Modular Properties of Superstring Amplitudes and Holography

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0. Higher-derivative interactions in closed superstring theory

Non-Perturbative features – S-Duality in superstring theory:

Older work with Pierre Vanhove, Sav Sethi, Michael Gutperle, Anirban Basu, ….

I. SL(2,Z) modular forms and U(1)-violation in IIB superstring

with Congkao Wen ArXiv: 1904.13394

First-order differential relations between coefficients in low energy expansion, which imply Laplace eigenvalue equations for low order terms.
Modular forms for coefficients of n-point Maximal U(1)-Violating Interactions

Predicts precise perturbative and non-perturbative (D-instanton) terms.

II. Motivation : Holographic connection of Type IIB superstring amplitudes with correlation functions of N=4 SUSY Yang-Mills

with Shai Chester, Silviu Pufu Yifan Wang, Congkao Wen (to appear)

Montonen-Olive SL(2,Z) duality of N=4 SUSY Yang-Mills ⟺ **SL(2,Z) S-duality of type IIB superstring**

The Low Energy Expansion of String Theory

- Lowest order term reproduces the results of classical supergravity

Einstein-Hilbert

$\alpha' = \ell_s^2$

ℓ_s - string length scale

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \ldots$$ Interactions of other supergravity fields

Metric – $G_{\mu\nu}$

Scalar field - dilaton

$e^{-\phi} = \frac{1}{g}$ ← string coupling constant

compactify space-time to dimensions D < 10

- Holographic Dictionary: $N^2\int d^{10}x\sqrt{-\det G}\, R + \ldots$

$\frac{\alpha'^2}{L^4} = \frac{1}{gN}, \quad g = \frac{1}{4\pi} g_{YM}^2$

Coefficient depends on moduli (scalar fields).
- constrained by **S-Duality**

- Higher order terms: $\frac{1}{\alpha'}\int d^{10}x\sqrt{-\det G}\, g^{-\frac{1}{2}}\,\mathcal{F}(\phi,\ldots)\, R^4$

(maximal supersymmetry)

Transformation to string frame

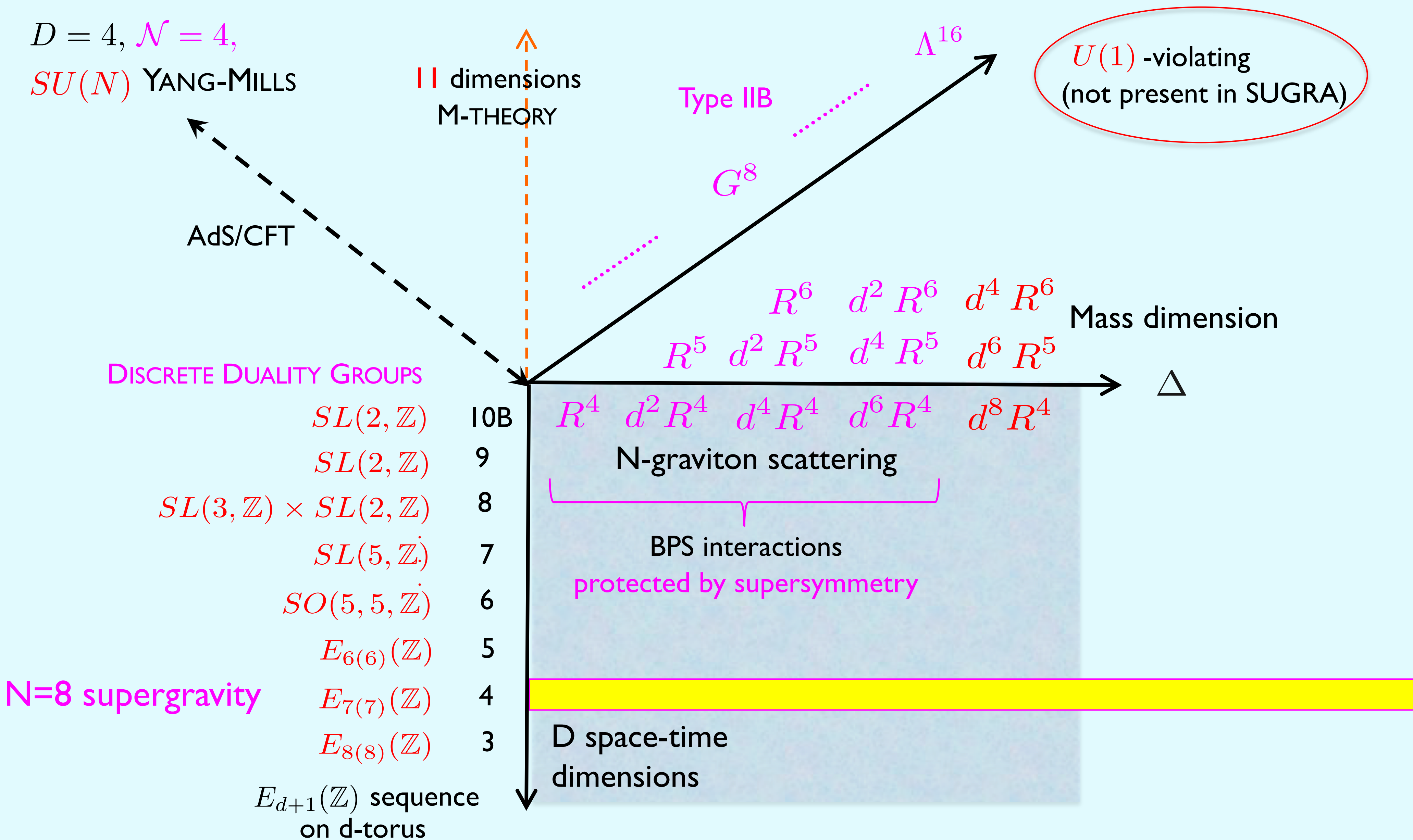
- More Generally: $(\alpha')^{n-1}\int d^{10}x\sqrt{-\det G}\, g^{-\frac{1}{2}+\frac{n}{2}}\,\mathcal{F}(\phi,\ldots)\, d^{2n}R^4$ (or $R^{4+n}\,, F_5^{2n}R^4\,, \ldots$)

Translates into $N^{\frac{1}{2}-\frac{n}{2}}\int d^{10}x\sqrt{-\det G}\,\mathcal{F}(\phi,\ldots)\, d^{2n}R^4$

(fixed α' and g - i.e $N \to \infty$ and fixed g_{YM})

THE LOW ENERGY EXPANSION OF (TYPE IIB) STRING THEORY

HIGHER DERIVATIVE CORRECTIONS to Einstein theory



Ten-dimensional Type IIB - maximal supersymmetry

32 supersymmetries

One complex modulus $\tau = \tau_1 + i\tau_2$ inverse string coupling constant $\tau_2 = \frac{1}{g} = e^{-\phi}$

S-Duality group $SL(2,\mathbb{Z})$ $\tau \to \frac{a\tau + b}{c\tau + d}$ $a,b,c,d \in \mathbb{Z}$, $ad - bc = 1$

How powerful are the constraints imposed by (maximal) SUSY and Duality ??

Investigate the exact moduli dependence of low lying terms in the low energy expansion.

Duality relates different regions of moduli space –

Connects perturbative and non-perturbative features in a highly nontrivial manner.

$SL(2,\mathbb{R})$ Transformations of massless Type IIB fields

- Coset space $SL(2,\mathbb{R})/U(1)$ $\quad U(1)$ gauge symmetry $\quad$ paramterise coset by complex scalar $\tau = \tau_1 + i\tau_2$
- Fix $U(1)$ gauge - embed the $U(1)$ in $SL(2,\mathbb{R})$.
- A $SL(2,\mathbb{R})$ transformation induces a compensating $U(1)$ transformation to preserve gauge condition.

$$e^{2i\phi} = \left(\frac{c\tau + d}{c\bar\tau + d}\right)$$

$U(1)$ charges of fields $\Phi \quad = q_\Phi$

Scalar bosons $\quad P_\mu = i\frac{\partial_\mu \tau}{2\tau_2} \quad q_P = -2 \qquad \bar P_\mu = -i\frac{\partial_\mu \bar\tau}{2\tau_2} \quad q_{\bar P} = 2$

Antisymmetric tensors $\quad G \quad q_G = -1 \qquad G \quad q_{\bar G} = 1$

Complex combinations of RR and NSNS three-form field strengths

Neutral bosons $\quad dC^{(4)}\,,\ R \quad q_{\mathcal{F}_5} = q_R = 0$

Self-dual five-form and curvature

Fermions $\quad \Lambda \quad q_\Lambda = -\frac{3}{2} \qquad \bar\Lambda \quad q_{\bar\Lambda} = \frac{3}{2} \qquad \psi_\mu \quad q_\psi = -\frac{1}{2} \qquad \bar\psi_\mu \quad q_{\bar\psi} = \frac{1}{2}$

Complex Dilatini $\qquad$ Complex Gravitini

Note: Chiral U(1) Anomaly in Type IIB Supergravity in D=10 Dimensions

breaks $SL(2,\mathbb{R})$ to $SL(2,\mathbb{Z})$. $\qquad$ **Moduli space** $\quad SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})/U(1)$

$SL(2,\mathbb{R})$ **not a symmetry in string theory**

Systematics of U(1) violation

Consider a *linearised* constrained SCALAR CHIRAL ON-SHELL SUPERFIELD describing fluctuations around $\tau = \tau_0$. Function of a single 16-component Grassman spinor, θ.

(Howe, West)

$$\Phi(x,\theta) = \delta\tau + \theta\,\Lambda + \theta^2\,G + \theta^3\,d\psi \;+ \theta^4\,(R^4 + dF_5) + \theta^5\,d^2\psi^* + \theta^6\,d^2\bar{G} + \theta^7\,d^3\Lambda^* + \theta^8\,d^4\bar{\tau}$$

U(1)-charge of superfield $= -2$ U(1)-charge of $\theta = -\frac{1}{2}$

Linearised action $$\int d^{16}\theta\, F[\tau_0 + \Phi(x,\theta)] \;= \int d^{16}\theta \sum_n \frac{\partial^n F(\tau_0)}{\partial\tau_0^n}\,[\Phi(x,\theta)]^n$$

$$= F_{(4)}(\tau_0)\,R^4 + F_{(5)}(\tau_0)\,G^2\,R^3 + \cdots + F_{(16)}(\tau_0)\,\Lambda^{16} \qquad F_{(n)}(\tau_0) = \frac{\partial^n F(\tau_0)}{\partial\tau_0^n}$$

- U(1) VIOLATION FOR N-POINT FUNCTIONS: $q = -2(n-4)$ All four-point functions conserve U(1)

  Maximal U(1) violation in n-particle amplitude

- These 8-derivative interactions are $\frac{1}{2} - BPS$

  $\frac{1}{4} - BPS$ $\frac{1}{8} - BPS$

- More generally consider derivatives on these interactions - e.g. d^4R^4, d^6R^4

- Note for example that $F_{\Lambda^{16}} = \dfrac{\partial^{12}F_{R^4}}{\partial\tau_0^{12}}$

Higher Derivative SL(2,Z)-Covariant Action

The linearised interactions fit into a $SL(2,\mathbb{Z})$ - invariant action of the form

$\kappa = (\alpha')^2\, g$

$$S_n^p = (\kappa)^{\frac{p-1}{2}} \int d^{10}x\, e\, F^{(p)}_{w\,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\})$$

Monomial in n fields

$q = -2(n-4)$
$= -2w$

Degeneracy in kinematic factors

R^4	$p = 0$
$d^4\, R^4$	$p = 2$
$d^6\, R^4$	$p = 3$

- Derivatives $d^{2p}_{(i)}$ (contractions suppressed) explicit in amplitude calculations

  e.g. for $n = 4,\ p = 3$ $\ d^6\, R^4 \sim (s^3 + t^3 + u^3)\, R^4$

- **Degeneracy** first arises for n=4, p=6; n=5, p=4; **n=6, p=3** e.g. $d^6\, G^4 R^2$

- The quantity $\mathcal{P}_n(\{\Phi\})$ is the product of n fields in linearised approximation with $q = -2(n-4)$

- Since $\mathcal{P}_n(\{\Phi\})$ carries a non-zero U(1) charge, the moduli-dependent coefficient $F^{(p)}_{w\,i}(\tau)$ must transform with a compensating charge.

Non-holomorphic modular form — modular weight w

- The complete nonlinear action is not known - even in the p=0 case (1/2-BPS).

  although it is known for backgrounds in which certain bosonic fields vanish

Non-Holomorphic Modular Forms

Consider a $SL(2,\mathbb{Z})$ transformation $\tau \to \dfrac{a\tau + b}{c\tau + d}$ $\quad a, b, c, d \in \mathbb{Z} \quad ad - bc = 1$

A non-holomorphic modular form with weight (w, w') transforms as

(holomorphic → w, anti-holomorphic → w')

$$f^{(w,w')}(\tau) \to (c\tau + d)^w \, (c\bar{\tau} + d)^{w'} \, f^{(w,w')}(\tau)$$

So if $w' = -w$

$$f^{(w,-w)}(\tau) \to \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^w f^{(w,-w)}(\tau)$$

Transforms with phase – U(1) charge $q = 2w$

$e^{2iw\phi} \qquad \phi = \dfrac{i}{2} \log\left(\dfrac{c\bar{\tau} + d)}{c\tau + d}\right)$

Covariant derivatives:

$$\mathcal{D}_w = i\tau_2 \frac{\partial}{\partial \tau} + \frac{w}{2} \qquad \bar{D}_{w'} = -i\tau_2 \frac{\partial}{\partial \bar{\tau}} + \frac{w'}{2}$$

$$\mathcal{D}_w \, f^{(w,-w)} = f^{(w+1,-w-1)}$$

Increases the U(1) charge

$$\bar{\mathcal{D}}_w \, f^{(w,-w)} = f^{(w-1,-w+1)}$$

Decreases the U(1) charge

Non-holomorphic Eisenstein series

$$E(s,\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

- $SL(2,\mathbb{Z})$ invariant (generalises to higher rank duality groups) - weight $(0,0)$ form
- Solution of Laplace eigenvalue eqn.

$$\left(\Delta - s(s-1)\right) E(s,\tau) = 0 \qquad \Delta = \tau_2^2 \left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right) = 4\partial_\tau \partial_{\bar\tau}$$

- Fourier series $\quad E(s,\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\tau_2)\, \cos(2\pi i k\tau_1)$
- Zero mode $k=0$ - TWO POWER-BEHAVED TERMS (perturbative) :

$$\mathcal{F}_0 = 2\zeta(2s)\,\tau_2^s + \frac{2\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\,\zeta(2s-1)}{\Gamma(s)}\,\tau_2^{1-s}$$

- Non-zero modes $k>0$ - D-INSTANTON SUM

K Bessel

$$\mathcal{F}_k = \frac{4\pi^s}{\Gamma(s)}\,|k|^{s-\frac{1}{2}}\sigma_{1-2s}(|k|)\,\tau_2^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi||k|\tau_2)$$

$$\sim \frac{2\pi^s}{\Gamma(s)}\,|k|^{s-1}\sigma_{1-2s}(|k|)\,(1+O(\tau_2^{-1}))$$

divisor sum

$$\sigma_n(k) = \sum_{p|k} p^n$$

Low order interaction coefficients

for U(1)-conserving four-point amplitudes - e.g. four gravitons

Laplace equations motivated by supersymmetry and various dualities

$(\alpha')^{-1}\, R^4$

$\frac{1}{2} - BPS$

$$\left(\Delta - \frac{3}{4}\right) F_0^{(0)}(\tau) = 0$$

solution $F_0^{(0)}(\tau) = E(\frac{3}{2}, \tau)$

no. of derivatives

weight w

Contains tree-level and genus-one together with D-instantons

Non-renormalisation beyond 1 loop for R^4

$\alpha'\, d^4\, R^4$

$\frac{1}{4} - BPS$

$$\left(\Delta - \frac{15}{4}\right) F_0^{(2)}(\tau) = 0 \qquad F_0^{(2)}(\tau) = E(\frac{5}{2}, \tau)$$

Contains tree-level and genus-two together with D-instantons

Non-renormalisation beyond 2 loops for $d^4 R^4$

- Similarly for other dimension-12 interactions $\alpha' d^2 R^5,\ \alpha' R^6,\ F_5^4 R^4, \ldots$

$AdS_5 \times S^5$

- $\frac{1}{N}$ correction to R^4

$$\frac{\alpha'}{L^2} g^{\frac{1}{2}}\, R^4 \sim N^{-\frac{1}{2}}\, R^4$$

$(\alpha')^2\, d^6\, R^4$
$\frac{1}{8} - BPS$

$$F_0^{(3)}(\tau) = \mathcal{E}_0^{(3)}(\tau)$$

NOT Eisenstein series but satisfies INHOMOGENEOUS Laplace equation

$$(\Delta - 12)\, \mathcal{E}_0^{(3)}(\tau) = -E(\tfrac{3}{2},\tau)E(\tfrac{3}{2},\tau) \longleftarrow$$

The square of the coefficient of R^4

THE SOLUTION OF THIS EQUATION HAS SOME WEIRD AND WONDERFUL FEATURES.

ZERO MODE OF SOLUTION (zero net D-instanton number):

$$g\, \mathcal{E}_0^{(3)}(\tau)\big|_{\substack{zero\\mode}} = \frac{2}{3}\zeta(3)^2\, g^{-2} + \frac{4}{3}\zeta(2)\zeta(3)\, g^0 + 4\zeta(4)\, g^2 + \frac{4}{27}\zeta(6)\, g^4 + \sum_k c_k\, e^{-\frac{4\pi k}{g}}(1 + O(g))$$

GENUS ZERO — GENUS ONE — GENUS TWO — GENUS THREE — SUM OF D-INSTANTONS

NON-RENORMALISATION BEYOND 3 LOOPS

PRECISE AGREEMENT WITH EXPLICIT PERTURBATIVE STRING THEORY MULTI-LOOP CALCULATIONS

[PARENTHETICAL COMMENT: THE NON-RENORMALISATION STATEMENTS IN MAXIMAL SUPERGRAVITY ARE IN AGREEMENT WITH THESE STRING THEORY RESULTS.]

First-Order Equations for U(1)-Violating Coefficients

The coefficient of a term violating the U(1) charge by $q = -2(n-4) = -2w$ units is given by

$$F^{(0)}_{n-4}(\tau) = c^{(0)}_n \, E_w(\tfrac{3}{2},\tau) \qquad E_0(s,\tau) \equiv E(s,\tau)$$

Non-holomorphic Eisenstein modular forms

Eisenstein series with holomorphic/anti-holomorphic weights $(w,-w)$ defined by

First-order equations $$E_{w+1}(s,\tau) = \frac{s+w}{2}\,\mathcal{D}_w E_w(\tau)$$ (arbitrary normalisation)

so $$E_w(s,\tau) = \frac{2^w (s-1)!}{s+w-1)!}\,\mathcal{D}_{w-1}\ldots\mathcal{D}_0\,E_0(s,\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{m+n\bar\tau}{m+n\tau}\right)^w \frac{\tau_2^s}{|m+n\tau|^{2s}}$$

Likewise, $$E_{w-1}(s,\tau) = \frac{s-w}{2}\,\bar{\mathcal{D}}_w E_w(\tau)$$

Laplace operators $$\Delta^w_+ = 4\,\bar{\mathcal{D}}_{w+1}\mathcal{D}_w \qquad \Delta^w_- = 4\,\mathcal{D}_{w-1}\bar{\mathcal{D}}_w \qquad \Delta^w_+ - \Delta^w_- = -2w$$

Laplace equations $$\Delta_- E_w = (s(s-1) - w(w-1))\,E_w$$

e,g, $$E_w(\tfrac{3}{2},\tau) = 2\zeta(3)\,\tau_2^{\frac{3}{2}} + \frac{4\zeta(2)}{1-4w^2}\,\tau_2^{-\frac{1}{2}} + \sum_{K=1}^{\infty}\left(C_{K,w}(\tau_2)e^{2\pi i K\tau_1} + C_{K,-w}(\tau_2)e^{-2\pi i K\tau_1}\right)$$

$\tau_2 = 1/g$ tree-level genus-one D-instantons anti-D-instantons

½-BPS and ¼-BPS U(1)-Violating Coefficients

Supersymmetry together with S-duality:

$n = 4 - w$	4	5	6		8	12
$(\alpha')^{-1}:$	R^4	$G^2 R^3$	$G^4 R^2$	$\dots$	G^8	$\dots \Lambda^{16}$

$$F^{(0)}_{n-4}(\tau) = c^{(0)}_n \, E_w(\tfrac{3}{2}, \tau)$$

$$d^4 \sim s^2 + t^2 + u^2 = \sum_{i<j} s_{ij}^2$$

$$\alpha' : \quad d^4 R^4 \quad d^4 G^2 R^3 \quad d^4 G^4 R^2 \ \dots \ d^4 G^8 \ \dots \ d^4 \Lambda^{16}$$

$$F^{(0)}_{n-4}(\tau) = c^{(2)}_n \, E_w(\tfrac{5}{2}, \tau)$$

- Satisfy sequence of Laplace eigenvalue equations.
- Coefficients determined by amplitude analysis

1/8-BPS U(1)-VIOLATING COEFFICIENTS

$n = 4+w$	4	5	6	8	12
$(\alpha')^2$:	$d^6_{(i)} R^4$	$d^6_{(i)} G^2 R^3$	$d^6_{(i)} G^4 R^2$ …	$d^6_{(i)} G^8$ …	$d^6_{(i)} \Lambda^{16}$

$$F^{(3)}_{n-4,i}(\tau) = c^{(3)}_{n,i}\, \mathcal{E}^{(3)}_w(\tau)$$

$n = 4+w$

TWO INDEPENDENT KINEMATIC STRUCTURES

$n=6$
$w=2$

$$d^6_{(1)} \sim \sum_{i<j} s^3_{ij} + \frac{3}{8} \sum_{i<j<k} s^3_{ijk}$$

Tree-level contribution

$$d^6_{(2)} \sim \sum_{i<j} s^3_{ij} - \frac{1}{2} \sum_{i<j<k} s^3_{ijk}$$

Does not contribute at tree-level

RECALL W = 0 CASE:

$$4\,\bar{\mathcal{D}}\mathcal{D}\,\mathcal{E}^{(3)}_0 = 12\mathcal{E}^{(3)}_0 - (E_0(\tfrac{3}{2}))^2$$

$\Delta = \Delta^0_+ = \Delta^0_-$

CONSIDER W = 1 CASE:

Define: $\mathcal{E}^{(3)}_1 = 2\mathcal{D}\mathcal{E}^{(3)}_0$ **FIRST-ORDER EQUATION**

Apply $\mathcal{D}$ to $w=0$ equation, $4\,\mathcal{D}\bar{\mathcal{D}}(\mathcal{D}\,\mathcal{E}^{(3)}_0) = 12(\mathcal{D}\mathcal{E}^{(3)}_0) - \mathcal{D}(E_0(\tfrac{3}{2}))^2$

$\longrightarrow$ $$\Delta_-\mathcal{E}^{(3)}_1 = 12\mathcal{E}^{(3)}_1 - 3\,E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})$$ $w=1$ **LAPLACE EQUATION**

Applying $\bar{\mathcal{D}}$ and requiring consistency with w=0 Laplace equation leads to

$\longrightarrow$ $$\bar{\mathcal{D}}\mathcal{E}^{(3)}_1 = \mathcal{E}^{(3)}_0 - \frac{1}{12}(E_0(\tfrac{3}{2}))^2$$

INHOMOGENEOUS
FIRST-ORDER EQUATION

1/8-BPS Coefficients - the w=2, n=6 Case

$i=1$ $\qquad \mathcal{E}^{(3)}_{2,i}\, d^6_{(i)}\,(G^4R^2+\Lambda^8R^2+\dots)$

$i=1.2$ Labels distinct kinematic structures – motivated by amplitude analysis

The factor $\mathcal{E}^{(3)}_{2,1}(\tau)\, d^6_{(1)}$ contains the tree-level contribution

Define: $\mathcal{E}^{(3)}_{2,1}=2\mathcal{D}\mathcal{E}^{(3)}_{1}$ Then consistency with $\mathcal{E}^{(3)}_{1}$ equation

Leads to **Laplace equation**

$$\Delta_-\mathcal{E}^{(3)}_{2,1}=10\mathcal{E}^{(3)}_{2,1}-\frac{15}{2}\left(E_0(\tfrac{3}{2})\,E_2(\tfrac{3}{2})+\frac{3}{5}E_1(\tfrac{3}{2})\,E_1(\tfrac{3}{2})\right)$$

and **first-order equation**

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,1}=5\mathcal{D}\mathcal{E}^{(3)}_{1}-\frac{3}{2}E_1(\tfrac{3}{2})\,E_0(\tfrac{3}{2})$$

$i=2$

The factor $\mathcal{E}^{(3)}_{2,2}(\tau)\, d^6_{(2)}$ does **not** have a tree-level contribution

first-order equation

$$\bar{\mathcal{D}}\mathcal{E}^{(3)}_{2,2}(\tau^0)=a\left(\mathcal{E}^{(3)}_{1}(\tau^0)-\frac{1}{2}E_0(\tfrac{3}{2},\tau^0)\,E_1(\tfrac{3}{2},\tau^0)\right)$$

Tree-level term cancels in this combination Leading term from one-loop contribution.

Fix the constant by one-loop calculation.

$$\mathcal{E}^{(3)}_{2,2}=\frac{a}{5}(\mathcal{E}^{(3)}_{2,1}-2E_1(\tfrac{3}{2})\,E_1(\tfrac{3}{2}))$$

Laplace equation

$$\Delta_-\mathcal{E}^{(3)}_{2,2}=10\mathcal{E}^{(3)}_{2,2}-\frac{5a}{12}\left(E_0(\tfrac{3}{2})E_2(\tfrac{3}{2})-E_1(\tfrac{3}{2})E_1(\tfrac{3}{2})\right)\,.$$

1/8-BPS Coefficients - the w>2, n>6 Cases

The extension to all terms of the form $\mathcal{E}^{(3)}_{2,i}(\tau)\, d^6_{(i)}\,\mathcal{P}_n(\{\Phi\})$

Superstring Scattering Amplitudes (very sketchy)

(MBG, Wen Arxiv: 1904.13394

Recall $$S_n^p = (\kappa)^{\frac{p-1}{2}} \int d^{10}x\, e\, F_{w\,i}^{(p)}(\tau)\, d_{(i)}^{2p}\, \mathcal{P}_n(\{\Phi\})$$

(i) Amplitudes with external Φ from $\mathcal{P}_n(\Phi)$ Background $\tau = \tau_0$

(ii) Amplitudes with fluctuations $\delta\tau = \tau - \tau_0$ not covariant

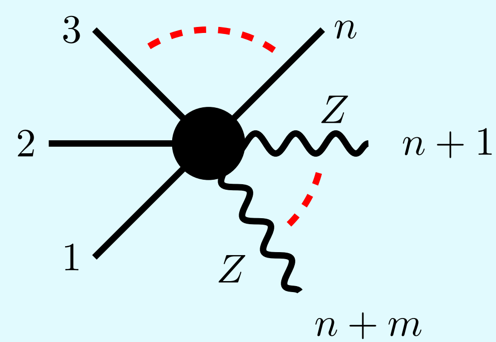
Redefine coordinate $\tau \to Z$ $\quad Z = \dfrac{\tau - \tau^0}{\tau - \bar{\tau}^0}$ Transforms with U(1) charge = -2 under $SL(2,\mathbb{Z})$

$$F_0(\tau) = \sum_{w=0}^{\infty} 2^w\, \mathcal{D}_{w-1} \ldots \mathcal{D}_0\, F_0(\tau)\Big|_{\tau=\tau^0} Z^w/w! + \cdots$$

Weight - w amplitude ~ covariant derivative on weight - $(w-1)$

Amplitudes with m Z s and n Φ s

$\mathcal{D}^m F_{n-4,i}^{(p)}(\tau^0)$



These are "**maximally U(1)-violating**" amplitudes – determined by contact interactions

No poles

Soft Limits

Function of Mandelsatam invarian

$$A_n(X,\mathbb{Z}_n)\big|_{p_n\to 0} = 2\,\mathcal{D}A_{n-1}(X)\,,$$

More explicitly

$$F^{(p)}_{n-4}(\tau^0)\,\mathcal{O}^{(p)}_{n,i}\big|_{p_n\to 0} = 2\,\mathcal{D}F^{(p)}_{n-5}(\tau)\big|_{\tau=\tau^0}\mathcal{O}^{(p)}_{n-1,i}$$

Low energy expansion

e.g. relates A_{ggggZ} to A_{gggg}

$n \le 5 \quad p = 2,\, 3$

$$\mathcal{O}^{(2)}_n = \frac{1}{2}\sum_{1\le<j\le n} s^2_{ij} \qquad \mathcal{O}^{(3)}_n = \frac{1}{2}\sum_{1\le i<j\le n} s^3_{ij}$$

$$\mathcal{O}^{(2)}_n\big|_{p_n\to 0} = \mathcal{O}^{(2)}_{n-1} \qquad \mathcal{O}^{(3)}_n\big|_{p_n\to 0} = \mathcal{O}^{(3)}_{n-1}$$

$n = 6, \quad p = 3$

$$\mathcal{O}^{(3)}_{6,1} = \frac{1}{32}\left(10\sum_{1\le i<j\le 6} s^3_{ij} + 3\sum_{1\le i<j<k\le 6} s^3_{ijk}\right) \qquad \mathcal{O}^{(3)}_{6,2} = \frac{1}{8}\sum_{\text{permutation}} s_{12}s_{34}s_{56}$$

$$\mathcal{O}^{(3)}_{6,1}\big|_{p_i\to 0} \to \mathcal{O}^{(3)}_5 \qquad \mathcal{O}^{(3)}_{6,2}\big|_{p_i\to 0} \to 0$$

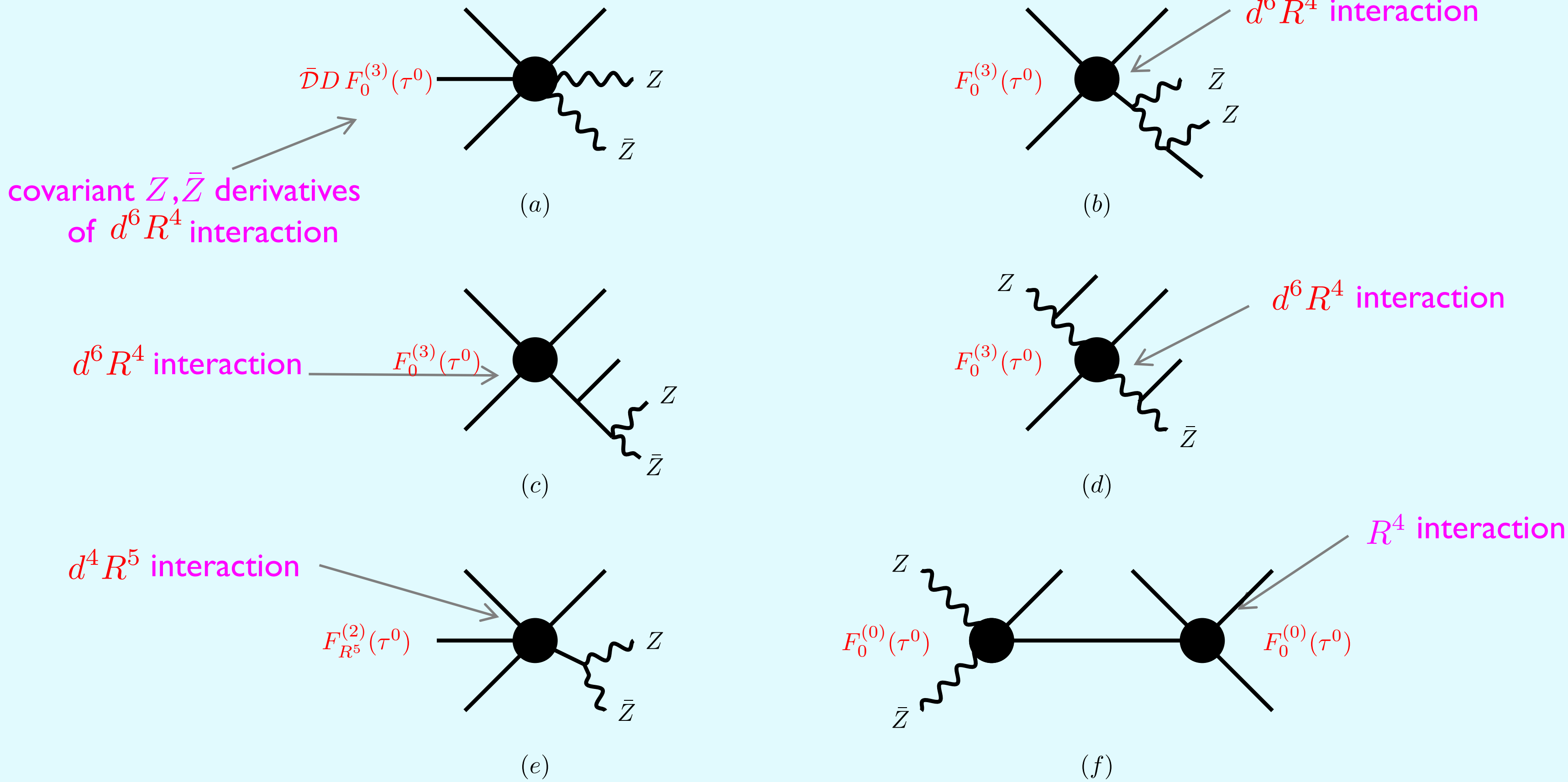
$n > 6, \quad p = 3$

$$\mathcal{O}^{(3)}_{n,1} = \frac{1}{32}\left((28-3n)\sum_{i<j} s^3_{ij} + 3\sum_{i<j<k} s^3_{ijk}\right) \qquad \mathcal{O}^{(3)}_{n,2} = (n-4)\sum_{i<j} s^3_{ij} - \sum_{i<j<k} s^3_{ijk}$$

$$\mathcal{O}^{(3)}_{n,1}\big|_{p_n\to 0} = \mathcal{O}^{(3)}_{n-1,1} \qquad \mathcal{O}^{(2)}_{n,1}\big|_{p_n\to 0} = \mathcal{O}^{(2)}_{n-1,1}$$

e.g. Amplitudes with a Z and a $\bar{Z}$ → Laplace equations

Viz. (Yin, Wang)



Above diagrams follow from interactions in low energy effective action

SUPERSYMMETRIC CONTACT TERMS FORBIDDEN FOR NON-MAXIMAL U(1) VIOLATING PROCESSES

Leads to Laplace eqn. $\bar{\mathcal{D}} \mathcal{D}\, \mathcal{E}_0^{(3)}(\tau^0) + a\mathcal{E}^{(3)}(\tau^0) + bE_0(\tfrac{3}{2}, \tau^0)\, E_0(\tfrac{3}{2}, \tau^0) = 0$

Coefficients may be fixed by comparison with tree-level amplitudes

This argument extends to all the Laplace equations (MBG, Wen)

Connection of SL(2,Z) in Type IIB superstring with Montonen-Olive duality in N=4 SUSY Yang-Mills

Flat-Space Limit of $\mathcal{N} = 4$ Yang-Mills Correlation Function

- Consider correlation function of four superconformal primariies **20'** operators
  Dual to four-graviton amplitude.

- The **Pestun partition function on** S^4 Supersymmetric Localisation
- $m = 0$ limit of $\mathcal{N} = 2^*$ theory (mass-deformed $\mathcal{N} = 4$ Super Yang-Millls on S^4)
- ***Integrated*** correlation function = $\left.\frac{\partial_m^2 \partial_\tau \partial_{\bar\tau} \log Z}{\partial_\tau \partial_{\bar\tau} \log Z}\right|_{m=0}$

  $Z = Z_{class}\ Z_{1-loop}\ |Z_{inst}|^2$

  Classical 1-loop Nekrasov partition function

- Would like to consider approach to flat space limit (Penedones, Polchinski,….)

  Perturbative part $Z_{pert} = Z_{class}\, Z_{1-loop}$ (Binder, Chester, Pufu, Wang)

- Technical issues: Transform from S^4 to R^4; Mellin transform; .;…
  (Gerchkovitz, Gomis, Ishtiaque, Karaa, Komargodski, Pufu)

AdS/CFT dictionary

$$g = \frac{g^2_{\mathsf{YM}}}{4\pi}$$

$$\left(\frac{\alpha'}{R^2}\right)^2 = \frac{1}{g^2_{\mathsf{YM}} N}$$

Large-N limit

(A) 'tHooft limit $N \to \infty$ with $\lambda = g^2_{\mathsf{Y}M}\, N$ fixed

- Leading contribution $\sim N^2$ - Supergravity
- R^4 coefficient $N^{\frac{1}{2}} E(\tfrac{3}{2}, \tau, \bar\tau) \sim N^2 \lambda^{-\frac{3}{2}} + \lambda^{\frac{1}{2}} + \sum_{k\neq 0} e^{-2\pi k \frac{N}{\lambda}} e^{2\pi i k\theta}$

Perturbative terms agree but instantons suppressed

UNCONVENTIONAL LIMIT OF $\mathcal{N}=4$ YANG-MILLS

(B) g_{YM} **FIXED AND** $N \to \infty$ **PRESERVES** $SL(2,\mathbb{Z})$ **MONTONEN-OLIVE DUALITY**

Expect to get $\sim N^{\frac{1}{2}}\, E(\frac{3}{2},\tau) \sim N^{\frac{1}{2}} \left(g_{\rm YM}^{-3} + g_{\rm YM} + g_{\rm YM}^{-1} \sum_{k\neq 0} K_1(8\pi^2 g_{\rm YM}^{-2}\, k) e^{2\pi i k\theta} \right)$

$$\tau = \theta + i\frac{4\pi}{g^2_{\rm YM}} \equiv \tau_1 + i\tau_2$$

$$g_{\rm YM} \to 0 \qquad K_1(8\pi^2 k g_{\rm YM}^{-2}) = g_{\rm YM}\, e^{-\frac{8\pi^2 k}{g^2_{\rm YM}}}(1 + O(g_{\rm YM}))$$

k D-instanton contribution must emerge from k Yang—Mills instanton sector

Localisation reduces functional integral on S^4 to integral over constant scalar expectation value a_{ij}.

The Pestun partition function:

$$Z(m,\tau,\bar\tau) = \int d^{N-1}a \prod_{i<j} \frac{a_{ij}^2 H^2(a_{ij})}{H(a_{ij}-m)H(a_{ij}+m)} e^{-\frac{8\pi^2}{g^2_{\rm YM}}\sum_i a_i^2} |Z_{\rm inst}(m,\tau,\bar\tau,a_{ij})|^2\,,$$

$$H(z) = G(1+z)G(1-z)$$

Barnes G-function

Perturbative

The Nekrasov instanton partition function: contribution from instantons at North Pole and anti-instantons at South Pole of S^4

- Leading order N^2 determined to all orders in λ in the ‘tHooft limit. (Binder, Chester, Pufu, Wang)

String tree level

UNCONVENTIONAL LIMIT OF $\mathcal{N}=4$ YANG-MILLS

Non-perturbative terms: (MBG, Chester, Pufu, Wang, Wen)

$$Z_{inst}(m,\tau,,a) = \sum_{k=0}^{\infty} e^{2\pi i k\tau}\, Z^{(k)}_{inst}(m,\tau_2,a)$$ Fourier sum (sum over instanton number)

So

$$\partial_m^2 \log Z_{inst}(m,\tau,,a)\big|_{m=0} = \sum_{k=1}^{\infty} 2(e^{ik\theta}+e^{-ik\theta})\, e^{-\frac{8\pi^2 k}{g^2_{\text{YM}}}} \langle \partial_m^2 Z^{(k)}_{inst}(m,\tau_2,a)\rangle\big|_{m=0}$$

where $\langle \mathcal{O} \rangle = \int d^N a\, e^{-\frac{8\pi^2}{g^2_{\text{YM}}}} \prod_{I<J} a^2_{IJ}\, \mathcal{O}$

Using Nekrasov's result as interpreted by Pestun we have

$$Z^{(k)}_{inst}(m,\tau_2,a) = \frac{1}{k!}\left(\frac{2m^2}{m^2+1}\right)^k \oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi} \prod_{i=1}^{N} \frac{(\phi_I-a_I)^2-m^2}{(\phi_I-a_i)^2+1} \prod_{I<J}^{k} \frac{\phi^2_{IJ}(\phi^2_{IJ}+4)(\phi^2_{IJ}-m^2)^2}{(\phi^2_{IJ}+1)((\phi_{IJ}-m)^2+1)((\phi_{IJ}+m)^2+1}$$

where the integration contour circles the poles in a particular (and complicated) manner.

- Reproduces factor of $|k|^{\frac{1}{2}}\,\sigma_2(|k|)\, g_{\text{YM}}^{-\frac{1}{2}}\, K_1(8\pi^2 g_{\text{YM}}^{-2}\, k)$ in k D-instanton contribution to $E(\frac{3}{2},\tau)$

- Asymptotic expansion of K_1 at small g_{YM} gives an infinite set of perturbative corrections to the k D-instanton contribution.

- The leading k D-instanton contribution of order g^0_{YM} was obtained from a ADHM construction for $SU(N)$ $\mathcal{N}=4$ super Yang-Mills by a large-N saddle point method.

  (Dorey, Khoze, Mattis, Hollowood)

Comments

- We have determined non-perturbative behaviour of all "protected" terms in the low-energy expansion of the form $\mathcal{E}^{(p)}_{w,i}(\tau)\, d^{2p}_{(i)}\, \mathcal{P}_n(\{\Phi\})$ - up to overall constants $c^{(p)}_{n,i}$ $\quad (n = 4 + w)$ that are determined from tree (or one-loop) amplitudes and (in principle) by supersymmetry.

- These violate the continuous U(1) R-symmetry in string theory by $q = -2w = -2(n-4)$ units.

  The $w \neq 0$ cases do not arise in maximal supergravity

- These interactions are related by FIRST-ORDER DIFFERENTIAL EQUATIONS – consequence of SUPERSYMMETRY as is apparent from the amplitude calculations.

- LAPLACE EQUATIONS follow as consequence of first-order equations.

  Leading to the same non-renormalisation conditions as in maximal supergravity.

- Generalisations to toroidally compactified theory – higher rank duality groups

- Four-point integrated correlator in $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills in ultra-strong coupling limit $N \to \infty$ fixed $g_{\rm YM}$: Leading term $\sim N^2$ reproduces ten-dim. IIB supergravity amplitude.

- Next term $\sim N^{\frac{1}{2}}$ reproduces $\alpha'^{-1}\, E(\tfrac{3}{2}, \tau)\, R^4$

- At higher orders there are ambiguities since we are looking at the integrated correlator.