Respectively have ($\forall\, V \in L_{+3} = s(\psi, \mathrm{ad}\,\cdot)$)

Involutivity $\Leftrightarrow L_V\psi = A(V)\psi$ F-terms $\Leftrightarrow L_V\psi = 0$

Again, F-term condition holomorphic in $\psi \Rightarrow$ weight $+3$

(c.f. $\mathcal{L}_v\Omega = 0 \quad \forall v \in T^{0,1}$)

D-terms

- D-terms are the rest of the moment map conditions $\mu = 0$
- Here they are implemented via complexified quotient
- SUSY moduli space $\sim$ "moduli space of F-terms" (Really J)

Moment map as extremisation of $\mathcal{K}$

Moment map condition revisited

Define infinitesimal action of $\mathrm{GDiff}_{\mathbb{C}}$ on $\mathcal{Z}$ via

$$\text{Action of } \mathcal{I} \cdot \rho_V \qquad V \in \Gamma(E_{\mathbb{C}})$$

where $\mathcal{I}$ is the complex structure on $\mathcal{Z}$. We find

$$\mu(V) = -\tfrac{1}{2} i_{\mathcal{I} \cdot \rho_V} \delta\mathcal{K}$$

Therefore

$$\boxed{\mu = 0 \quad \Leftrightarrow \quad \text{critical point of } \mathcal{K} \text{ under } \mathrm{GDiff}_{\mathbb{C}} \text{ action}}$$

This $\Rightarrow$ Hitchin's variational principle for G_2 manifolds !

Hitchin's variational principle for G_2

Hitchin's statement

Say have closed G_2 structure $\mathrm{d}\varphi = 0$.

Vary within its class in $H^3(M,\mathbb{R})$, i.e. $\varphi' = \varphi + \mathrm{d}\beta$

$$\mathcal{H} = \int_M \mathrm{vol}_\varphi \qquad \text{then} \quad \delta\mathcal{H} = 0 \Leftrightarrow \mathrm{d} * \varphi = 0$$

G_2 holonomy $\Leftrightarrow$ critical point of $\mathcal{H}$ within $[\varphi] \in H^3(M,\mathbb{R})$

$E_{7(7)} \times \mathbb{R}^+$ version

Under infinitesimal imaginary GDiff on $\psi = \mathrm{e}^{A+\tilde{A}}\mathrm{e}^{\mathrm{i}\varphi}1$ have

$$\begin{aligned} i_{\mathcal{I}\cdot\rho_V}\delta\psi = \mathrm{i}L_V\psi &= -\mathrm{i}(\mathrm{d}\omega' + \mathrm{d}\sigma')\cdot\psi + (\text{"real diff"})\cdot\psi \\ &= \mathrm{e}^{A+\tilde{A}}\mathrm{e}^{\mathrm{i}(\varphi+\mathrm{d}\omega')}(1 + j\mathrm{d}\sigma' + \dots) \end{aligned}$$

I.e. Shift φ within $[\varphi]$ ($j\mathrm{d}\sigma$ contribution cancels from $\delta\mathcal{K}$)

Conclusion

- $\mathcal{N} = 1$ Minkowski backgrounds

  $\leftrightarrow$ Involutive subbundle $L \subset E$ and moment map $\mu = 0$
- Moduli space given by $\hat{\mathcal{Z}}/\mathrm{GDiff}_{\mathbb{C}}$ and further quotient by $\mathbb{C}^*$
- Possible to calculate in examples with "nice" parameterisation of deformations of the structures
- Found full moduli space of GMPT solutions
- $E_{7(7)}$ Kähler potential and superpotential
- Moment map extremises $\mathcal{K}$ over $\mathrm{GDiff}_{\mathbb{C}}$

Comments

- Really complexified (GIT) quotient is of "polystable points"

  $\rightarrow$ Notion of stability for G_2 structures?

  $\rightarrow$ Yau-type theorem for existence of G_2 metric?

- Hull-Strominger system via same treatment

  Gen. geom $\Rightarrow$ holomorphic Courant algebroid approach

  Also $\Rightarrow$ moment map $\leftrightarrow$ extremisation of volume

  [Garcia-Fernandez, Rubio, Shahbazi & Tipler]

- AdS?

- $\mathcal{N} \geq 2$?

- Higher derivative corrections?

The End

Thanks for your attention!