Waveform accuracy requirements for LISA

LISA Waveform Working Group

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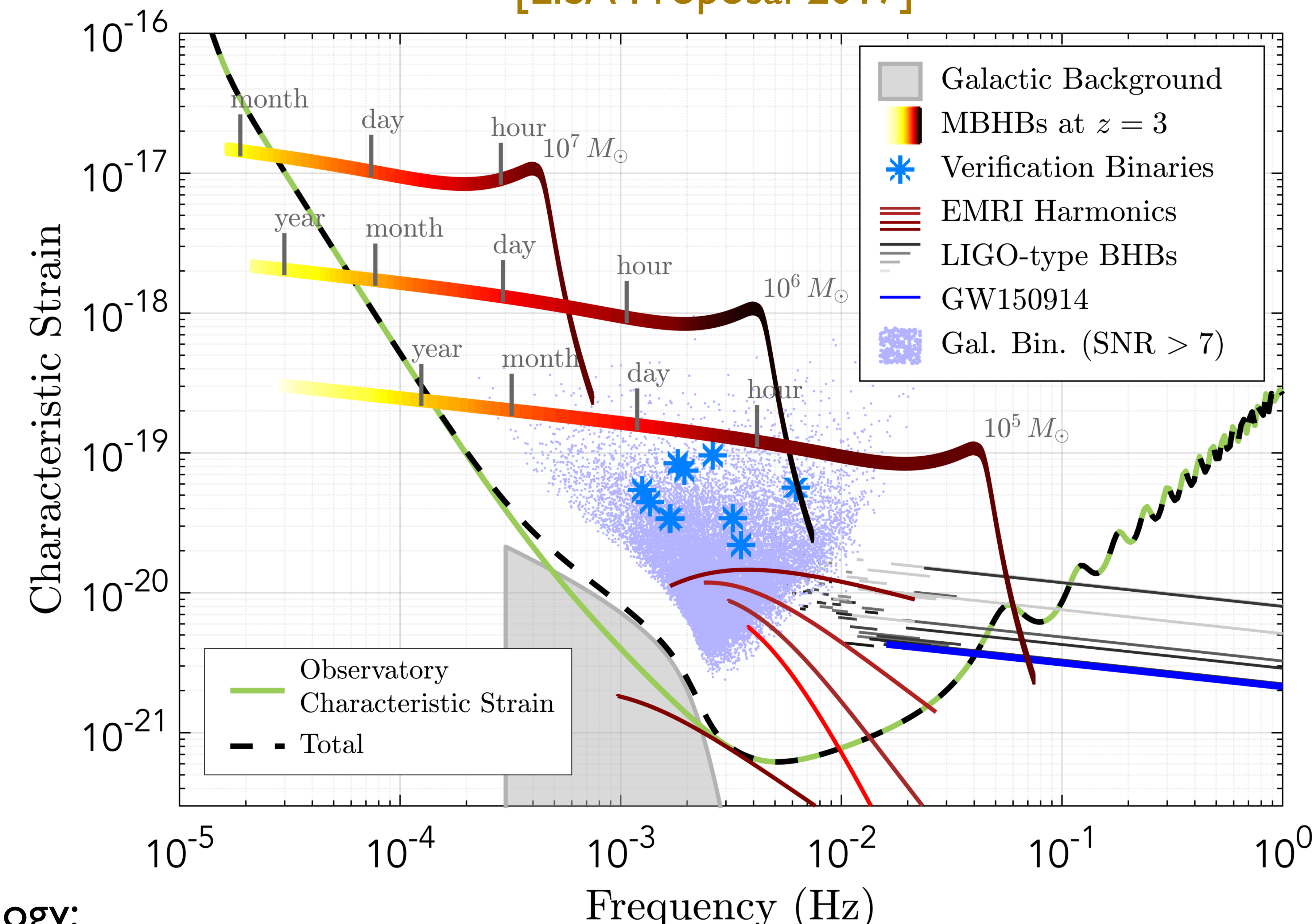
LISA Sources

[LISA Proposal 2017]

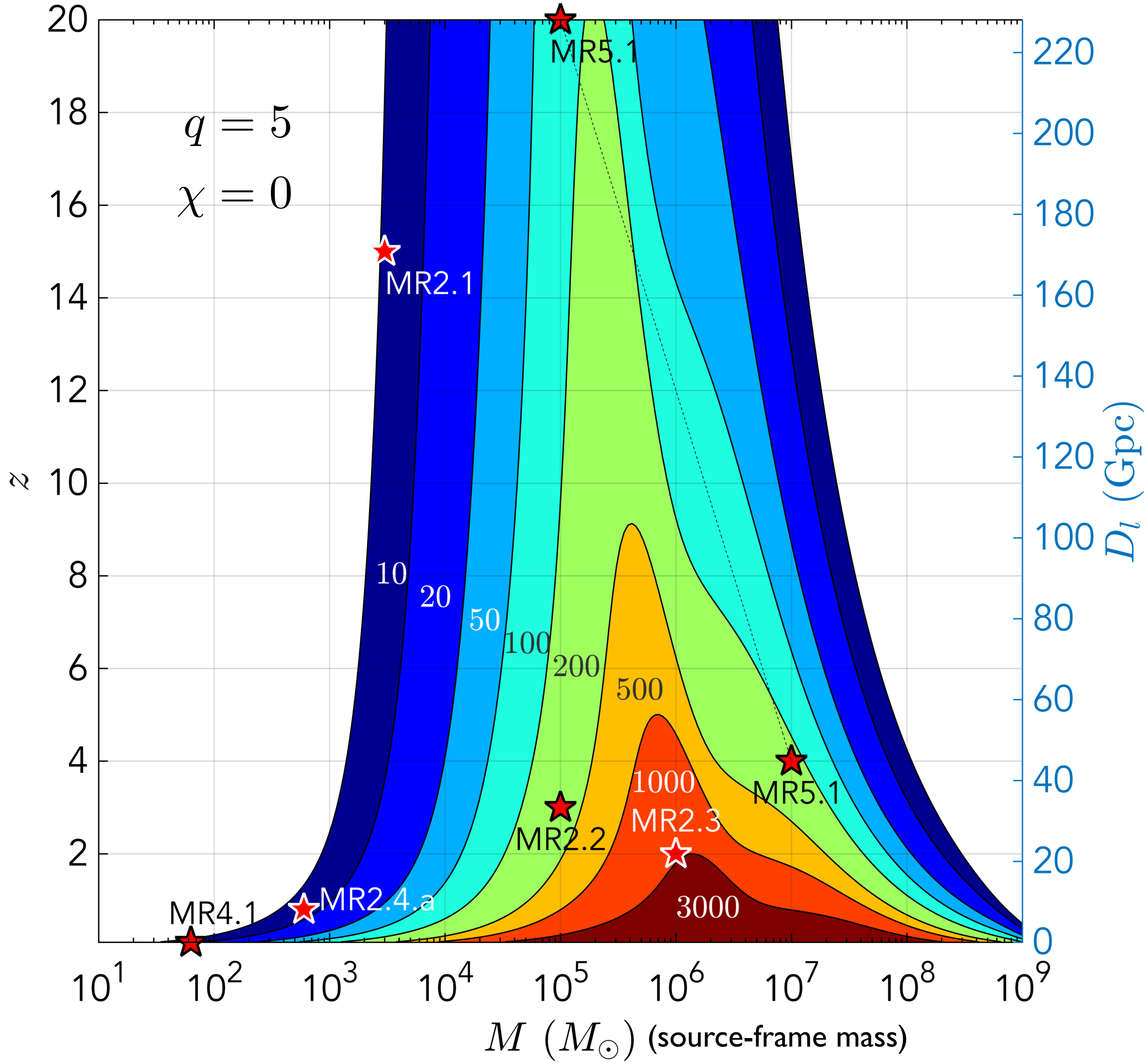


Terminology:

- Supermassive black holes binaries (**SMBHBs**)
- Stellar black hole binaries (**SBHBs**): multiband ?
- Extreme mass ratio inspirals (**EMRIs**)
- Galactic binaries (**GBs**)
- Other sources: cosmic strings, …

LISA sources and SNR: comparable-mass systems

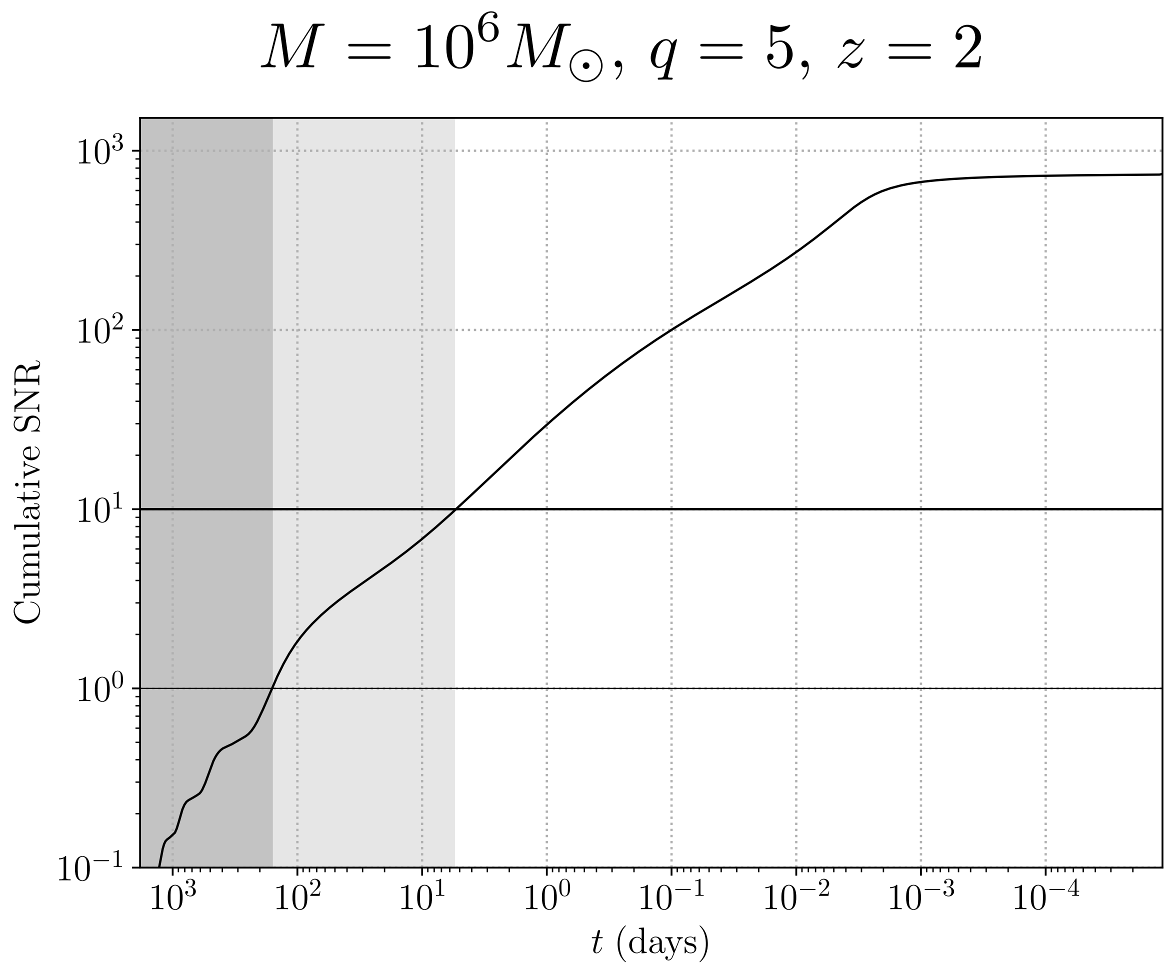
- Very high SNR for SMBHBs
- Challenge for waveforms: keep systematic errors smaller than statistical errors

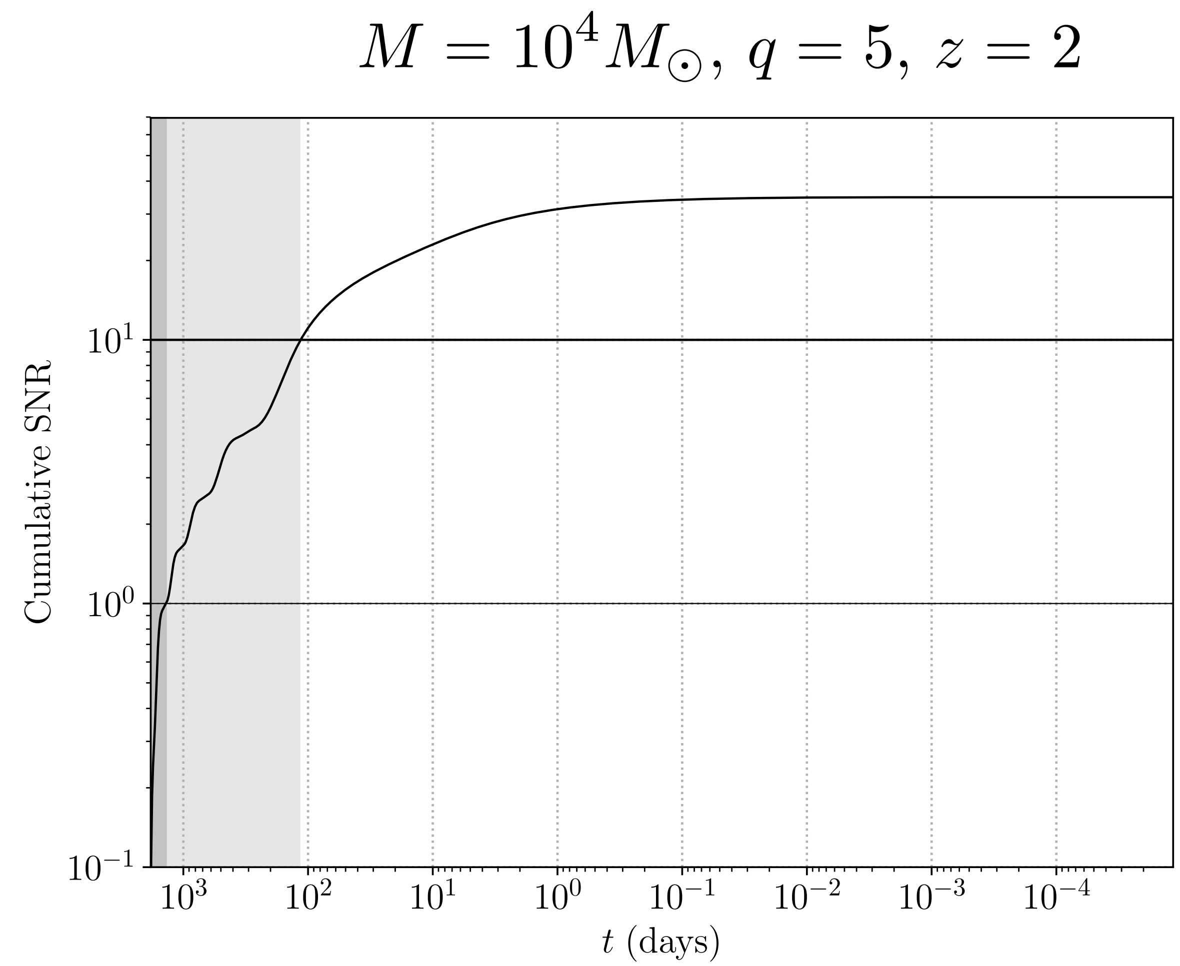


[LISA Proposal 2017]

Accumulation of SNR with time for SMBHB/IMBHB

Accumulation of SNR as time left before merger diminishes
Shaded areas: thresholds SNR=1 and SNR=10





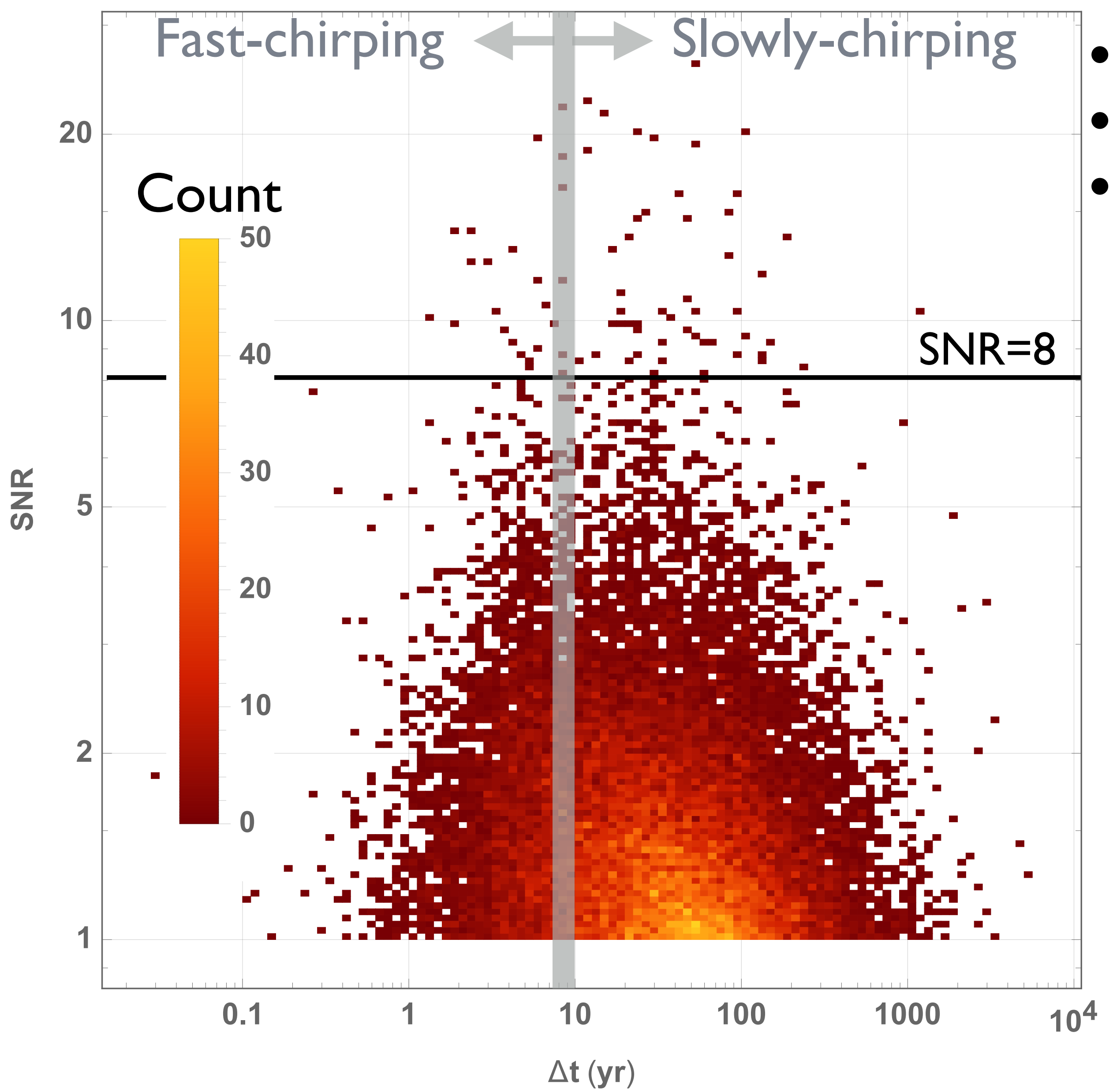
Two different definitions of "signal duration":

- Looking back in time from merger, when is the signal negligible ? Here SNR=1
- Accumulating signal towards merger, when is the signal detected ? Here SNR=10

For SMBHBs, SNR accumulates shortly before merger (days)

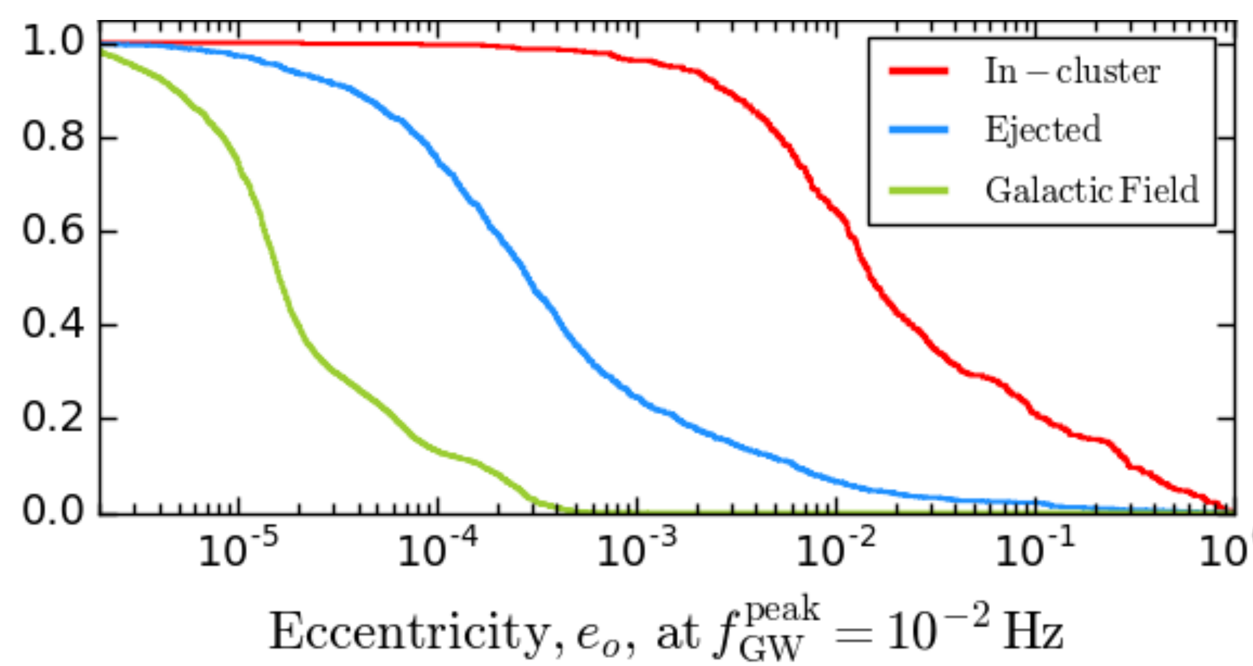
LISA sources and SNR: stellar black hole binaries

SNR vs time-to-merger
Catalog: [Sesana 2016]



- Deep inspiral, can use analytic PN models
- Possible significant eccentricity in the LISA band
- Two populations: chirping, exiting the band / slowly-chirping, staying in band

[Kremer&al 2018]



Waveform accuracy: goals

- **Detection:** templates enabling the detection of all signals (effectualness)
- **Characterisation:** waveform models enabling the recovery of source parameters without bias (faithfulness)

LISA: Global fit
sources and instrument

Explorations of data analysis (5 yrs?)

Case for approximate waveforms in injection/recovery

- LISA science case: how well are we going to detect and characterise sources ?
- Playground for data analysis techniques
- Explore tests of GR

Challenges :

- Need waveforms realistic enough
- Physical effects: merger, higher harmonics, spin, precession, eccentricity, astro. effects
- Computational efficiency required

Interactions:

- EM observations: advance warning for EM partners ? Localisation of sources ?
- Instrument: impact of instrumental design choices on LISA science ?

Data analysis for the mission (10 yrs?)

- Provide "final" waveforms with low enough systematic errors

- Residuals low enough to enable high-accuracy tests of GR

- Computational costs: integration with data analysis pipeline, coexistence of slow/fast models for low-latency ?
- HPC resources in ~2035 ?

- Demonstrate feasibility of waveform developments in advance ?

Waveform accuracy: tools

Bayesian formalism

- Matched-filtering overlap: $(h_1|h_2) = 4\mathrm{Re}\int df\, \frac{\tilde{h}_1(f)\tilde{h}_2^*(f)}{S_n(f)}$
- **Likelihood for Gaussian, stationary noise**, for independent channels: $d = s(\theta_0) + n$

$$\ln\mathcal{L}(d|\theta) = -\sum_{\text{channels}} \frac{1}{2}(h(\theta) - d|h(\theta) - d)$$

- Bayes theorem defines the posterior: $p(\theta|d) = \frac{\mathcal{L}(d|\theta)p_0(\theta)}{p(d)}$
- Fisher matrix (high SNR limit): $\ln\mathcal{L} \sim -\frac{1}{2}\Delta\theta_i(h_{,i}|h_{,j})\Delta\theta_j$

h template
θ parameters
d data
s signal
θ_0 signal params.
n noise
S_n noise PSD
$p_0(\theta)$ prior
$p(d)$ evidence

Faithfulness / effectualness

$\hat{h} = h/\sqrt{(h|h)}$

- Effectualness of a template bank:
  $\mathrm{FF} = \max_{h\in\text{bank}}(\hat{h}|\hat{s})$
  Need to study full search, false-alarm rate
- Faithfulness: $\mathrm{MM} = \min_{\delta t, \delta\varphi, \ldots}(1 - (\hat{h}|\hat{s}))$
  Max/averaged for different sets of params.
  Depends on detector

Which waveform accuracy ?

$$\boxed{\text{systematic} < \text{statistical}}$$

- **Conservative** criterion for bias, from unfaithfulness: $\mathrm{MM} < \frac{D}{2\mathrm{SNR}^2}$
- Golden standard : Bayesian parameter estimation injection/recovery studies (costly)
- Intermediate approaches ?

Solving two-body problem in General Relativity

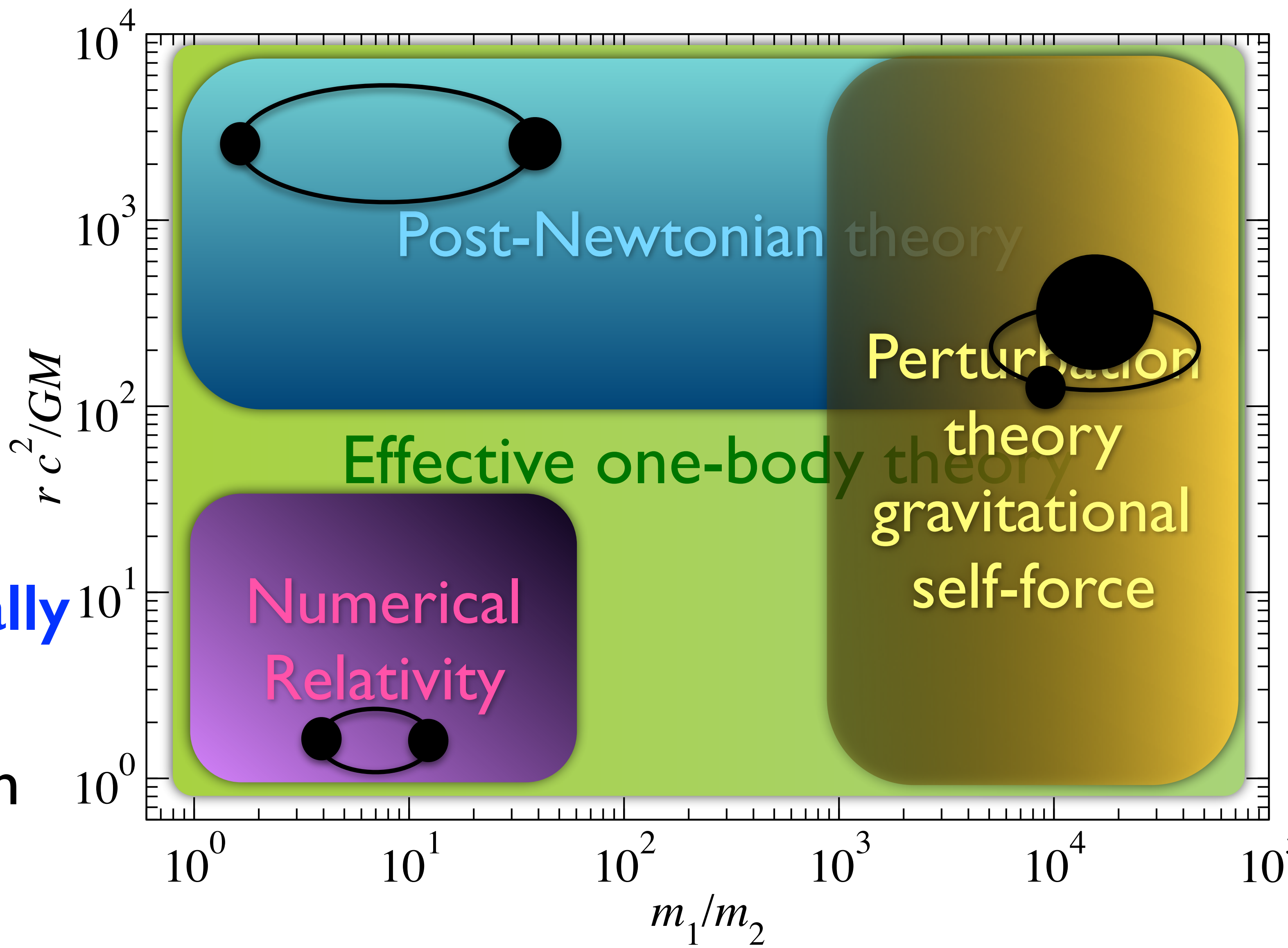
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **GR** is **non-linear theory**.

- Einstein's field equations can be solved:

- **approximately,** but **analytically** (**fast** way)
- **"exactly"**, but **numerically** on supercomputers (**slow** way)

$v^2/c^2 \sim GM/rc^2$



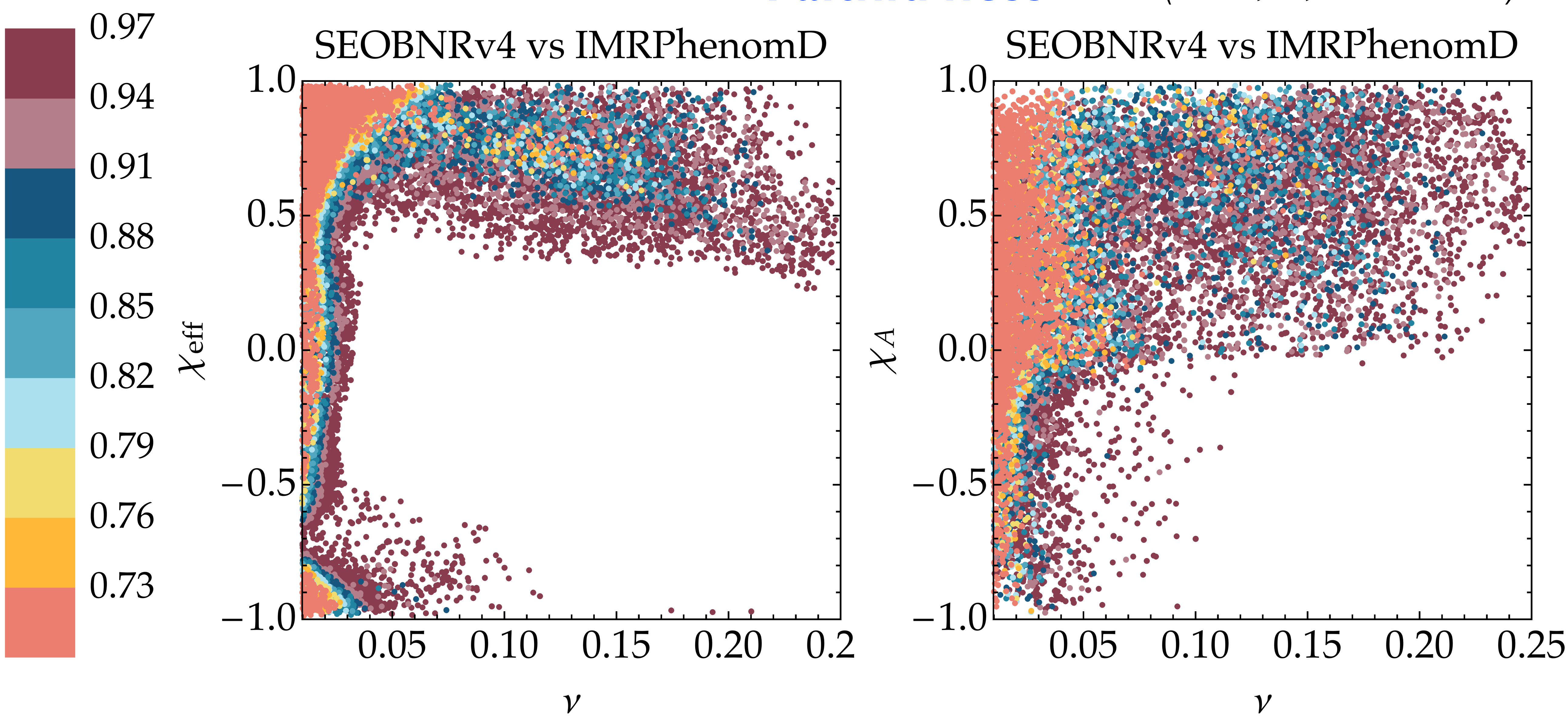
(AB & Sathyaprakash 14)

- **Synergy** between **analytical** and **numerical relativity** has been and will continue to be **crucial.**

Comparing EOBNR & IMRPhenom models: inferring parameters

- **Aligned/anti-aligned** waveform models. Only dominant **(2,2) mode**.
- Differences for **large mass ratios** (> 4) **and large spins** (> 0.8).

Faithfulness *(Bohe',…, AB et al. 16)*

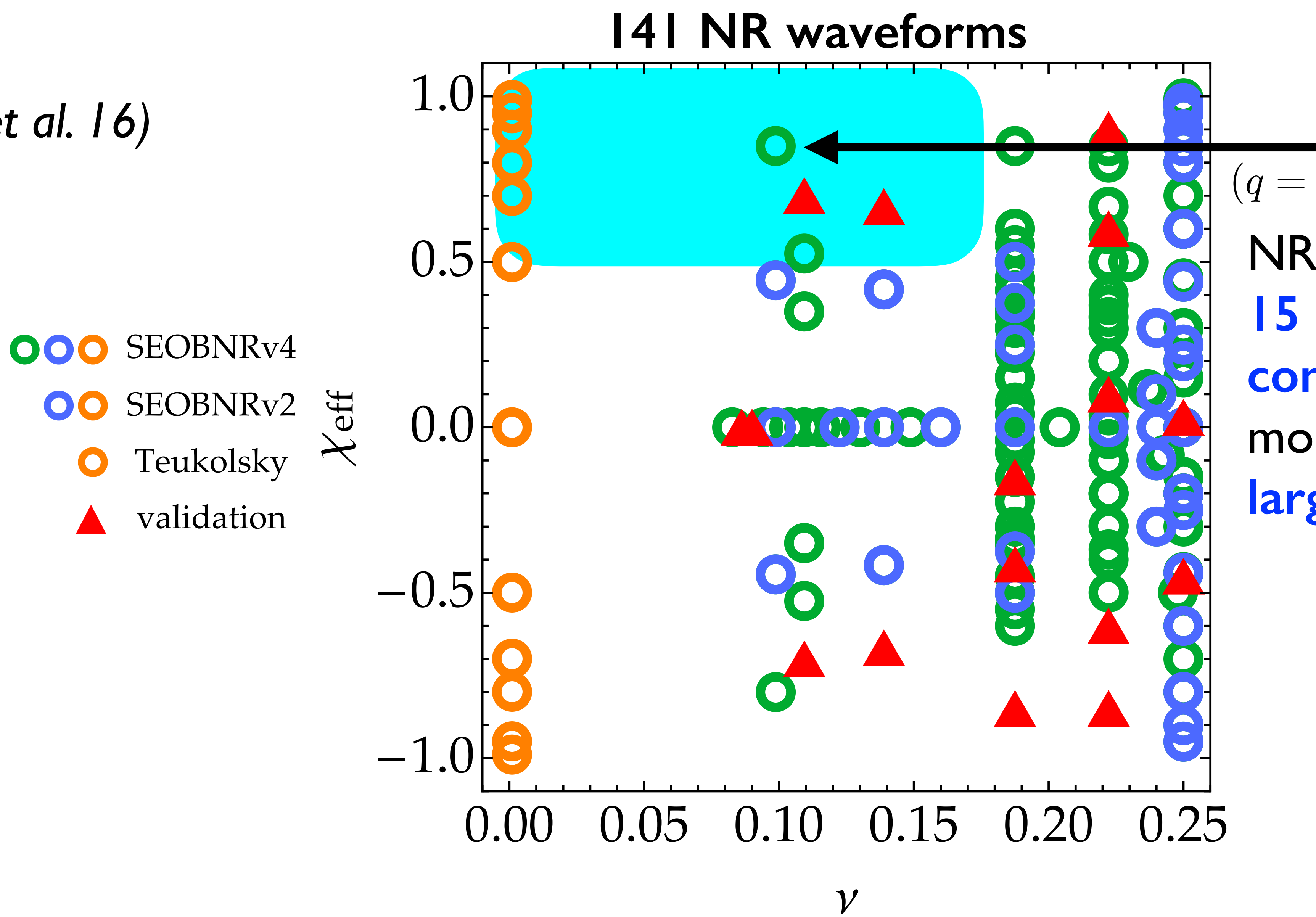


[Note that only 7% of 200,000 points have matches < 97%.]

Extending waveform model in all BBH parameter space

- Difficult to run **NR simulations** for **large mass ratios** (> 4) **and large spins** (> 0.8), with **large number** of GW **cycles** (> 50).

(Bohe',…, AB et al. 16)



NR waveform with only **15 GW cycles, it constrains** EOBNR model only for **masses larger than 150 Msun.**

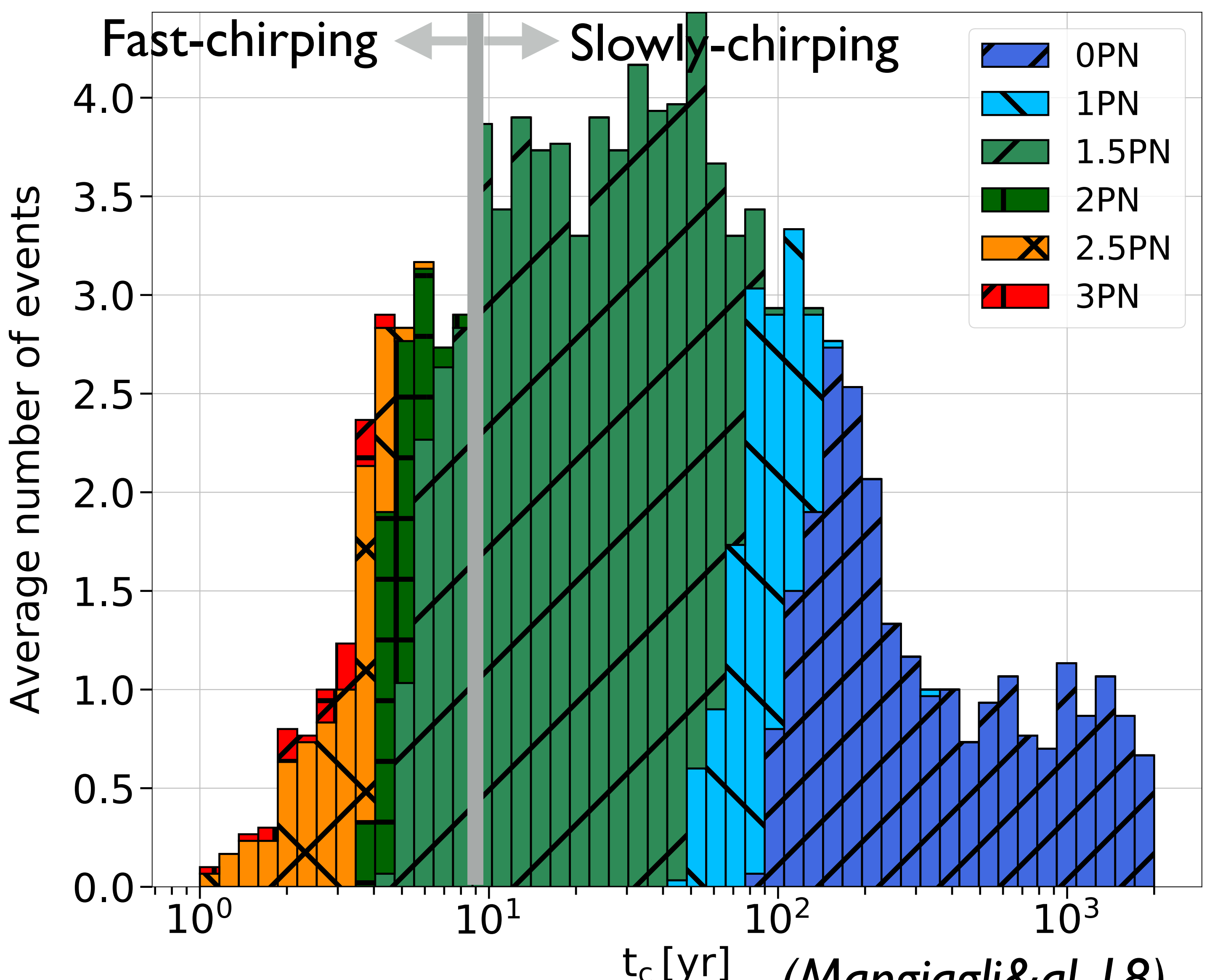
- For large mass ratios (> 4) combine **PN & GSF** results in **EOB** framework.

  (Damour 09; Barausse et al. 12, Le Tiec et al. 12, Bini et al. 12-16, Antonelli et al. in progress)

- Inclusion of **GSF** also important for **EMRIs** (LISA) and **IMRIs** (3G detectors).

Assessing accuracy requirements for SBHBs

- Compare different PN truncation orders in the phase
- No eccentricity yet in this analysis



(Mangiagli&al 18)

Faithfulness criterion:

$$\mathrm{MM} < \frac{D}{2\mathrm{SNR}^2}$$

- The 3PN order is sufficient for all sources, 2PN sufficient for slowly-chirping

Open problems, questions & challenges

- **Current** waveform **models** for SMBHs, IMBHs, SBHBs **do not contain all relevant** physics. **Which physics** is needed for **exploratory studies** and **by 2034?**

- **How** do we **assess waveform accuracy?** Can we use **approximate criteria?**

- Which **waveform accuracy** is **required** for SMBHs, SBHBs, EMRIs, IMRIs for **exploratory studies** and **by 2034?**

- Do **accuracy requirements** change for **overlapping signals**?

- Which **accuracy** is required for **multi-band sources** (LISA-3G)?

Open problems, questions & challenges

- What are the **best strategies** for building **fast waveforms** for **exploratory studies** and **by 2034?** How much **accuracy** we can sacrifice **for speed**?

- What are **efficient strategies** for **parameter estimation** of SMBHs, SBHBs, EMRIs, IMRIs?

- Can we **forecast accuracy** of future waveform models, and **computational resources**?

- Do we need **novel** and efficient **(analytical and/or numerical) methods to solve** 2-body problem?

- Several waveform modeling and data analysis **challenges of LISA** are also **shared by 3G detectors**.

Current waveform models for SMBHs/IMBHs

- **PN theory** (Taylor-models) [only inspiral stage; fast; freq & time domain]

- **Numerical relativity** (NR waveforms) [IMR stages; slow; limited in length and parameter space; time domain]

- **Effective-one-body theory** (EOBNRvN models, TEOBResumS model), builds on PN, GSF, PM and NR [IMR stages; not sufficiently fast; time domain]

- **Phenomenological framework** (IMRPhenom models), builds on EOB and NR [IMR stages; fast; frequency domain]

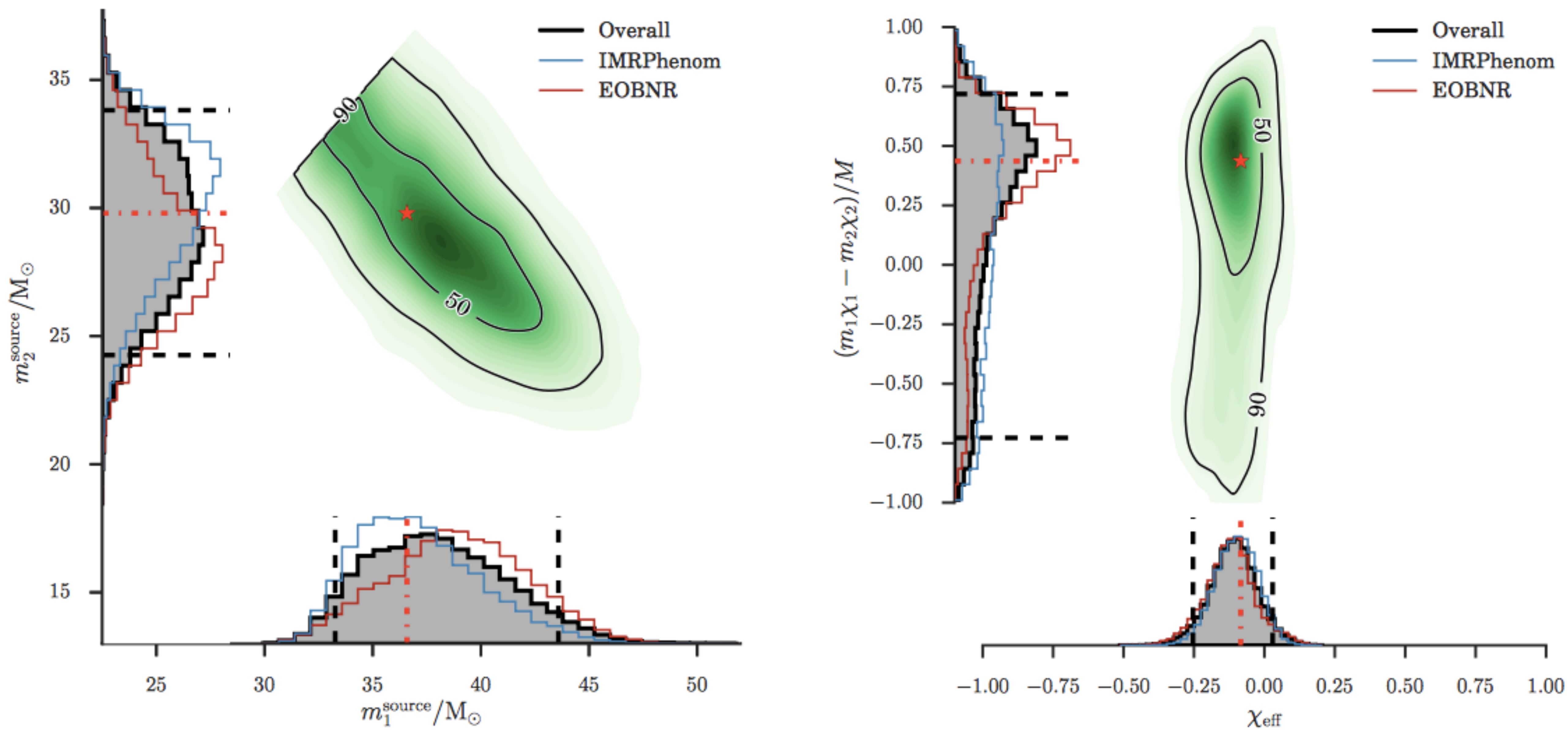
- **EOBNR reduced order models** (EOBNR_ROM models) [IMR stages; fast; frequency domain]

- **NR surrogate models**, build on analytic IMR and NR [IMR stages; fast, but limited in length and parameter space; time domain]

Systematics of current waveforms used in LIGO & Virgo

- **Mock signal** from NR simulation with parameters close to GW150914.

(Abbott et al. CQG 34 (2017) 104002)



- **Overall, no evidence for systematic bias** relative to the statistical error of original parameter recovery of GW150914.

Systematics of current waveforms used in LIGO & Virgo (contd.)

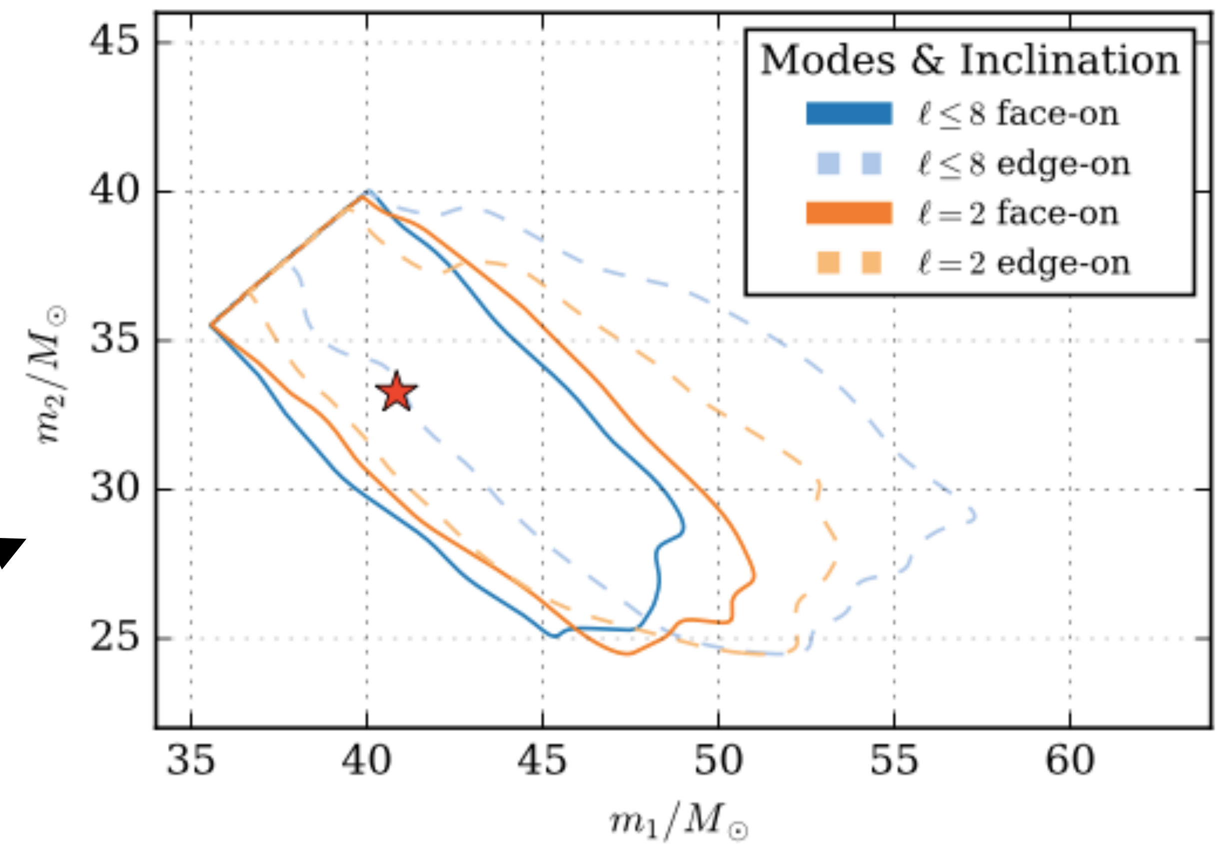
(Abbott et al. CQG 34 (2017) 104002)

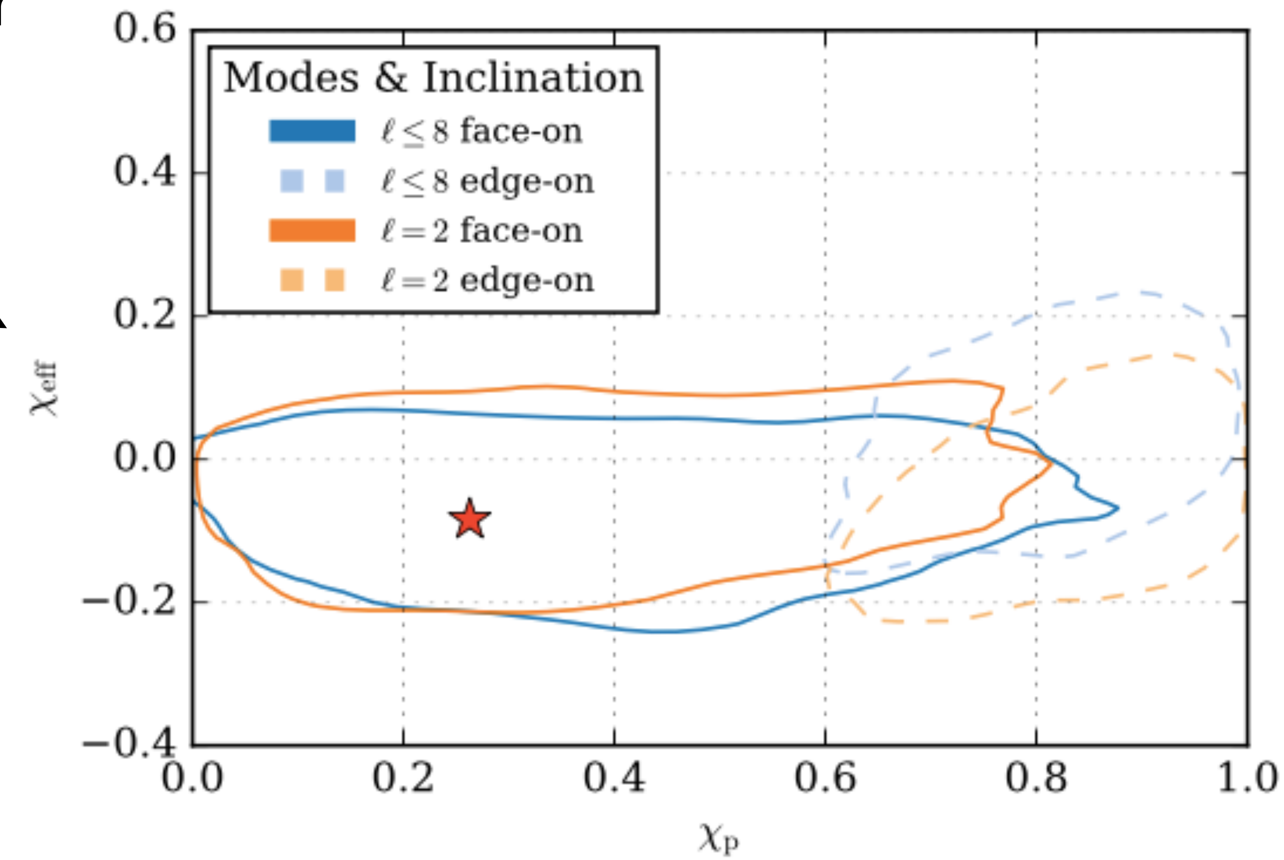
- **Parameter biases** are found to occur for some **configurations disfavored by data** of GW150914.

- E.g., biases are present for **binaries inclined edge-on to the detector** over a small range of choices of polarization angles.

(see also Williamson et al. 2017)

- Biases can be present for **binaries with eccentricity > 0.05**.

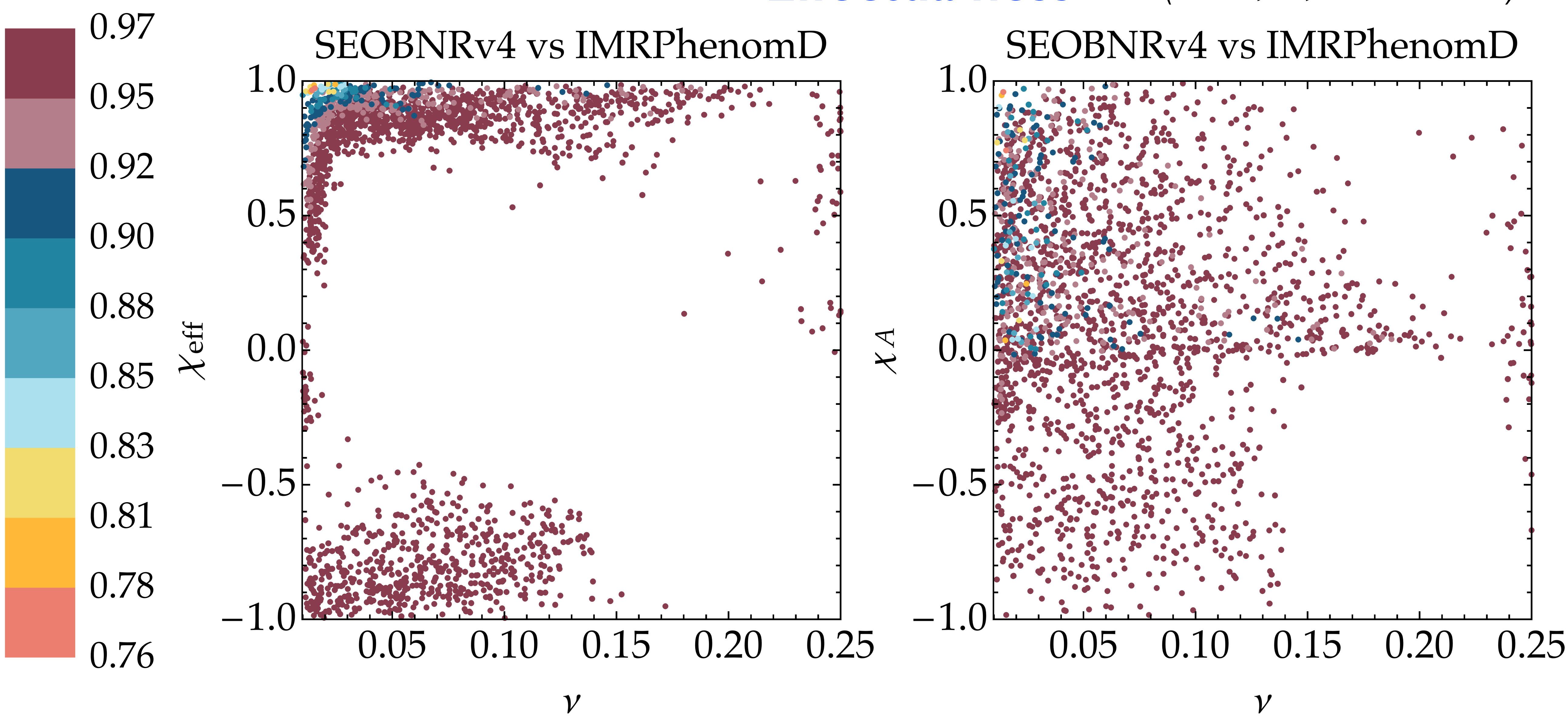




Comparing EOBNR & IMRPhenom models: detection

- **Aligned/anti-aligned** waveform models. Only dominant **(2,2) mode**.

Effectualness *(Bohe',..., AB et al. 16)*



[Note that only 2.1% of 100,000 points have matches < 97%.]

PN versus PM expansion for conservative two-body dynamics

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}} + \frac{1}{c^2}\left[\ldots\right] + \frac{1}{c^4}\left[\ldots\right] + \ldots$$

$$E(v) = -\frac{\mu}{2}\,v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	$\ldots$
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	$\boxed{v^{12}}$	$\ldots$
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	$\boxed{v^{10}/r}$	$\ldots$
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	$\boxed{v^8/r^2}$	$\ldots$
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	$\boxed{v^6/r^3}$	$\ldots$
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	$\ldots$
$\ldots$						$\ldots$	$\ldots$	

(credit: Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

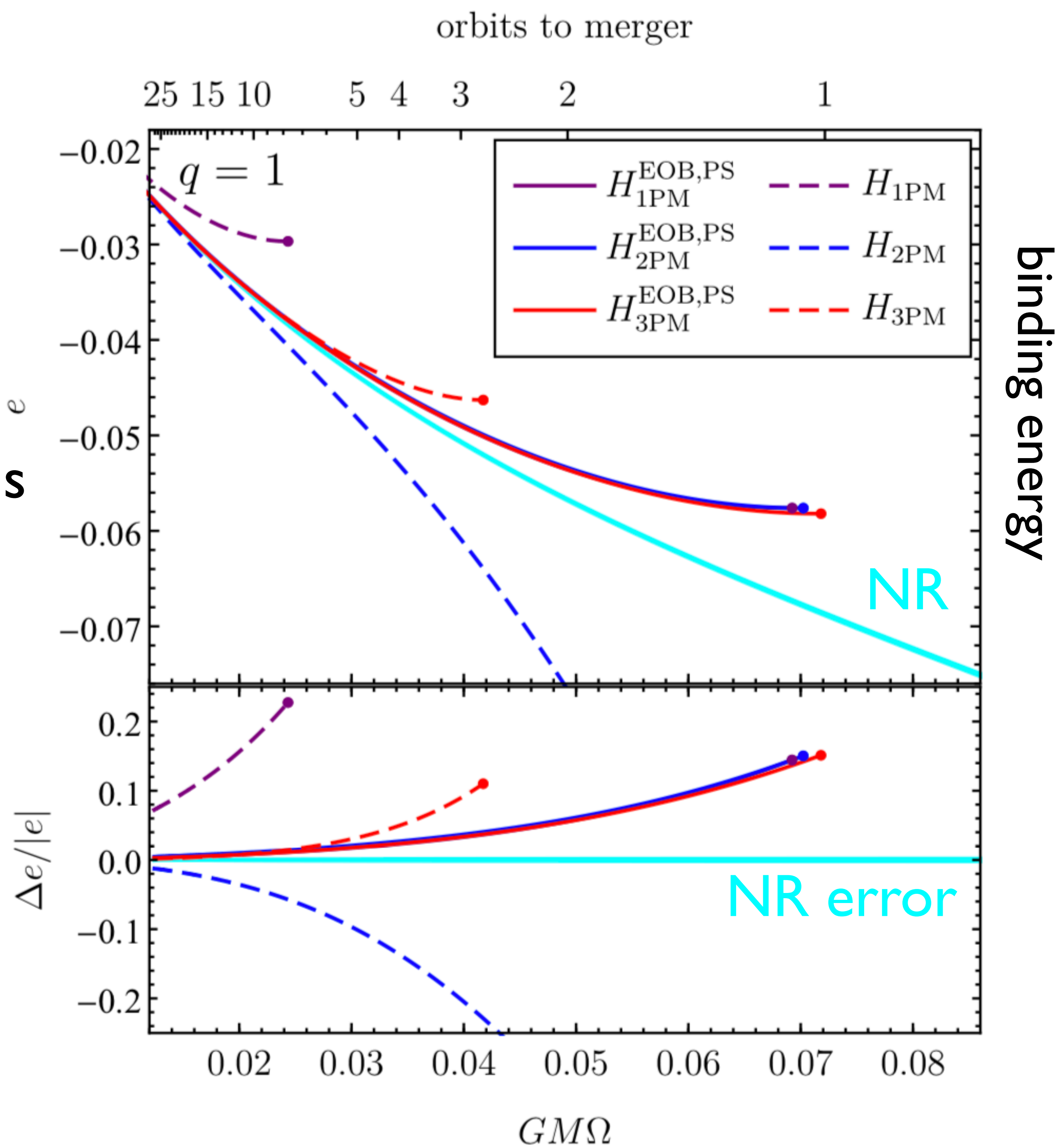
- **PM results** *(Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines, Steinhoff & AB 18)*

Comparison between 3PM and NR binding energies

- 2-body **Hamiltonian at 3PM order** computed using scattering-amplitude methods

(Cheung et al. 18, Bern et al. 19)

- **Crucial** to **push PM** calculations at **higher order**.

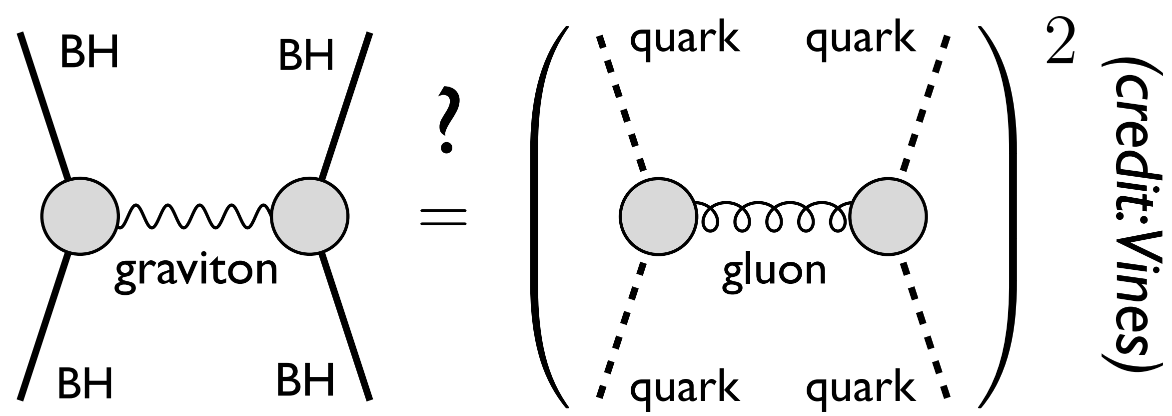


(Antonelli, AB, Steinhoff, van de Meent & Vines 19)

New ideas, new methods to solve 2-body problem

- **Post-Minkowskian results** through **modern scattering-amplitude** calculations **may help** improving accuracy.

- For PM results to have **"real" phenomenological impact** (LIGO-Virgo-LISA-3G), we **need conservative** and **dissipative results** (i.e., also **waveforms**).



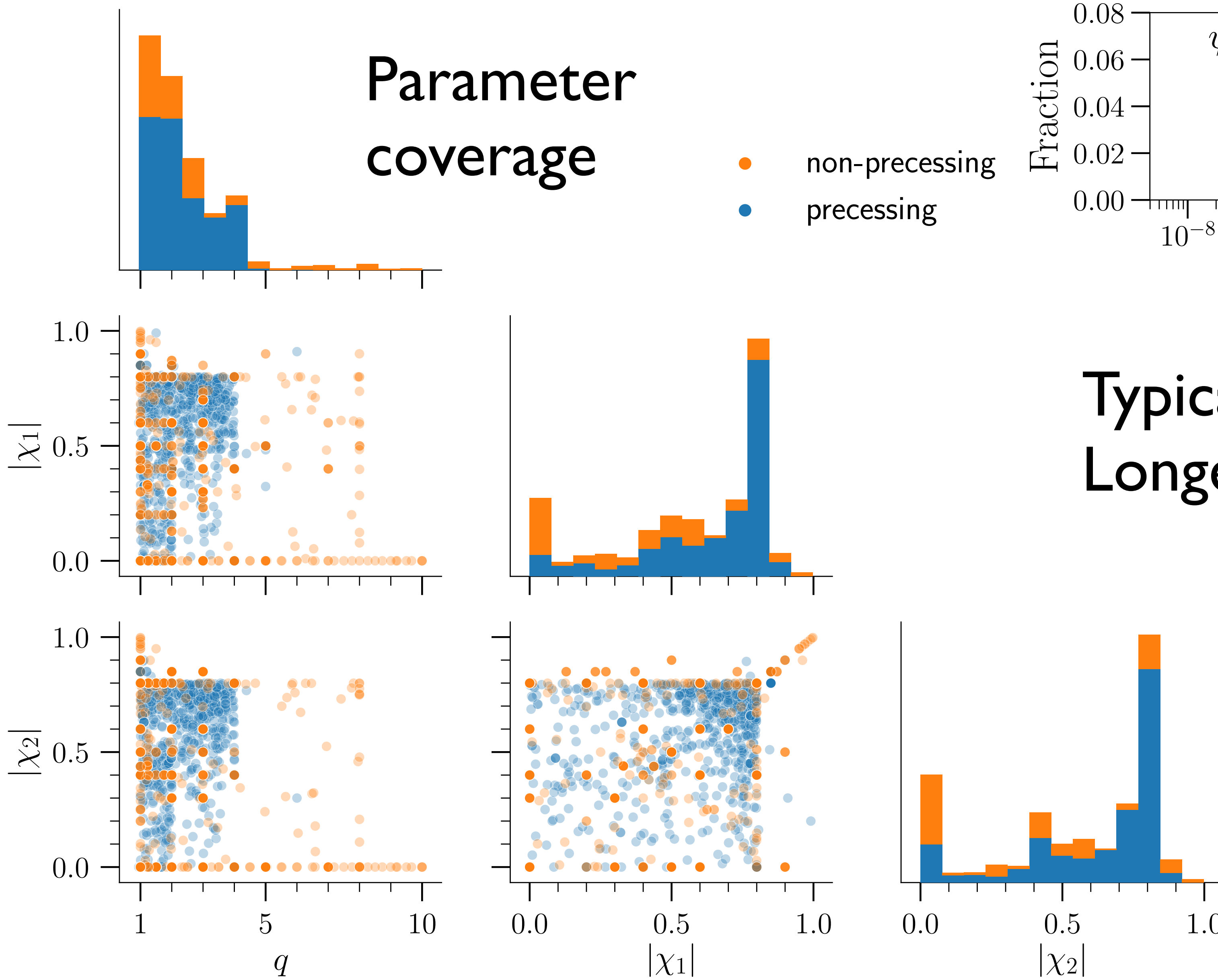
- Need of **more efficient resummation** of 2-body problem for **entire parameter space.**

- Could **2-body problem** be obtained **from 1-body problem exactly**?

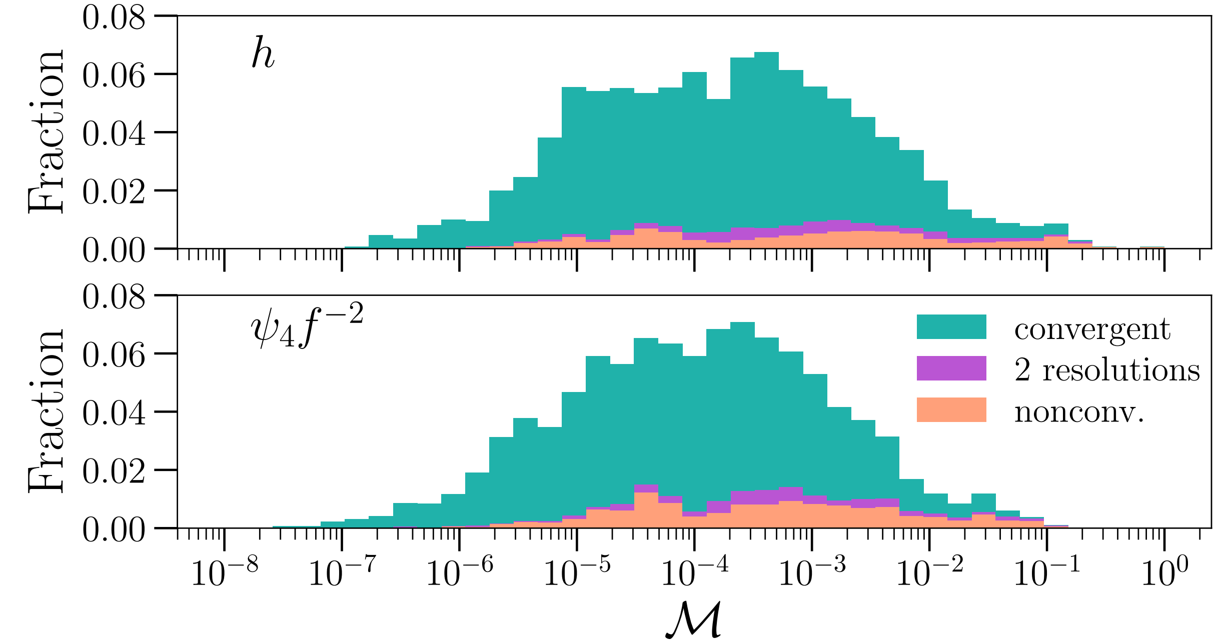
Illustration of NR status: SXS catalog 2019

(Boyle&al 19)

Parameter coverage



Unfaithfulness distribution

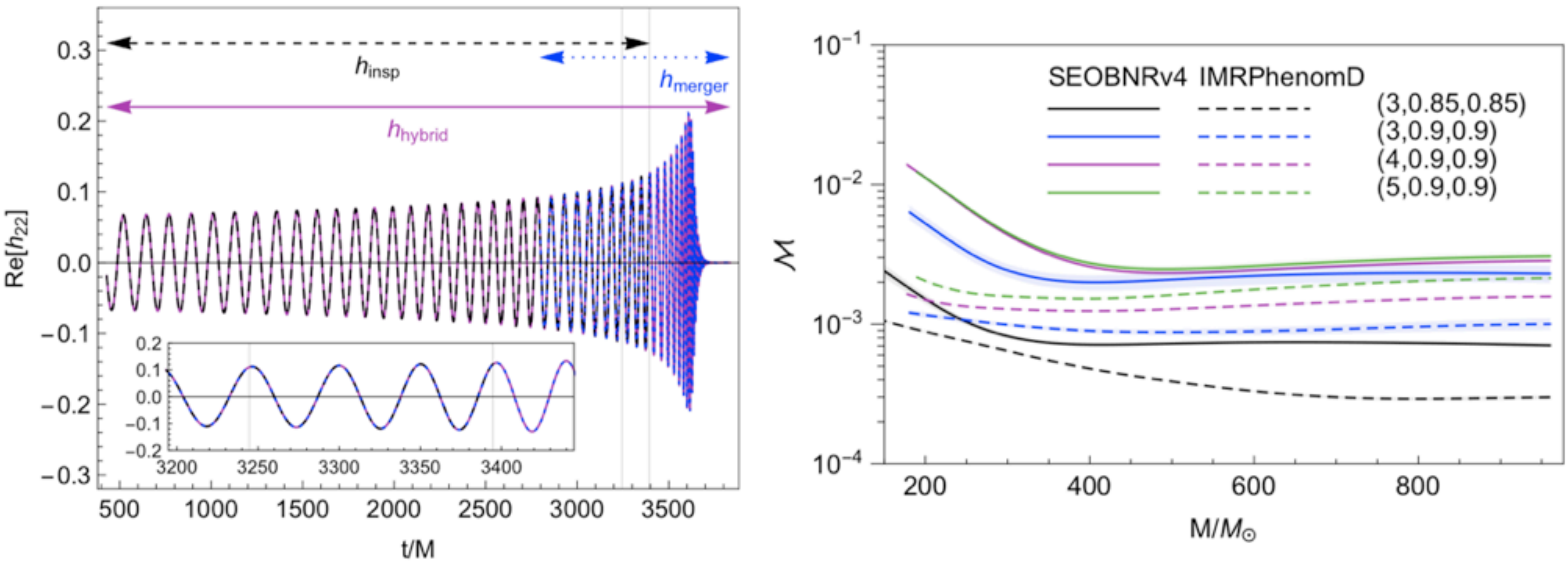


Typical length: $\Delta t < 5000 - 10000 M$

Longest: $\Delta t = 10^5 M$

“Temporary” solution? Waveforms combining NR codes

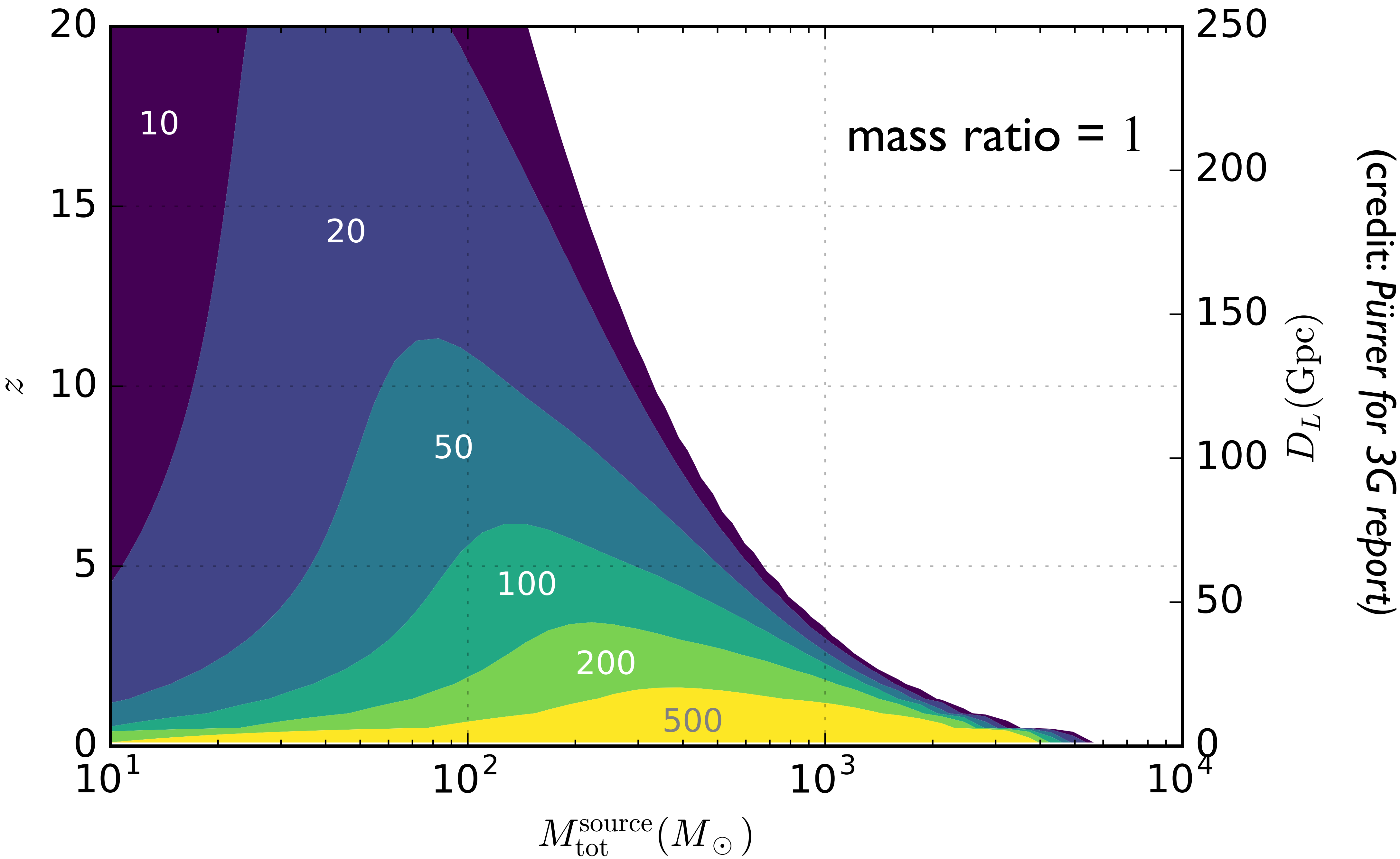
- **Synergistic** use **of finite-difference** (Einstein Toolkit, ET) & pseudo **spectral** (SpEC) **NR codes.**



(Hinder, Ossokine, Pfeiffer & AB 18)

- **Is it worth** to invest on **this strategy?**

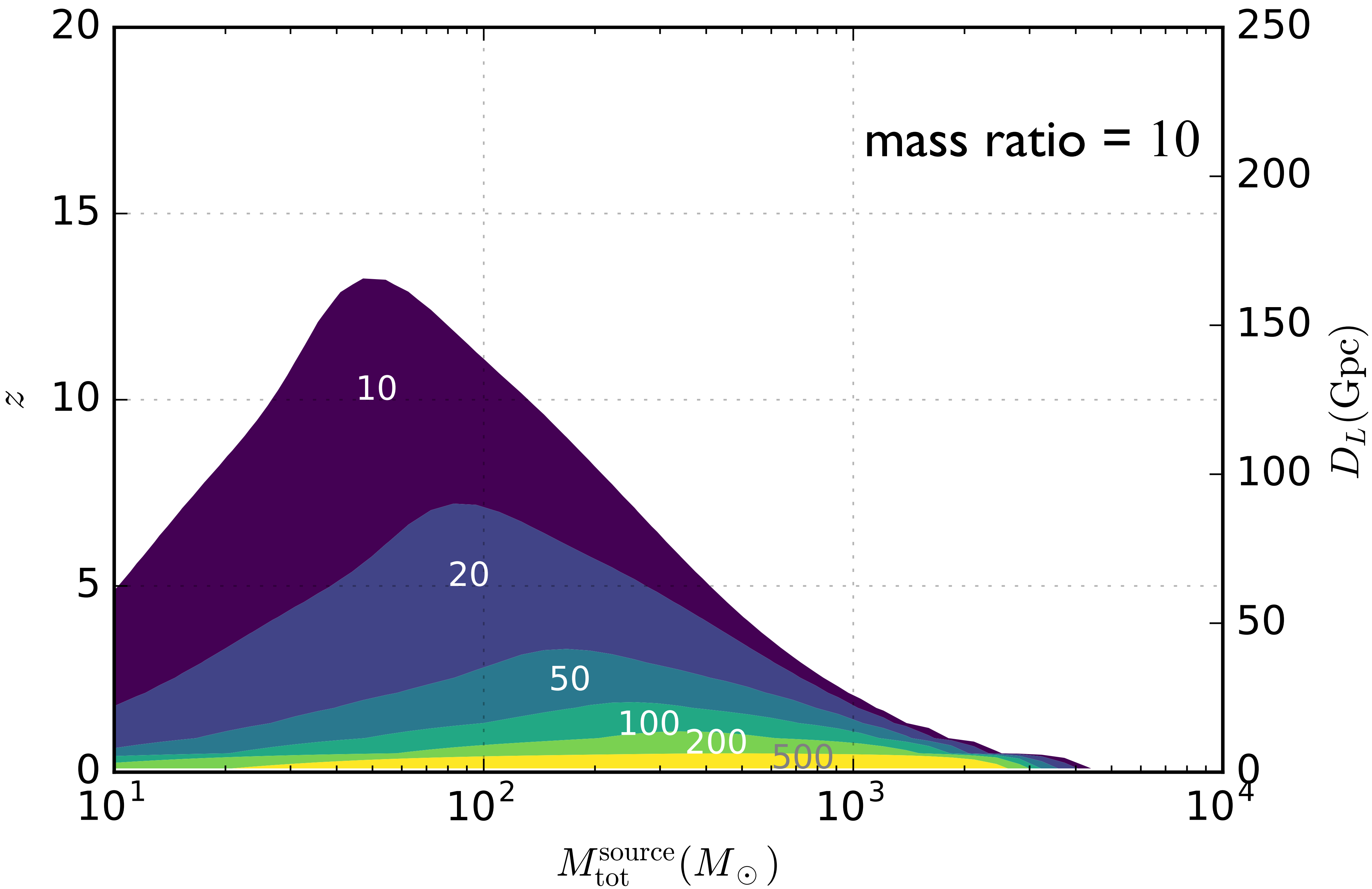
Binary's masses/distance spanned by 3G detectors



- 3G detectors will observe **binary coalescences with SNR** ($\sim$20) even **at high redshift** (z $\sim$10-15), and with **SNR** > 100 at z < 5.

- Demands on waveform **accuracy** are **higher,** modeling is more **challenging.**

Binary's masses/distance spanned by 3G detectors



(credit: *Pürrer for 3G report*)

- 3G detectors will observe **binary coalescences with SNR** (~ 10) up to **redshift** (z ~ 12), and with **SNR** > 100 at z < 2.

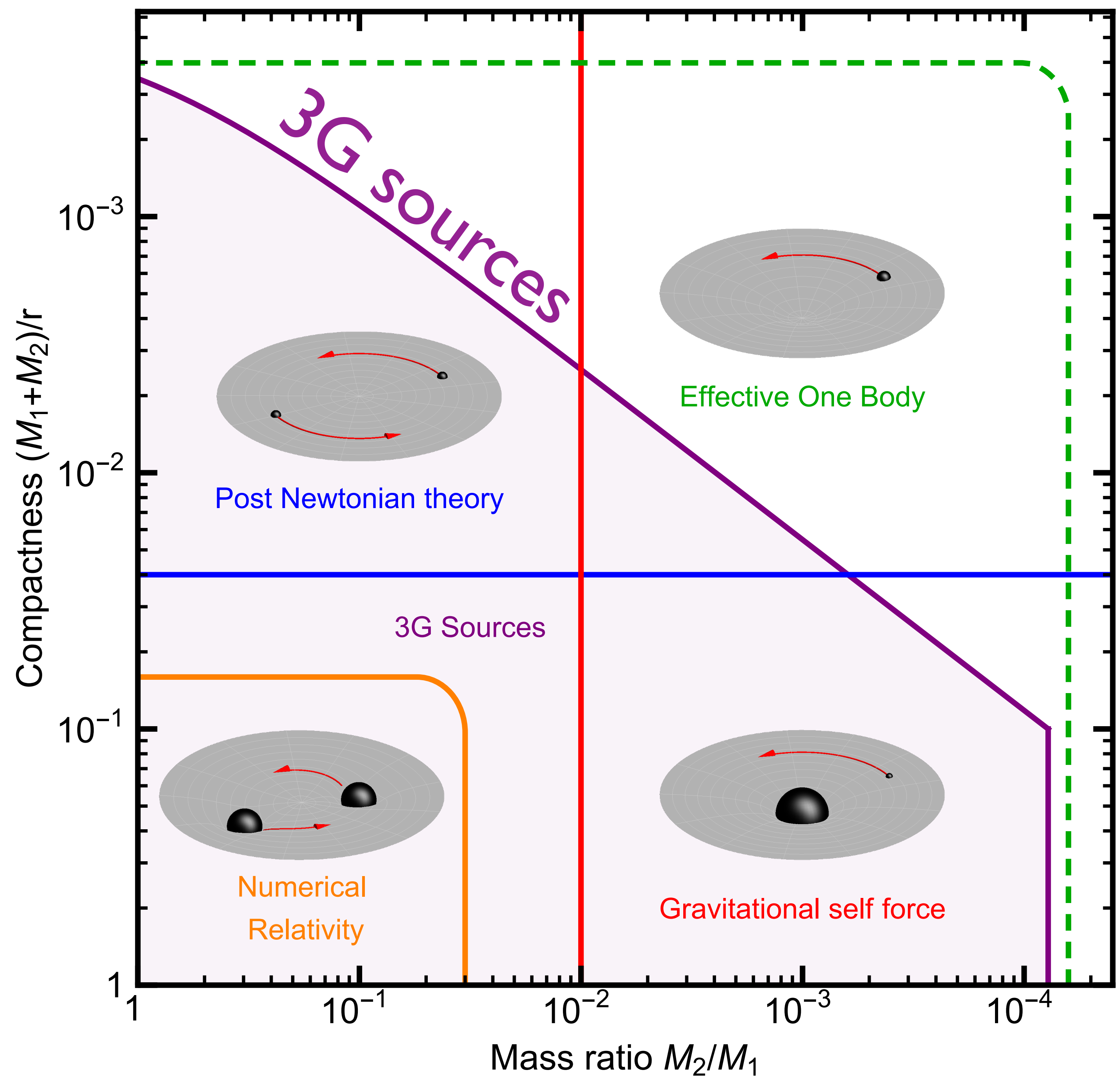
- Demands on waveform **accuracy** are **higher,** modeling is more **challenging.**

Need to solve 2-body problem in larger region of parameter space with 3G

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **GR** is **non-linear theory**.

- Einstein's field equations can be solved:
  - **approximately,** but **analytically** (**fast** way)
  - **"exactly"**, but **numerically** on supercomputers (**slow** way)



(credit: Van de Meent *for 3G report*)

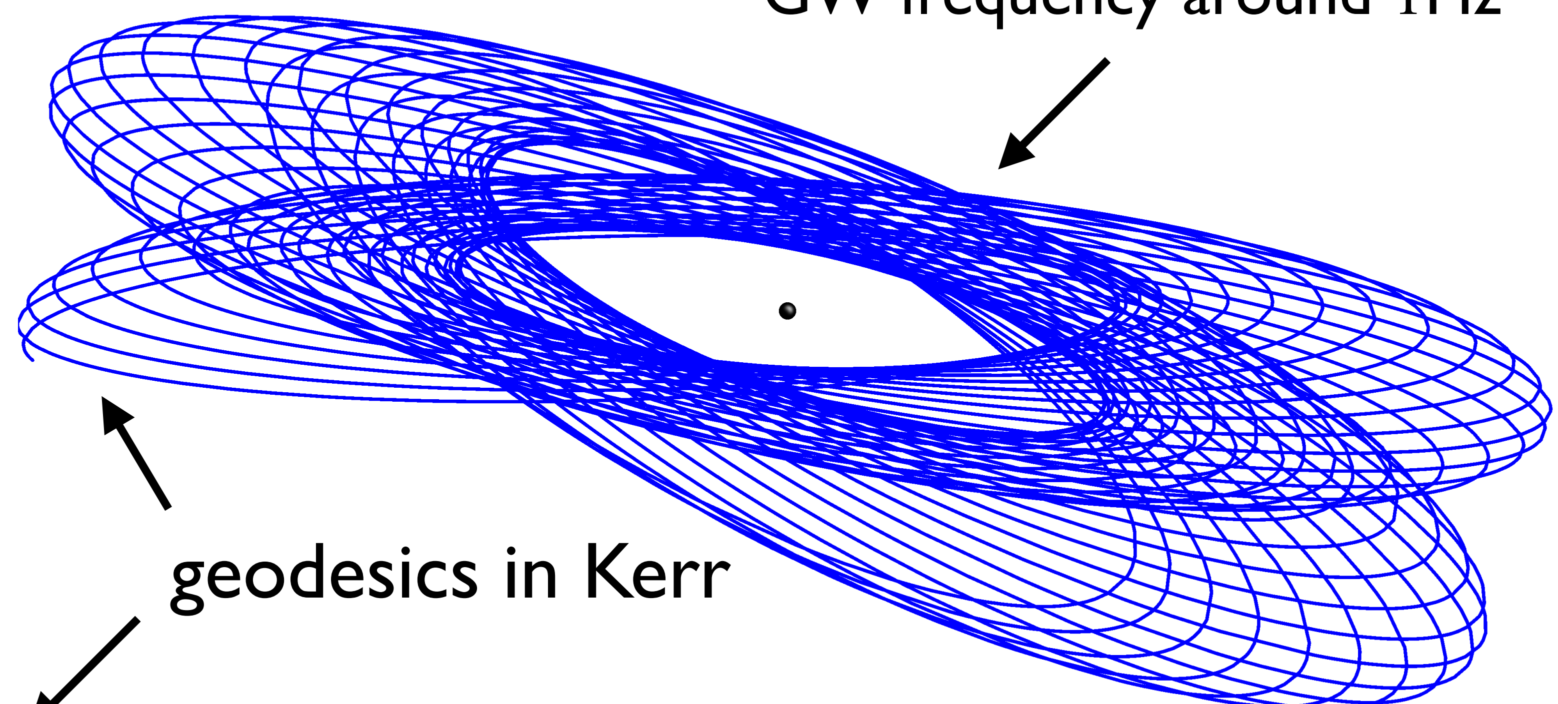
- **Synergy** between **analytical** and **numerical relativity** is **crucial.**

New sources with 3G detectors: intermediate-mass black-hole inspirals

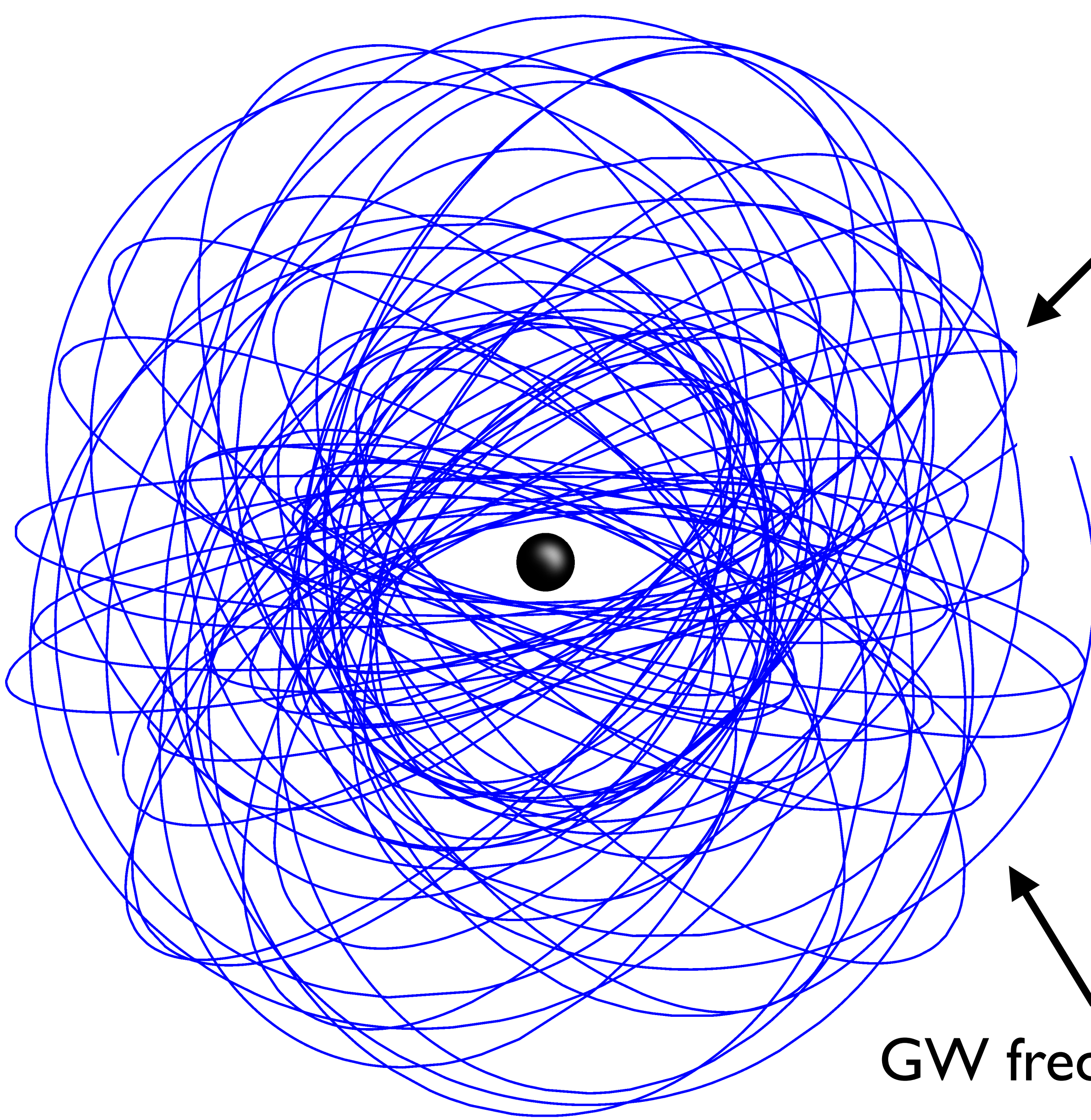
- central BH's spin = 0.9
- eccentricity = 0.5

$$M = 1000 M_{\odot}$$

GW frequency around 1Hz



geodesics in Kerr



GW frequency around 10 Hz

(credit: Van de Meent *for 3G report*)

- Sweeping in band for a **few thousand GW cycles,** probing **strong-field** gravity.
- **GSF** is likely to be important, we **need** to develop **accurate waveform models**.

SBHB Waveforms

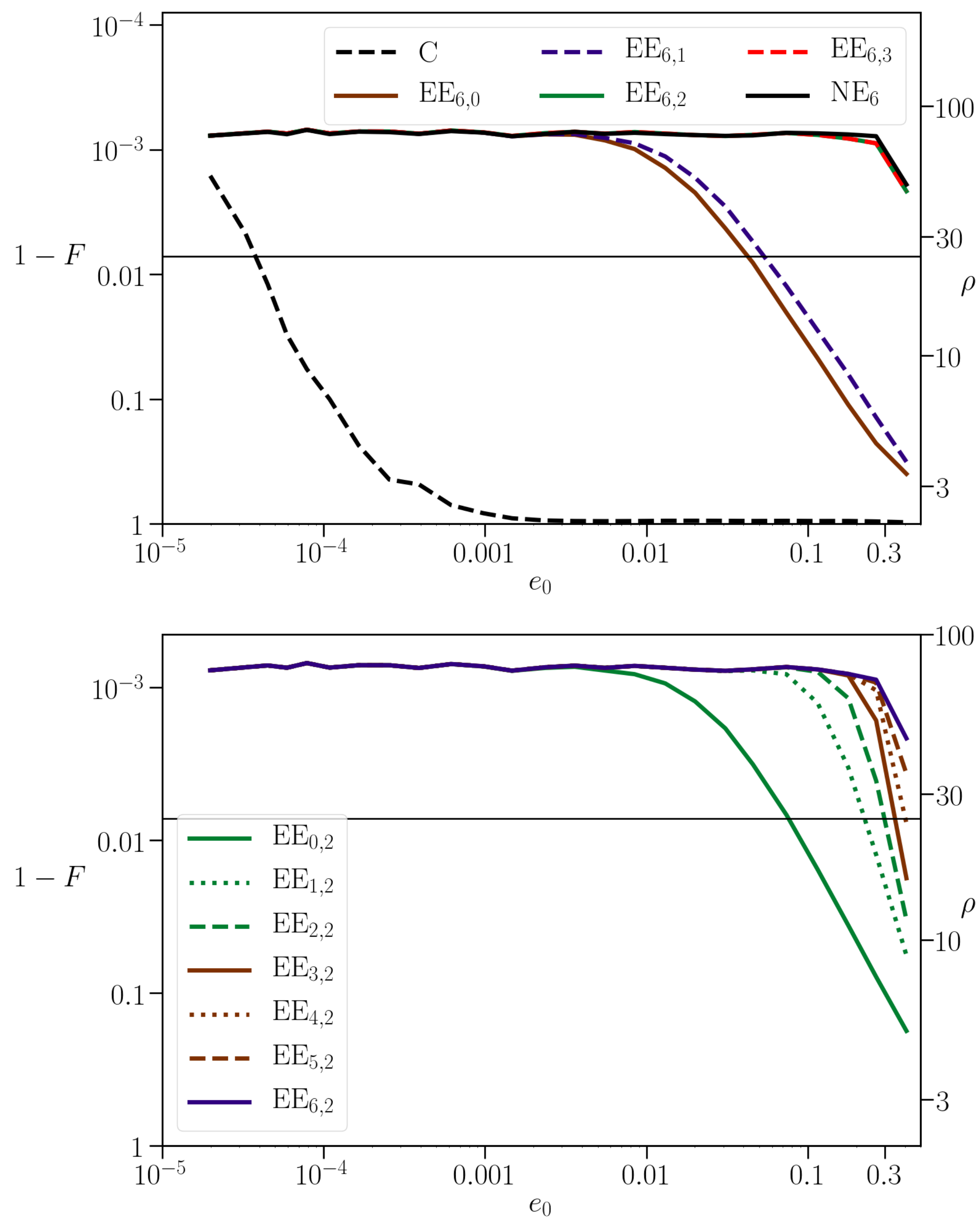
Several flavours of models have been proposed that can describe stellar black hole binary inspirals.

- Taylor F2-like eccentric models (e.g. Tanay+ 1602.03081), based on expanding PN phase and amplitudes for low eccentricities.
- Taylor T4-like eccentric models (e.g. Huerta+ 1609.05933, AK+ 1801.08542), based on PN expanding the phase evolution equation, and expanding amplitudes for low eccentricities.
- Moderately eccentric PN model (Moore & Yunes 1903.05203), based on solving for phase functions as a function of eccentricity.
- EOB eccentric model (Hinderer & Babak 1707.08426), based on a reparametrization of the equations of motion.
- EOB eccentric model (Cao & Han 1708.00166), based on treating eccentricity as a perturbation to the circular EOB evolution equations.

Since those signals lie in the high-frequency range of the LISA band, all those models need to be adapted to take into account the full LISA response.

SBHB Waveform Accuracy

Studies have looked at the convergence of the eccentricity expansion of the amplitude series, and compared circular to eccentric phasing.



(AK+ 1801.08542)

SBHB Waveform Accuracy

- Can we compare the convergence properties of different PN phasing flavours? Compare with EOB models? Are they all equivalent at 3PN order?
- What can we say about the convergence of PN eccentric phasing? What can we say about the convergence of the eccentricity expansion in the phase?
- What can we say about the PN requirements for multiband binaries? Can we achieve phase-locking between LISA and 3G detectors?

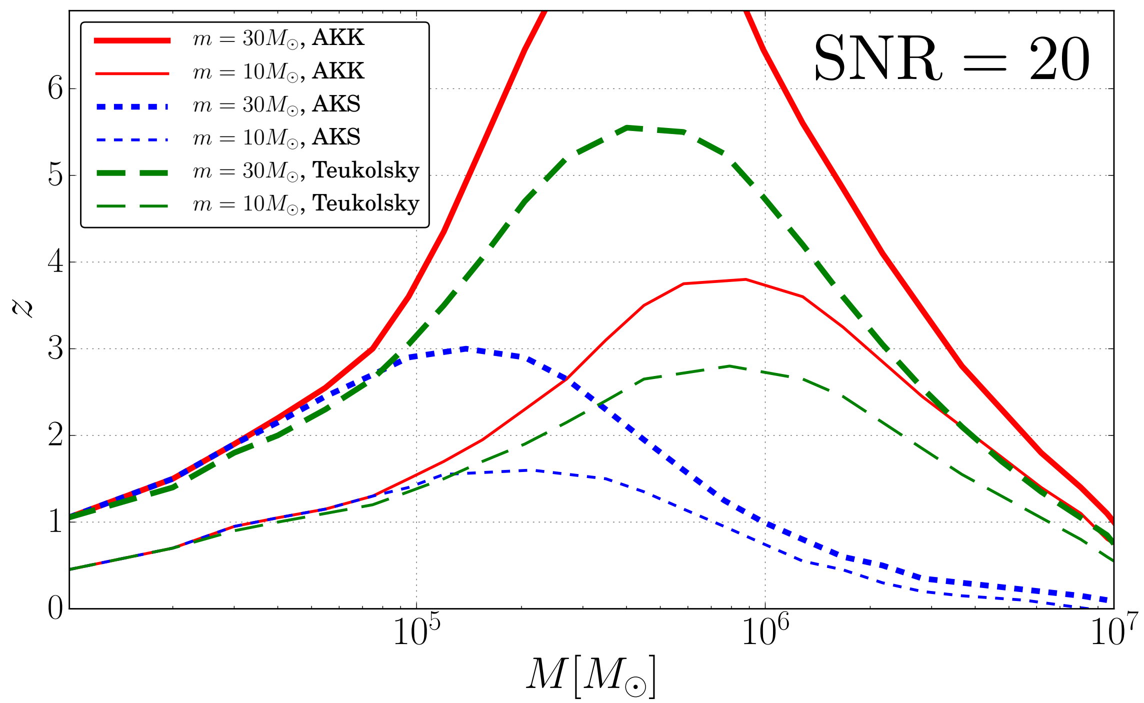
EMRI/IMRI Waveforms

Currently, three types of waveforms are being used to study extreme mass ratio inspirals.

- Analytic Kludge (Barack & Cutler gr-qc/0310125), based on treating the orbit as a Newtonian one with parameters adiabatically evolving.
- Numerical Kludge (Glampedakis+ gr-qc/0205033, Gair & Glampedakis gr-qc/0510129), based on combining relativistic orbits with PN fluxes.
- Fast Self-Forced Inspirals (van de Meent & Warburton 1802.05281), based on reformulating self-force corrected equations of motion using near identity transforms.

See high mass ratios panel tomorrow for more details.

LISA sources and SNR: extreme mass ratio inspirals



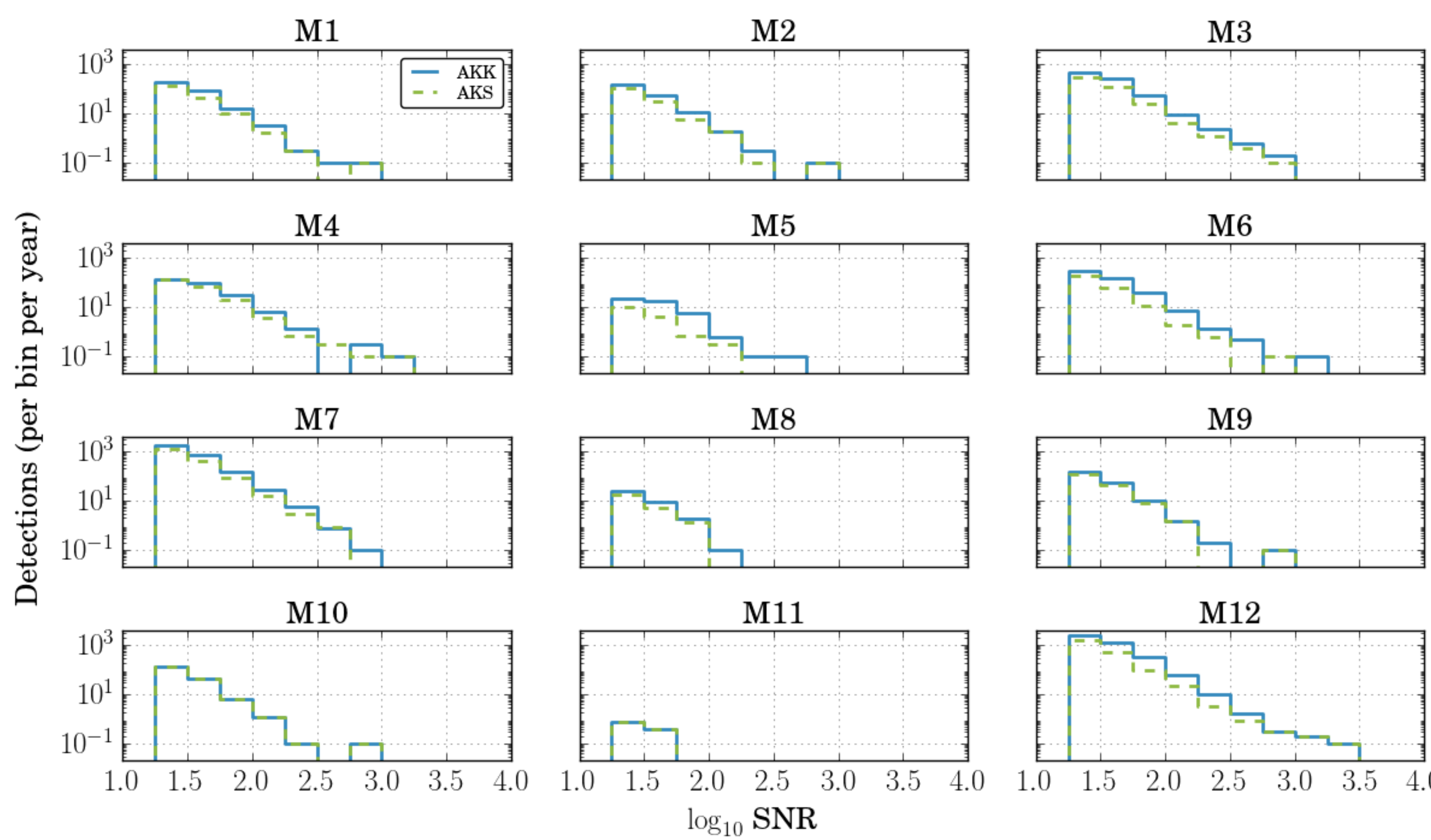
SNR=20 horizons for waveforms:

- Analytic Kludge, termination at Schwarzschild ISCO
- Analytic Kludge, termination at Kerr ISCO
- Teukolsky fluxes

[Babak&al 2017]

Rate and SNR for12 different astrophysical models:

[Babak&al 2017]



LISA sources and SNR: extreme mass ratio inspirals

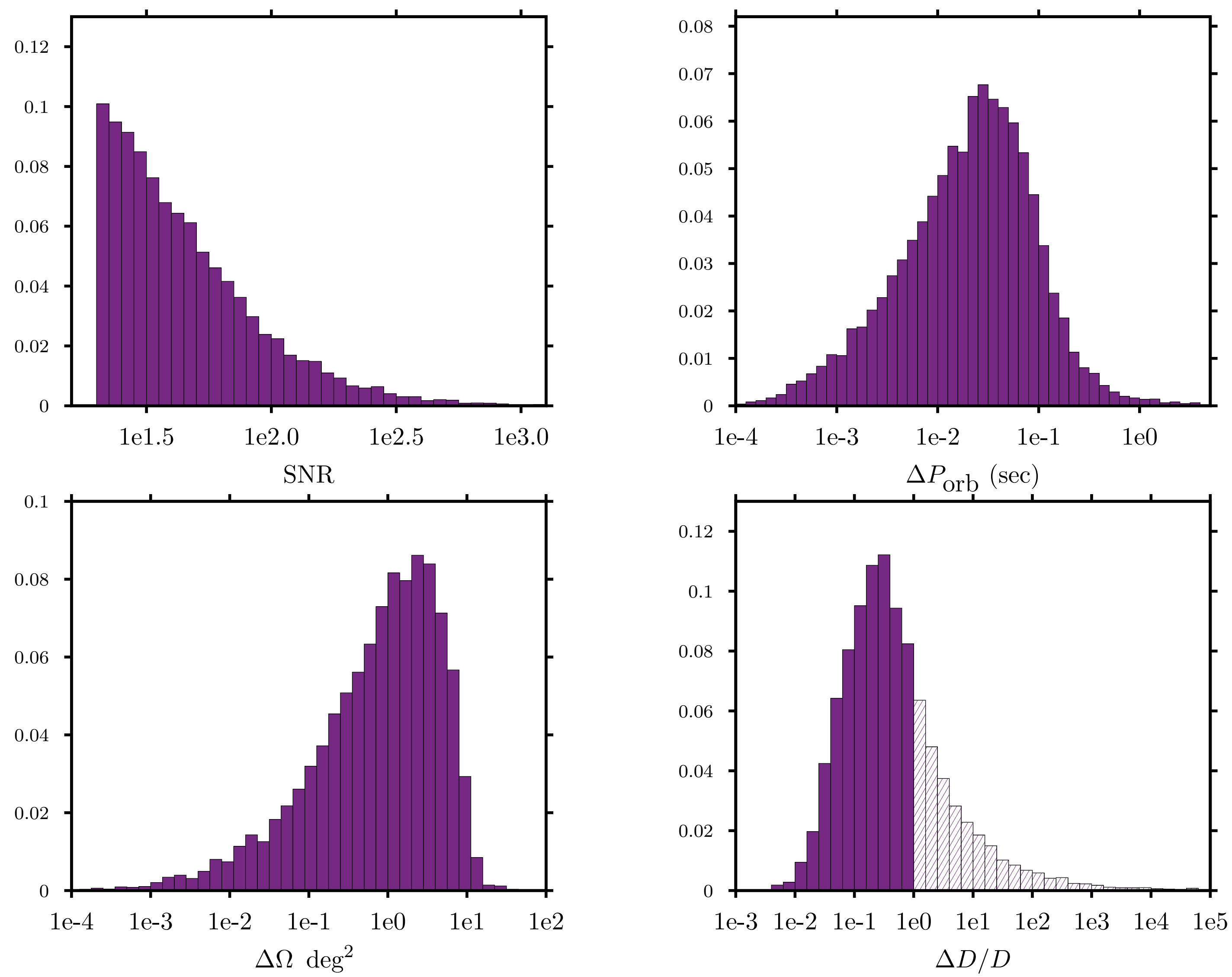
[Babak&al 2017]

Model	Mass function	MBH spin	Cusp erosion	M–σ relation	$N_{\rm p}$	CO mass $[M_\odot]$	EMRI rate $[\mathrm{yr}^{-1}]$ Total	Detected (AKK)	Detected (AKS)
M1	Barausse12	a98	yes	Gultekin09	10	10	1600	294	189
M2	Barausse12	a98	yes	KormendyHo13	10	10	1400	220	146
M3	Barausse12	a98	yes	GrahamScott13	10	10	2770	809	440
M4	Barausse12	a98	yes	Gultekin09	10	30	520 (620)	260	221
M5	Gair10	a98	no	Gultekin09	10	10	140	47	15
M6	Barausse12	a98	no	Gultekin09	10	10	2080	479	261
M7	Barausse12	a98	yes	Gultekin09	0	10	15800	2712	1765
M8	Barausse12	a98	yes	Gultekin09	100	10	180	35	24
M9	Barausse12	aflat	yes	Gultekin09	10	10	1530	217	177
M10	Barausse12	a0	yes	Gultekin09	10	10	1520	188	188
M11	Gair10	a0	no	Gultekin09	100	10	13	1	1
M12	Barausse12	a98	no	Gultekin09	0	10	20000	4219	2279

TABLE I. List of EMRI models considered in this work. Column 1 defines the label of each model. For each model we specify the MBH mass function (column 2), the MBH spin model (column 3), whether we consider the effect of cusp erosion following MBH binary mergers (column 4), the M–σ relation (column 5), the ratio of plunges to EMRIs (column 6), the mass of the COs (column 7); the total number of EMRIs occurring in a year up to $z = 4.5$ (column 8; for model M4 we also show the total rate per year up to $z = 6.5$); the detected EMRI rate per year, with AKK (column 9) and AKS (column 10) waveforms.

LISA sources and SNR: galactic binaries

SNR and Fisher PE [Robson-Cornish 2017]



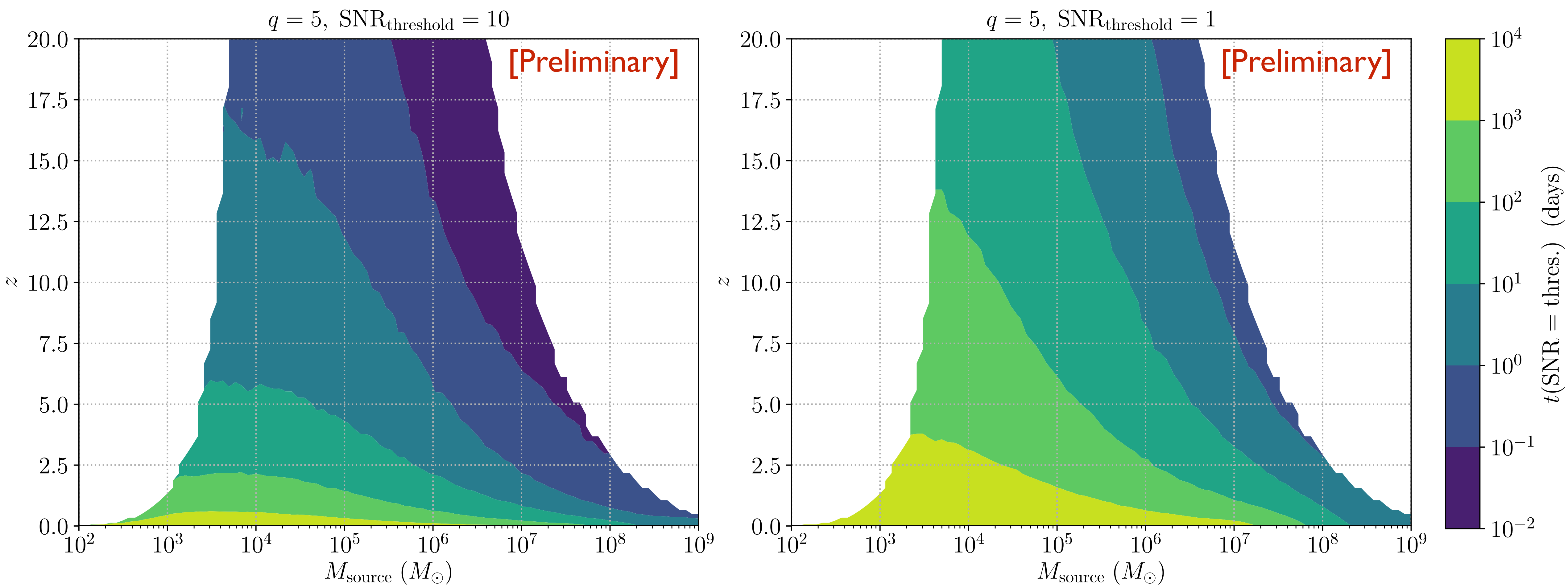
	6 mo	1 yr	2 yr	4 yr
# detected	6,590	11,142	18,281	29,059
2D mapped	104	1,065	4,138	6,304
3D mapped	19	129	1,010	2,373
$\mathcal{M}$ measured	233	737	4,432	10,770

Waveforms

- Quasi-monochromatic with $f,\ \dot{f},\ \ddot{f}$
- Astrophysical effects: mass transfer, perturbing third body

Length of LISA signals: for the observer

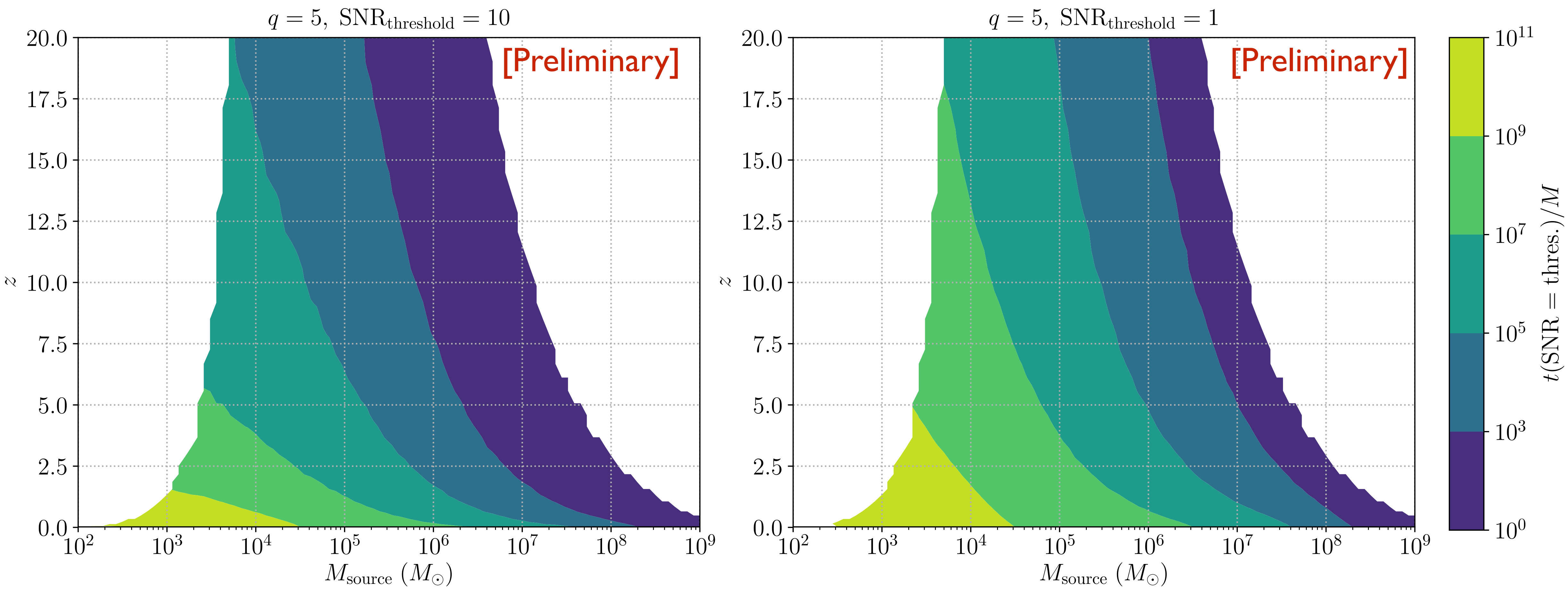
t(SNR): time to merger left when the signals has accumulated a given SNR



- SNR=10 as the time to merger left when we can claim detection
- SNR=1 assuming everything before that point can be neglected in PE

Length of LISA signals: for waveform models

t(SNR)/M: same length of signal, but seen in geometric units for waveforms models



- SNR=10 as the time to merger left when we can claim detection
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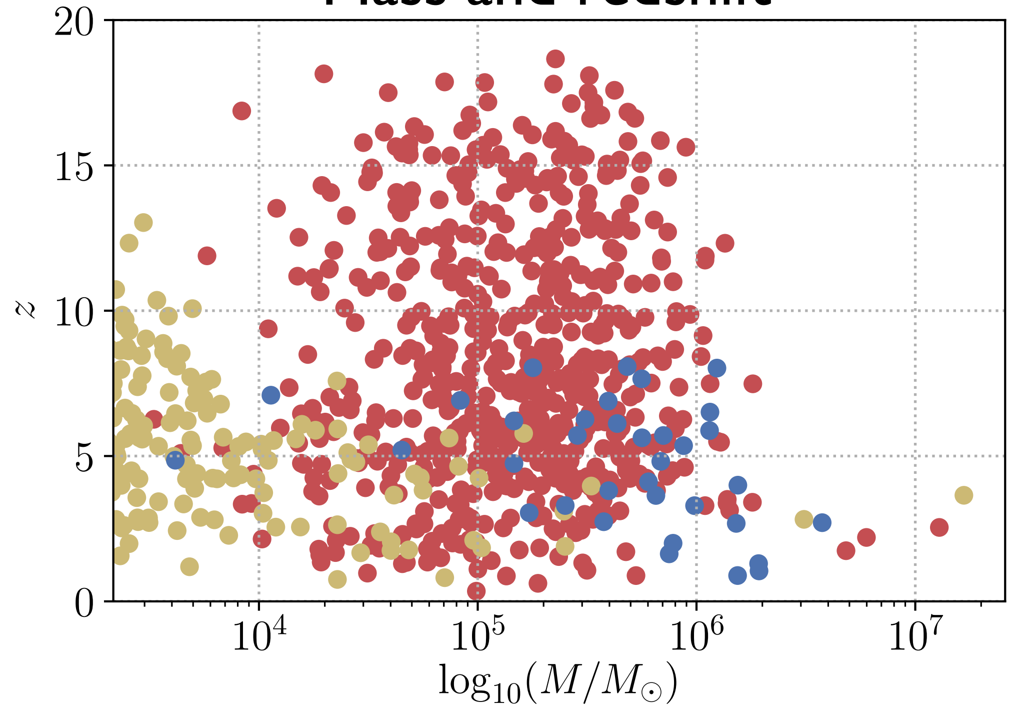
LISA: simulated catalog for MBHB astrophysical models

[Barausse 2012]

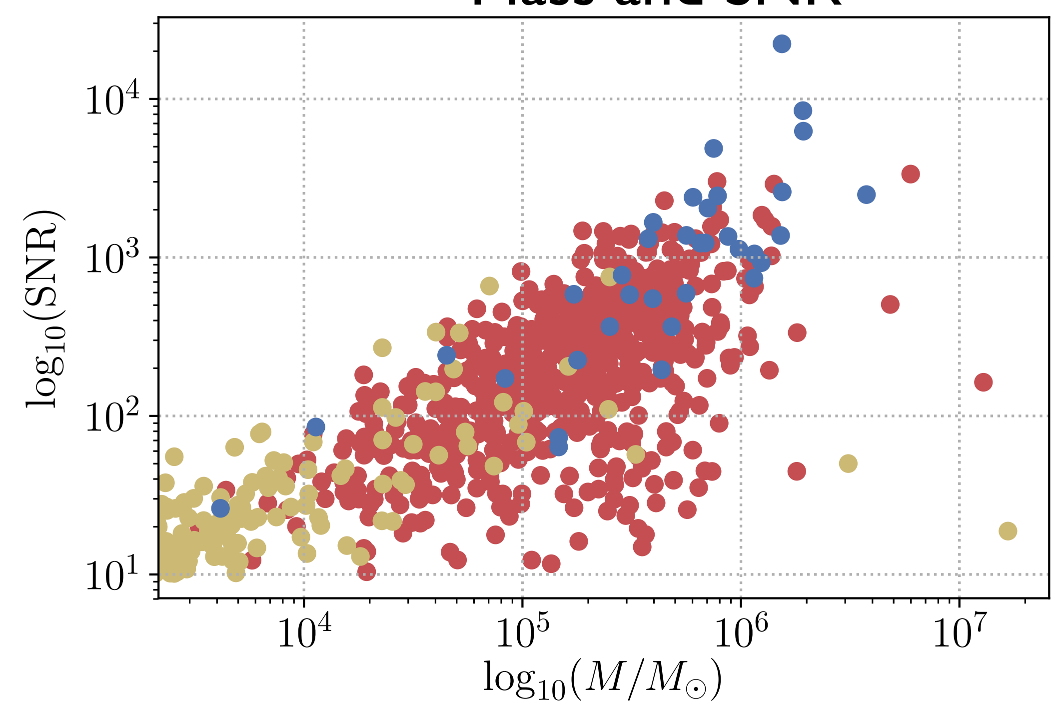
Astrophysical models:

- Heavy seeds - delay (35 / 5yrs)
- Light seeds - no delay (627 / 5yrs)
- PopIII seeds - delay (189 / 5yrs)

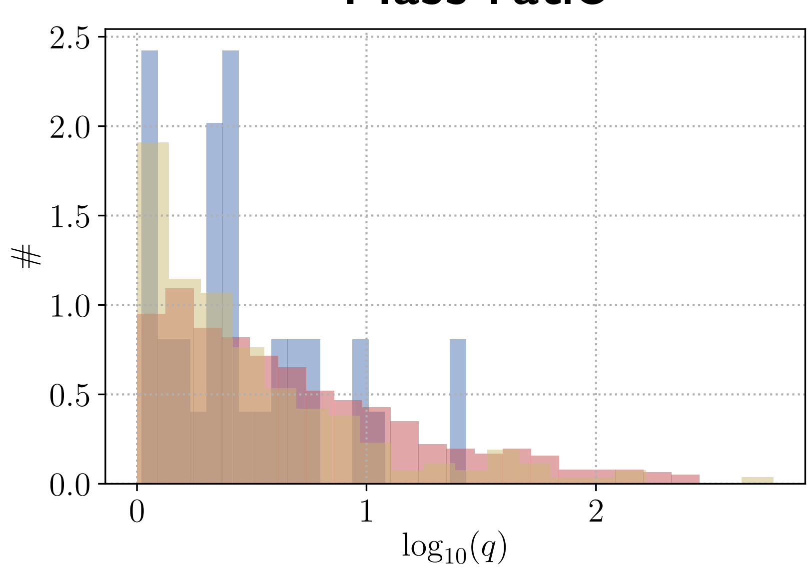
Mass and redshift



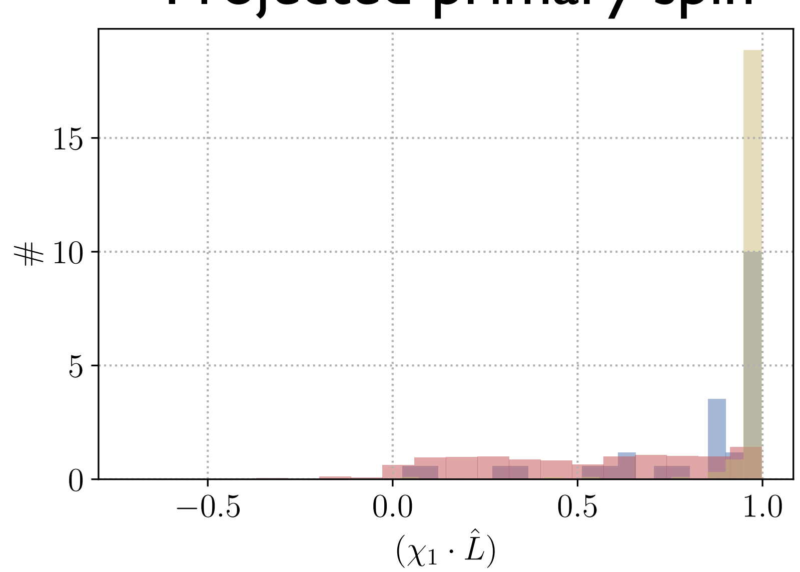
Mass and SNR



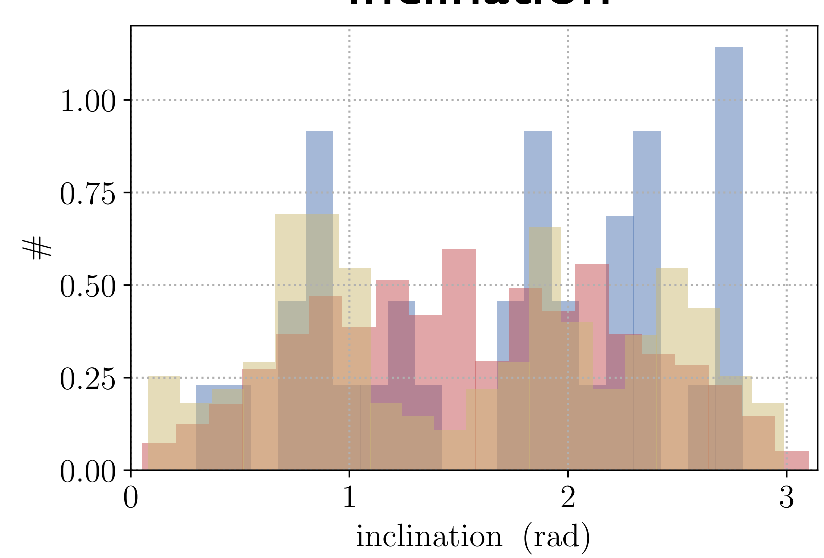
Mass ratio



Projected primary spin



Inclination



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[Barausse 2012]

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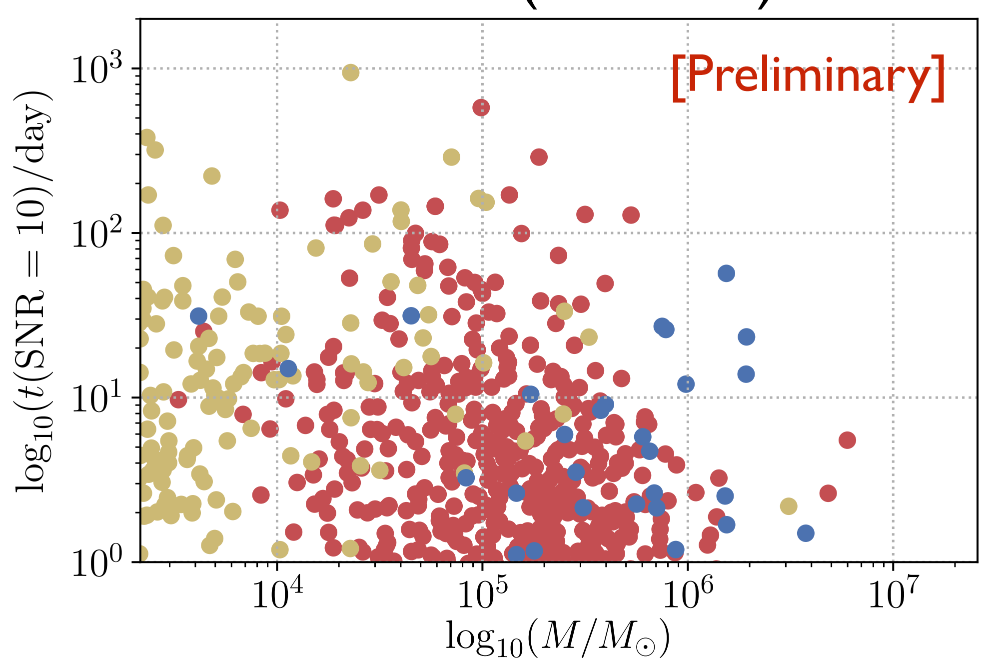
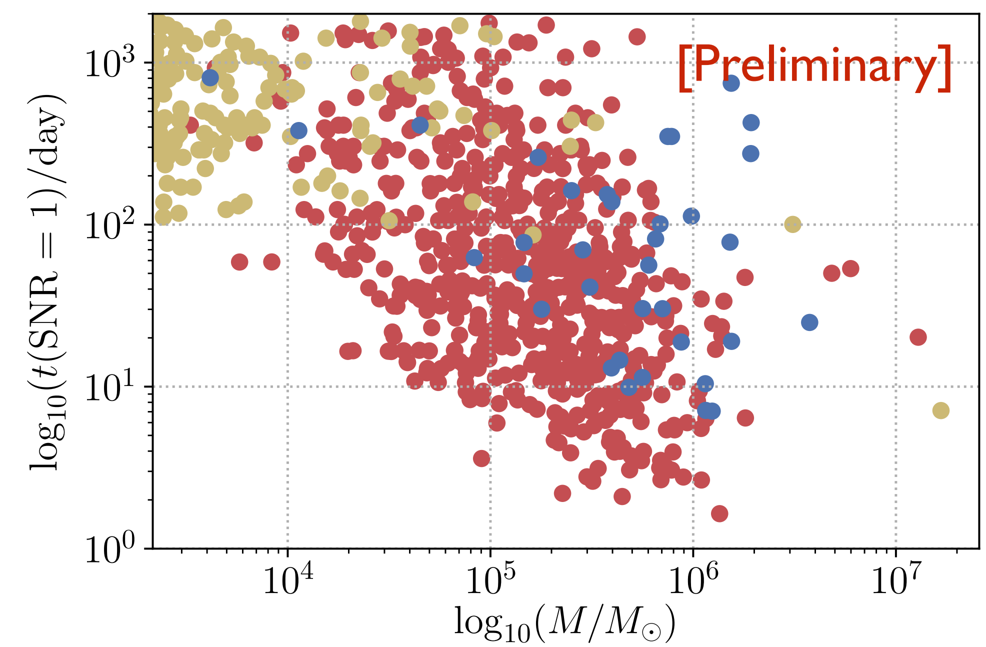
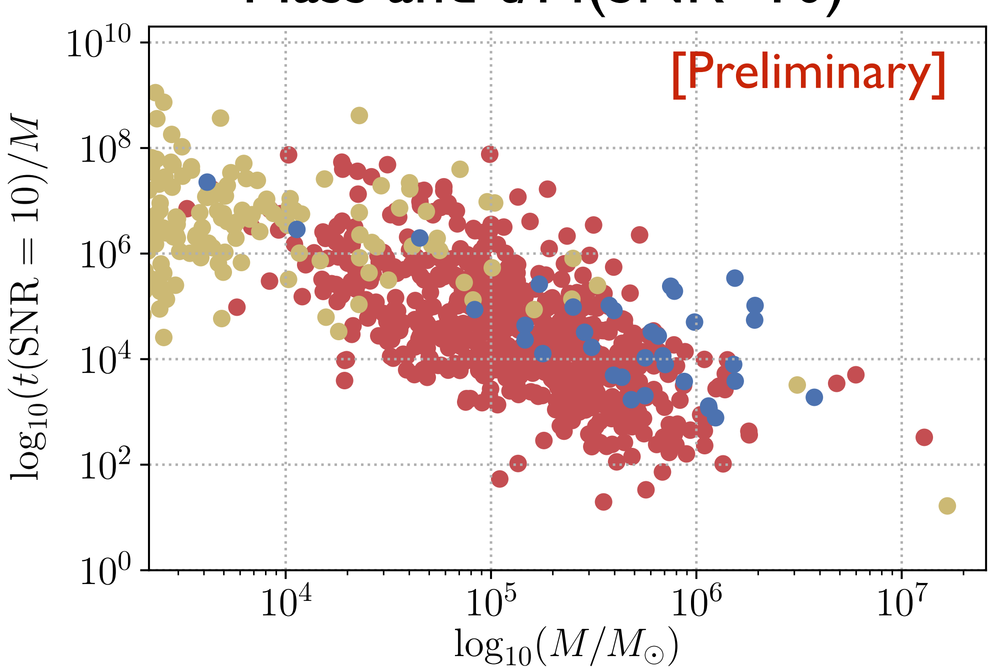
Mass and t(SNR=10)



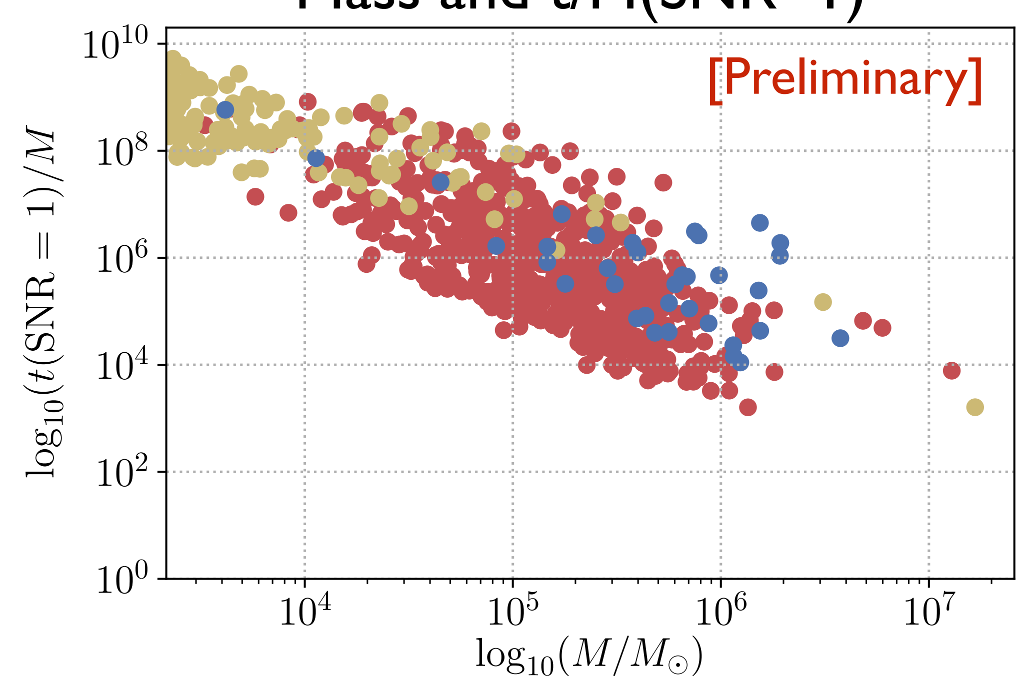
Mass and t(SNR=1)



Mass and t/M(SNR=10)



Mass and t/M(SNR=1)



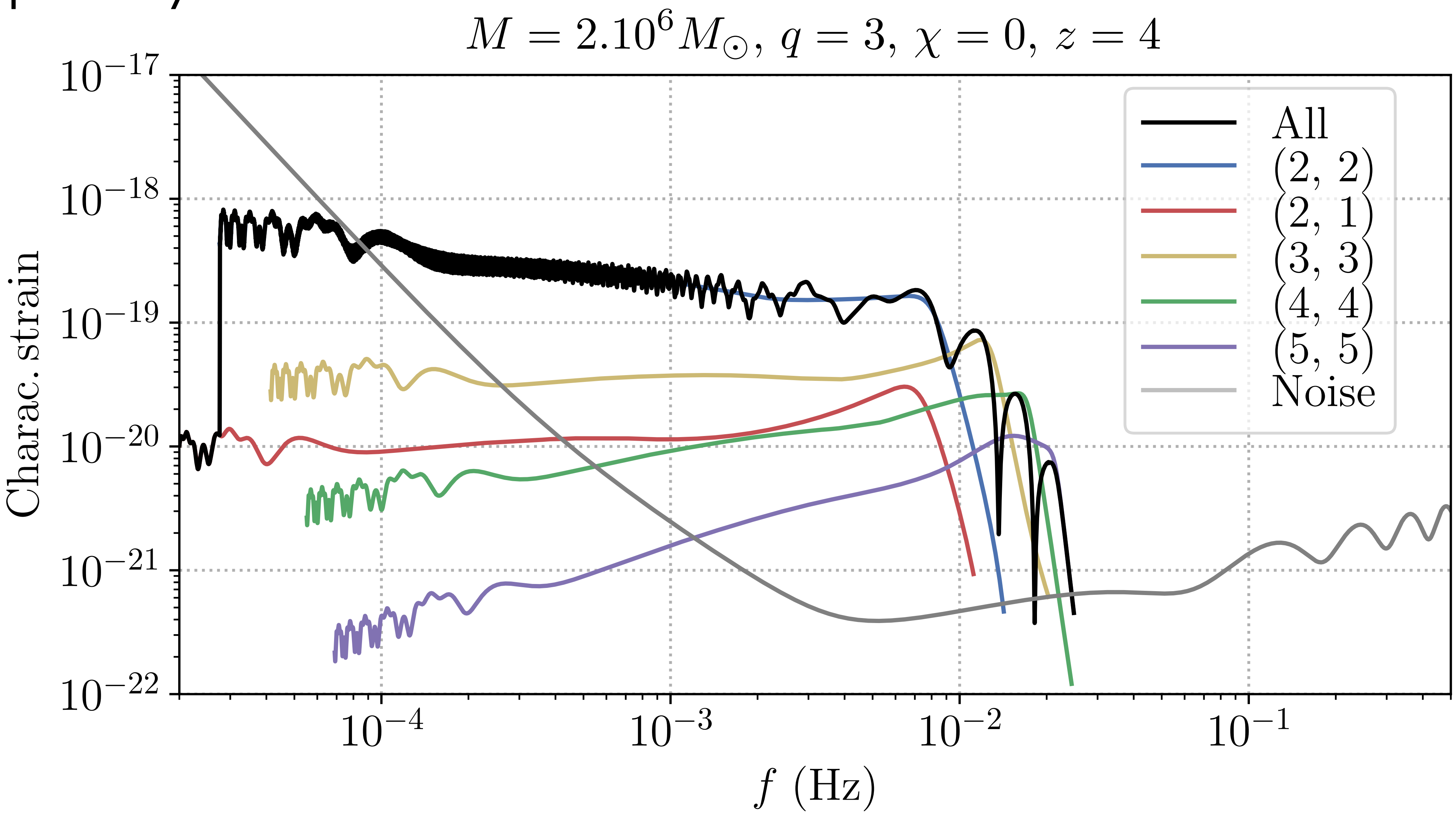
Higher harmonics in the waveform

The role of higher harmonics

$$h_+ - ih_\times = \sum {}_{-2}Y_{\ell m}(\iota, \varphi) h_{\ell m}$$

$${}_{-2}Y_{\ell m}(\iota, \varphi) \propto e^{im\varphi}$$

- Distance/inclination degeneracy broken
- Phase independently measured



Bayesian analysis

Response Code

- Pre-LDC FD LISA response C code, implementing [Marsat&Baker arXiv/1806.10734]
- Same as pyFDresponse in LDC, used here at leading order

Waveforms

- Non-spinning waveforms with higher modes: EOBNRv2HM
- Reduced order model

Likelihood cost
Single mode h22: 1-3ms
5 modes hlm: 15ms

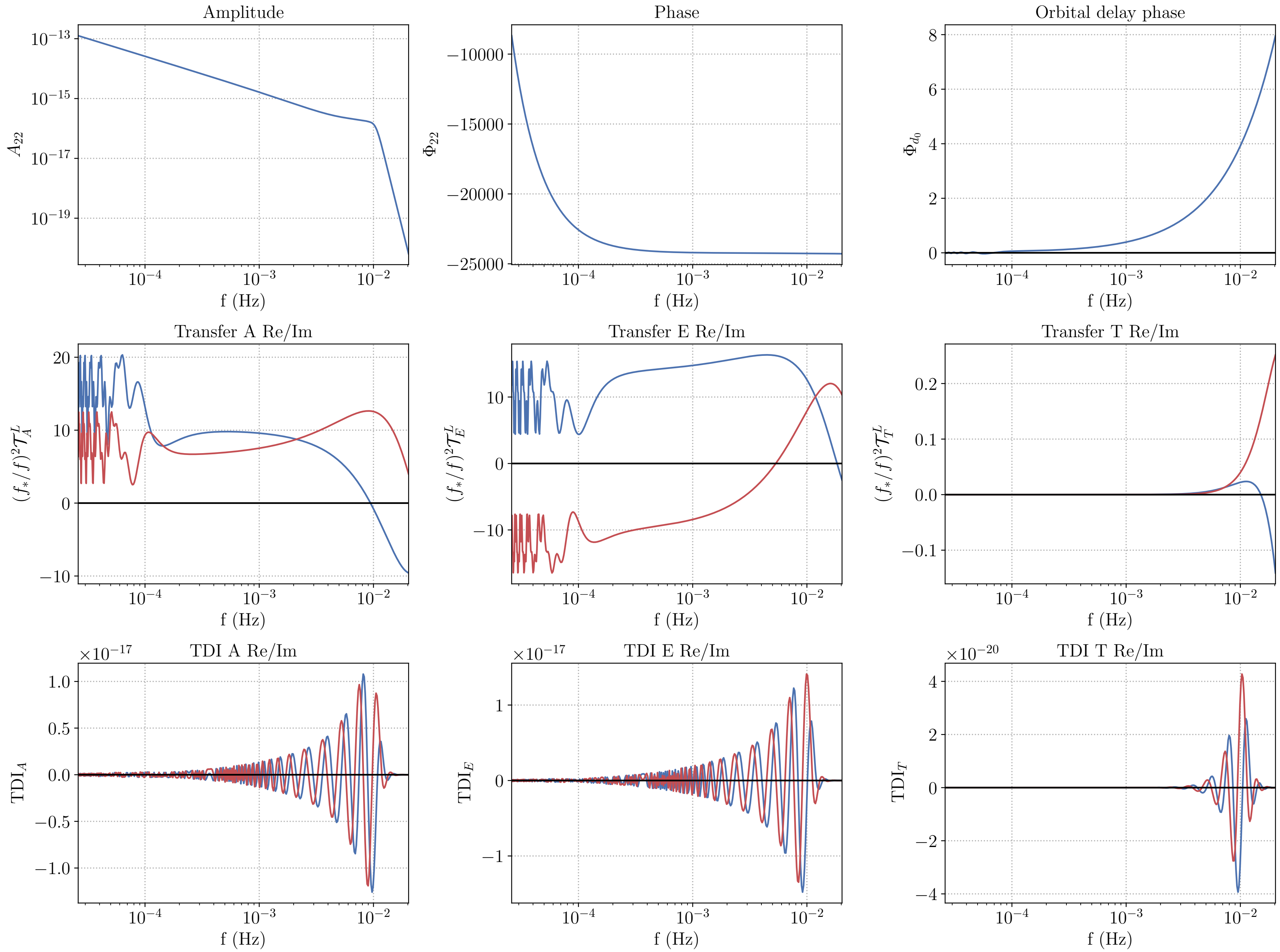
Likelihood evaluation

- Setting the noise realization to 0
- Amplitude/Phase sparse representation, inner products mode-by-mode

Bayesian samplers

- MultiNest (Bambi implementation) [Feroz&al 2009]
- Parallel-tempering MCMC with differential evolution [Baker]

SMBH analysis setting



Sources

- Plausible SMBH sources at z=4
- Masses $M = 2 \cdot 10^6\ M_\odot,\ q = 2$
- Vary orientation

SNR

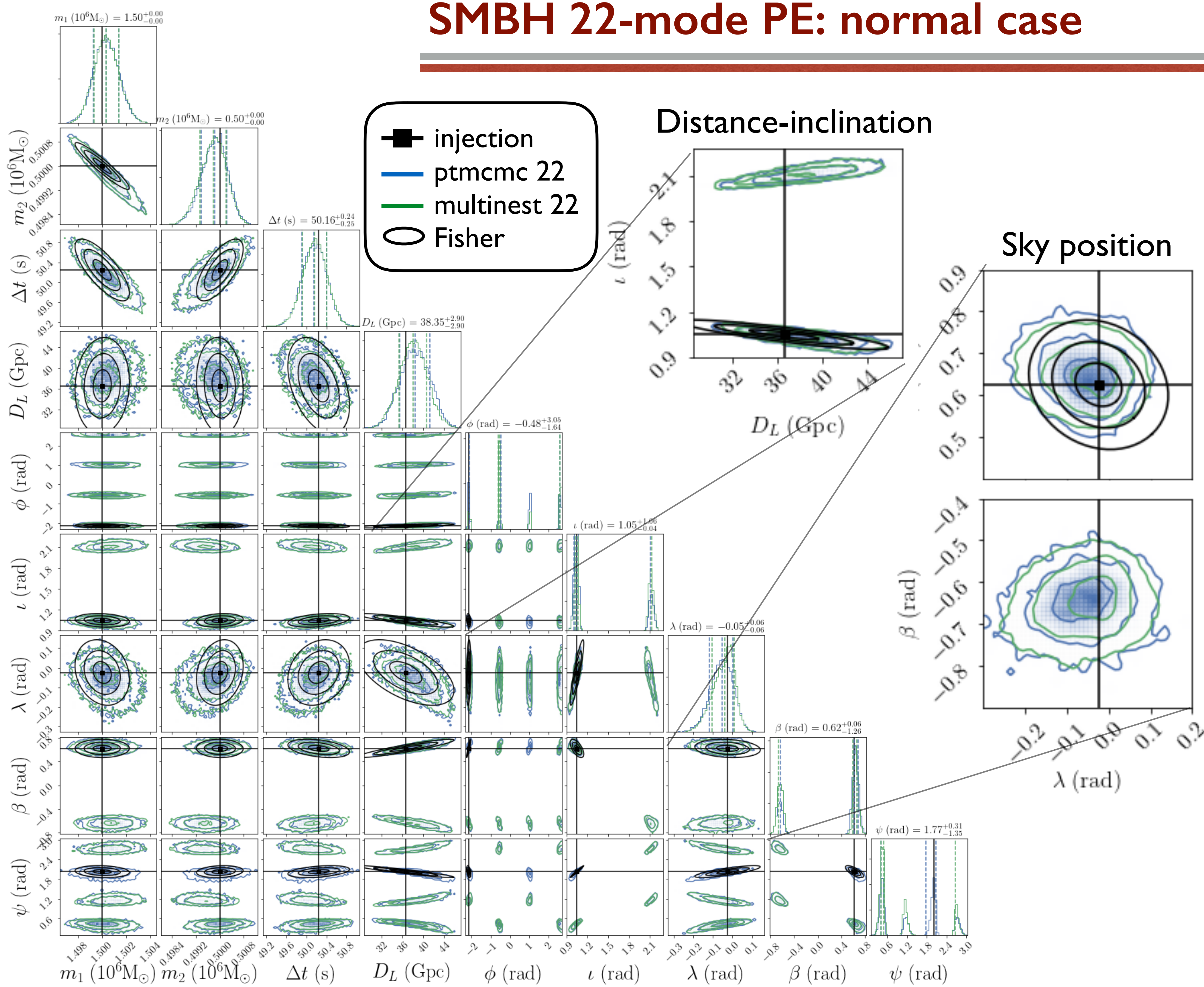
	I	II
22	857	645
HM	945	666

Evaluations

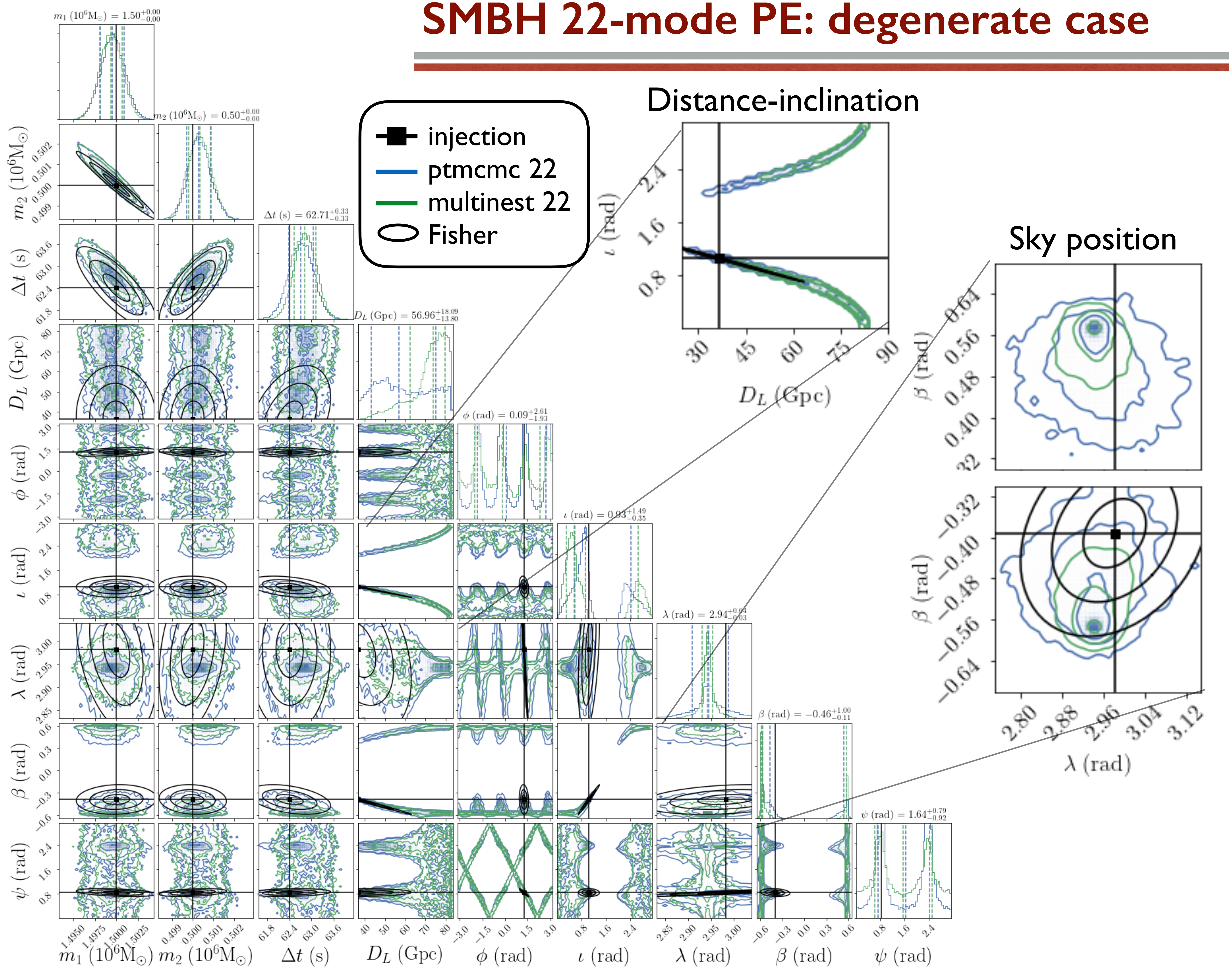
Multinest: $120 \cdot 10^6$

PTMCMC: $400 \cdot 10^6$

SMBH 22-mode PE: normal case



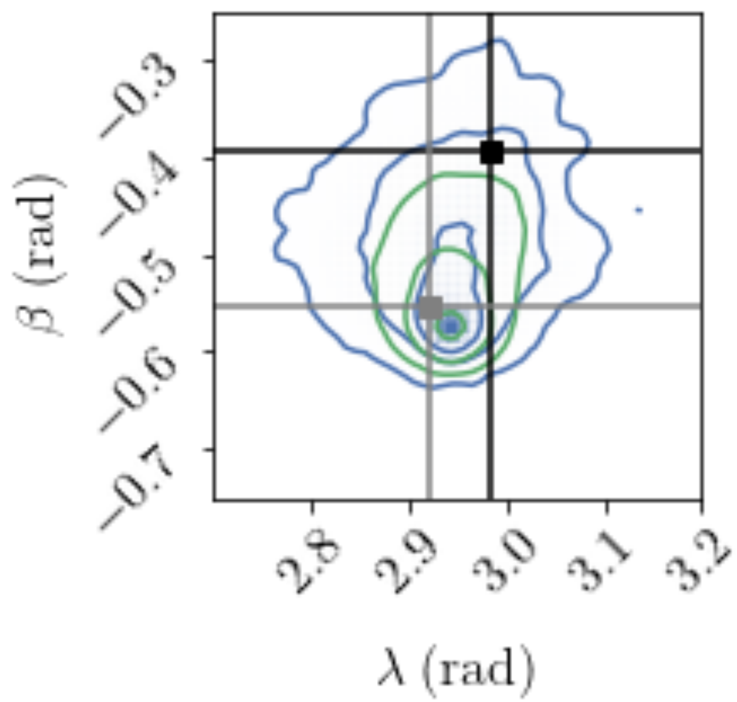
SMBH 22-mode PE: degenerate case



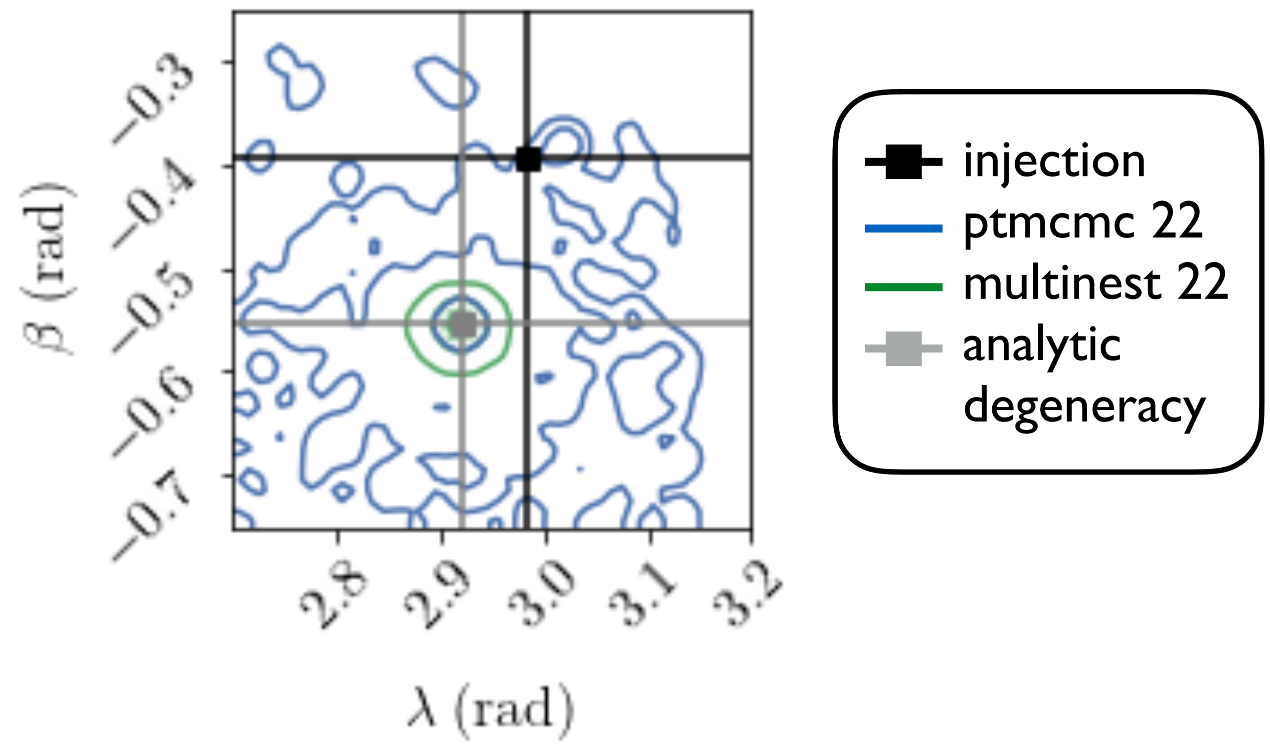
Understanding degeneracies

A projection effect for the marginal posterior

Sky, full likelihood 22-mode



Sky, 22-mode, ignoring LISA motion and pinning masses and time



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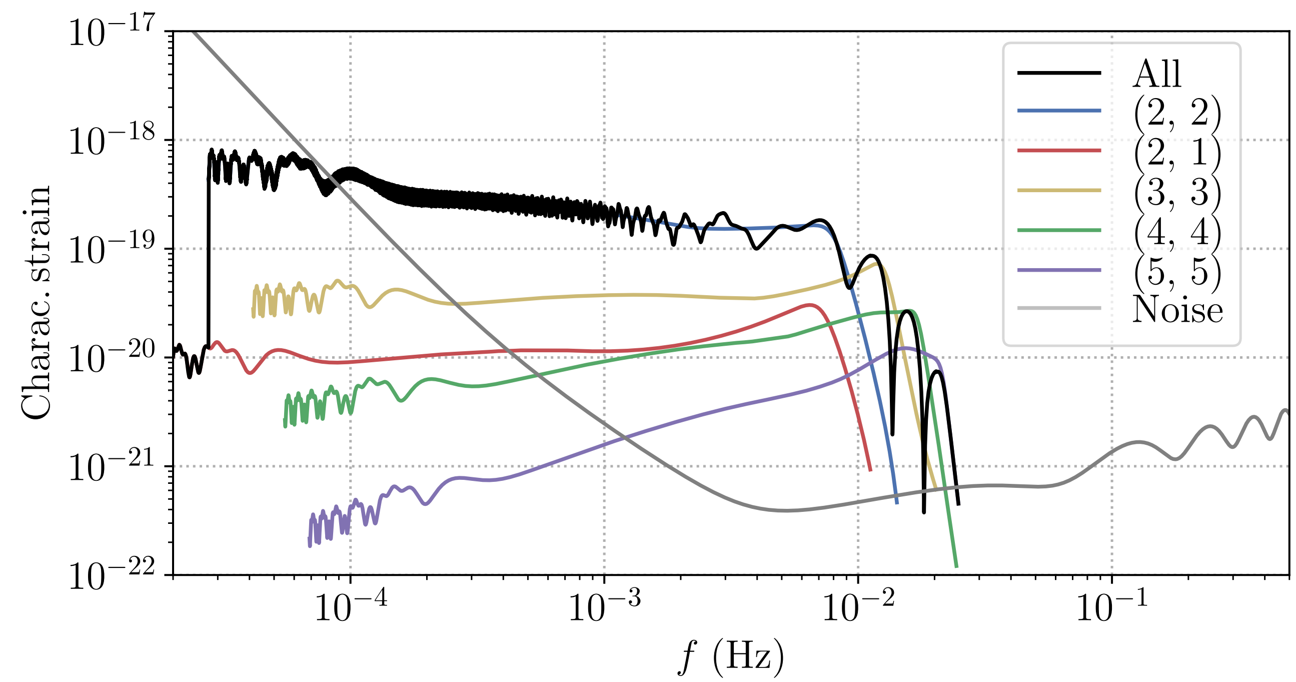
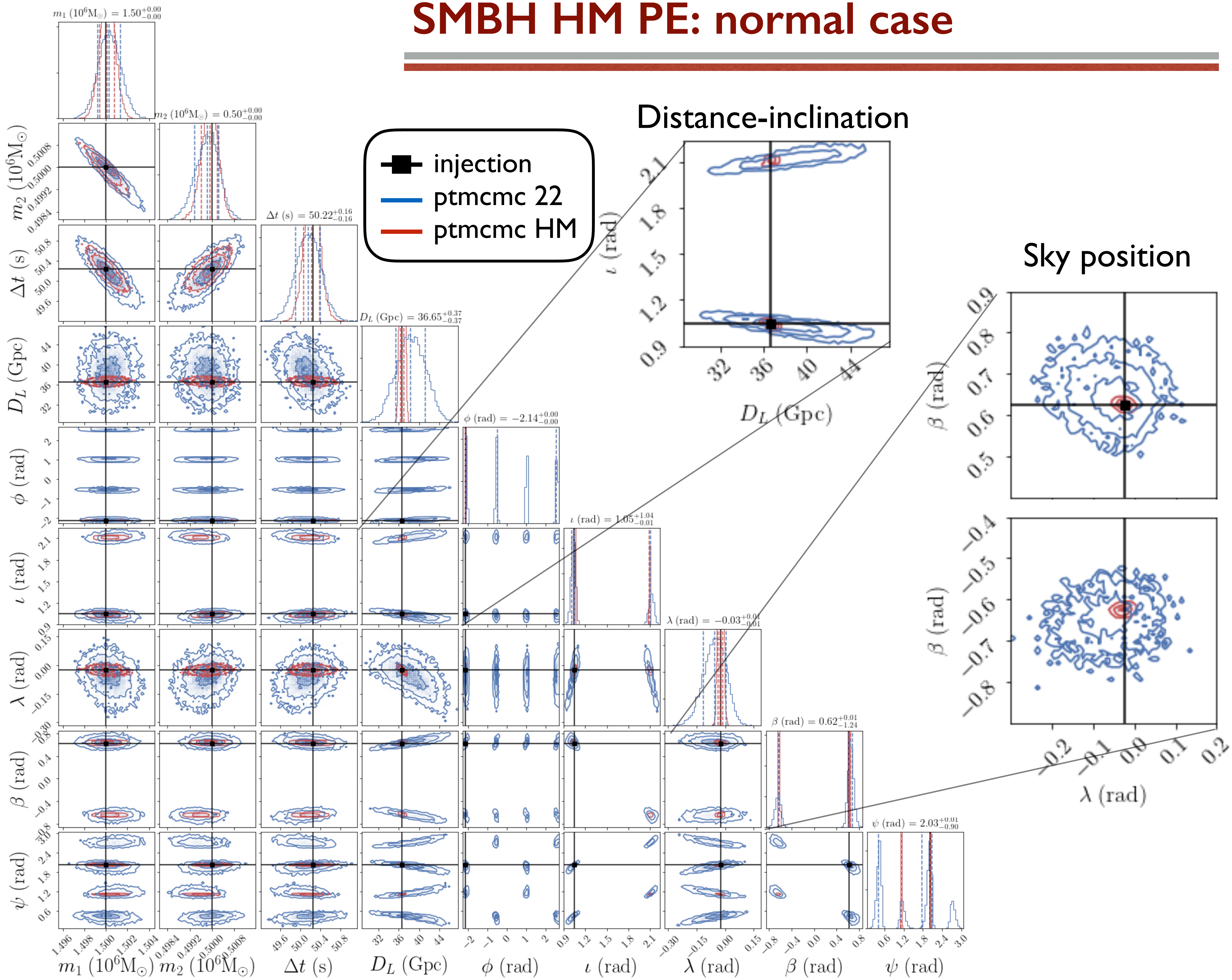
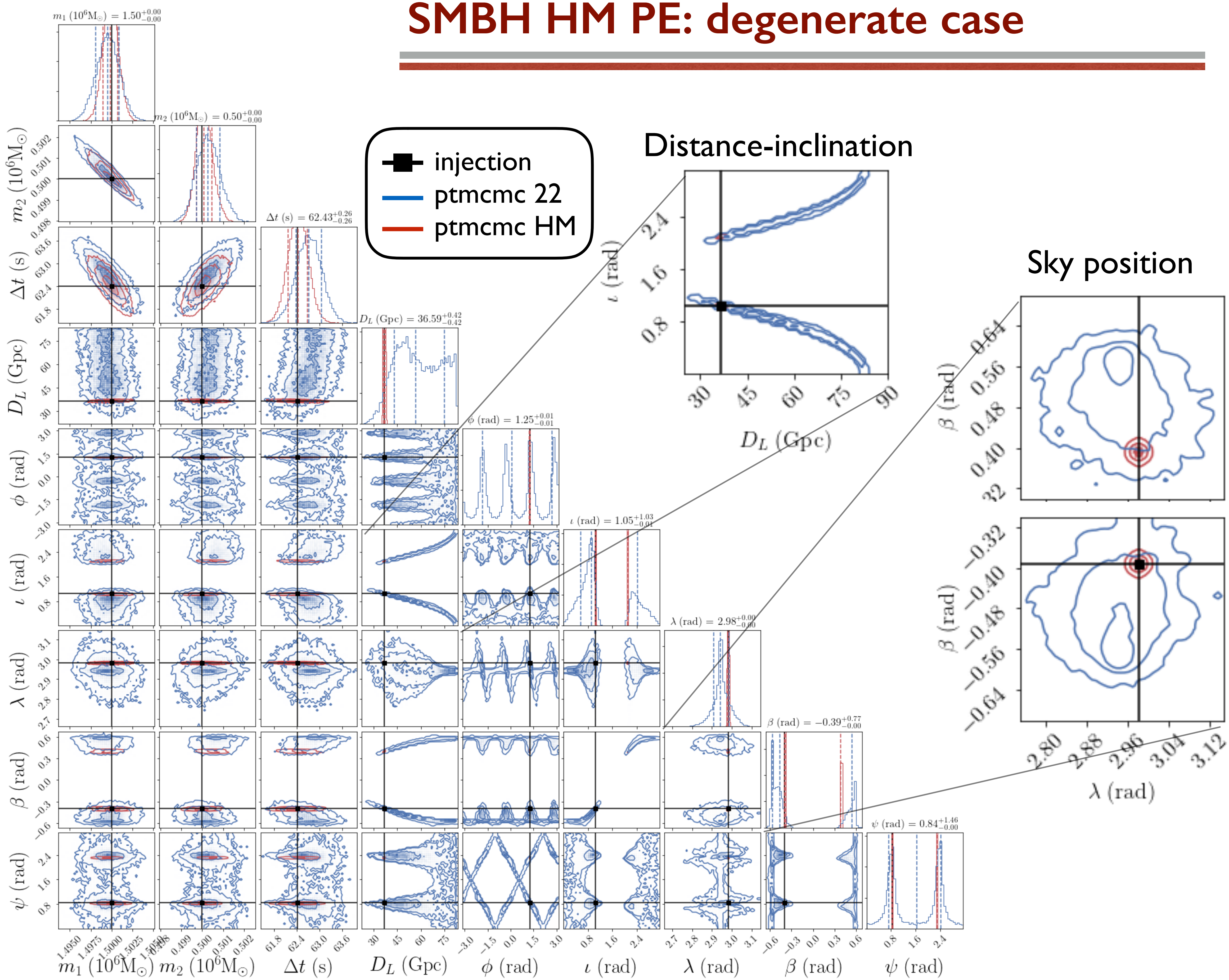


SMBH HM PE: normal case



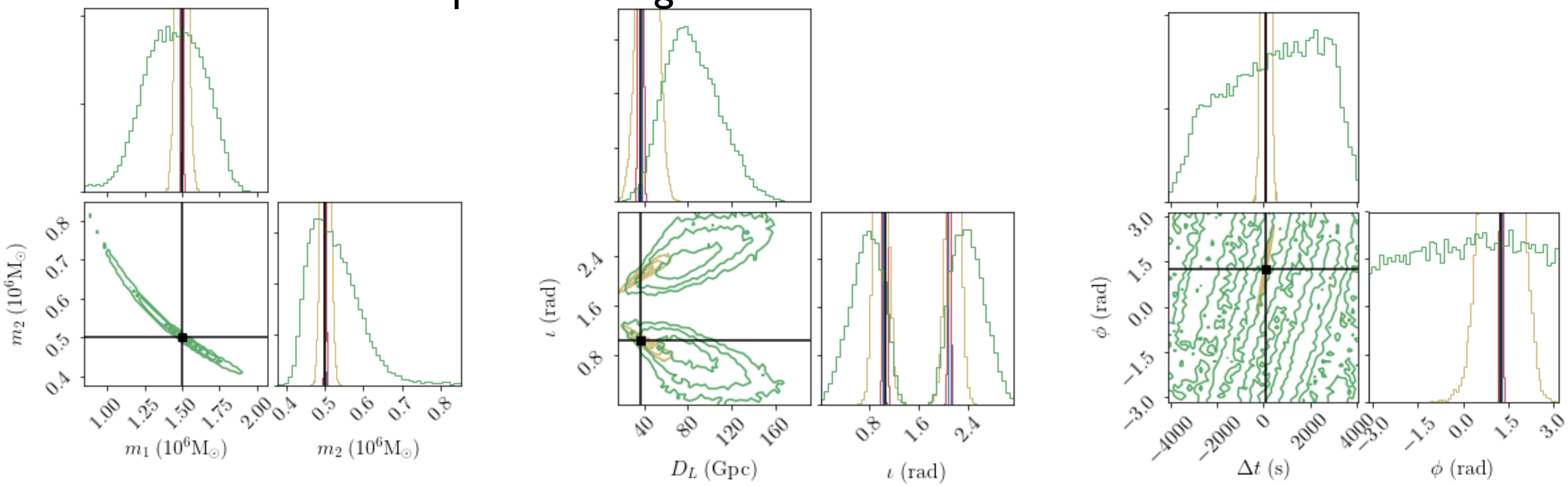
SMBH HM PE: degenerate case



SMBH PE: accumulation of information with time

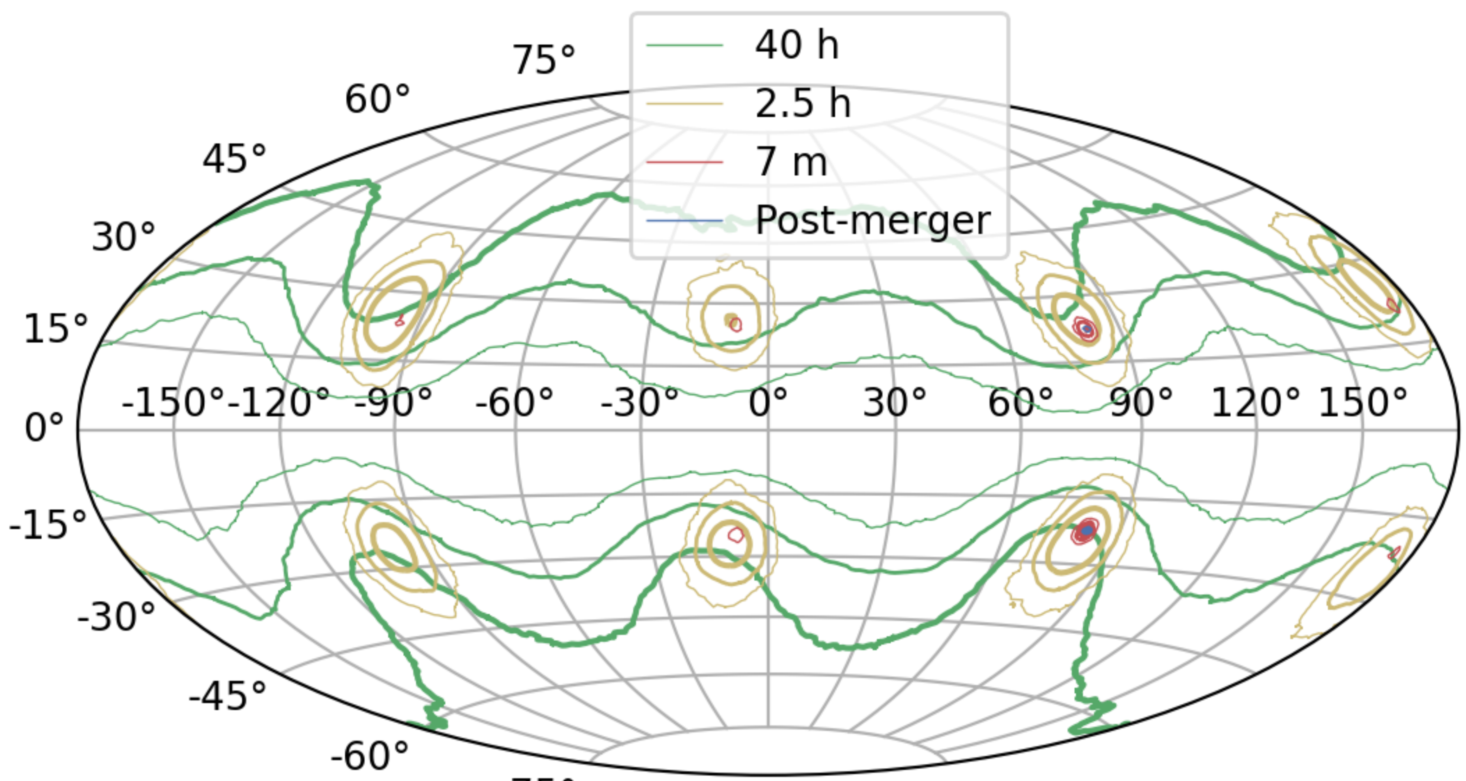
Method

- Represent a cut in time-to-merger by a cut in frequency, becomes inaccurate at merger
- Use PTMCMC sampler with higher modes



LISA-frame sky position:

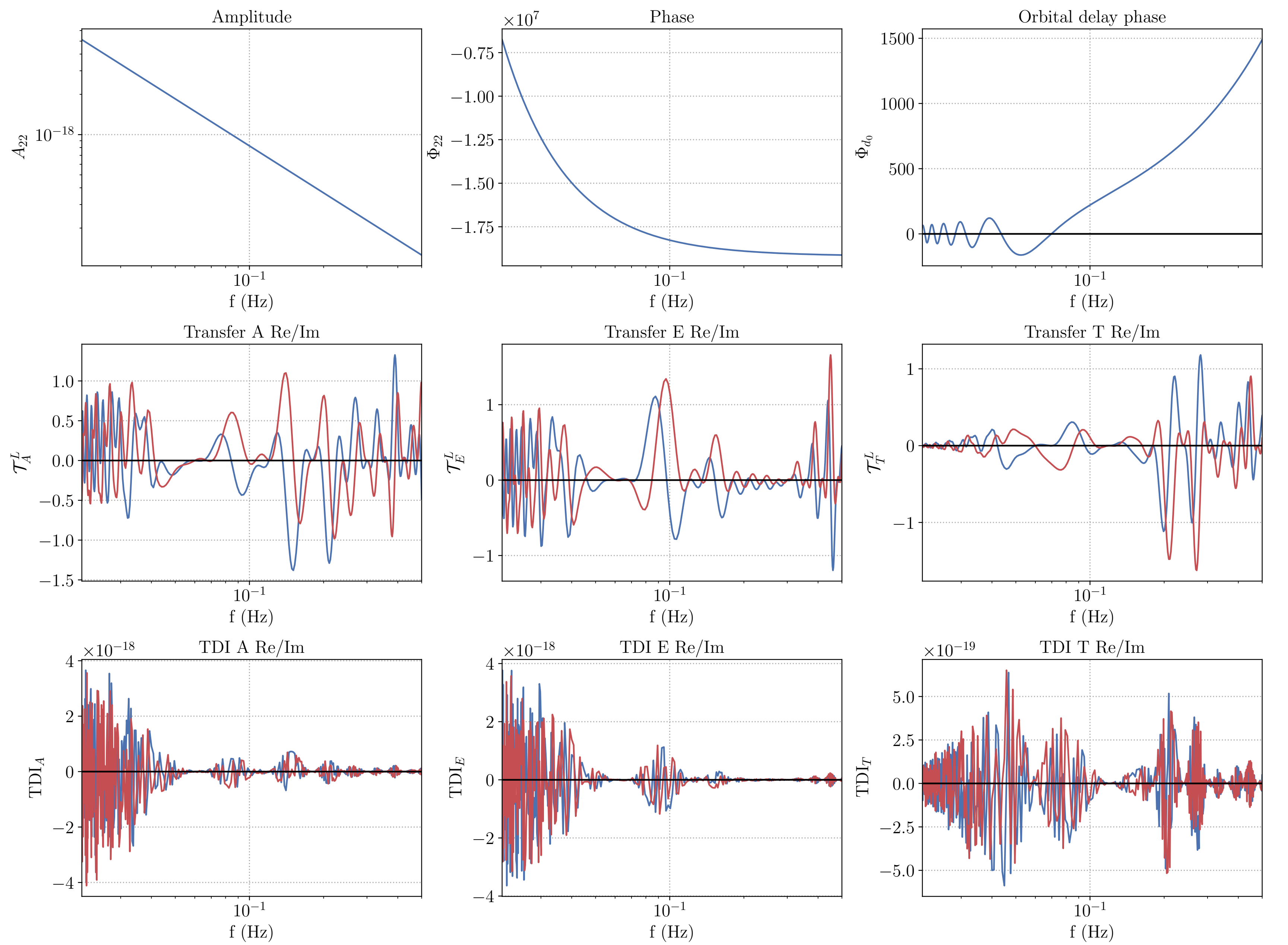
For this system, 8-maxima sky degeneracy only disappears at merger



SNR-based time cuts:

SNR	DeltaT
10	40h
42	2.5h
167	7min
666	-

SOBH analysis setting



Sources

- Plausible SOBH sources at low z
- Masses $M = 41\ M_\odot,\ q = 1.05$
  $M = 108\ M_\odot,\ q = 1.3$

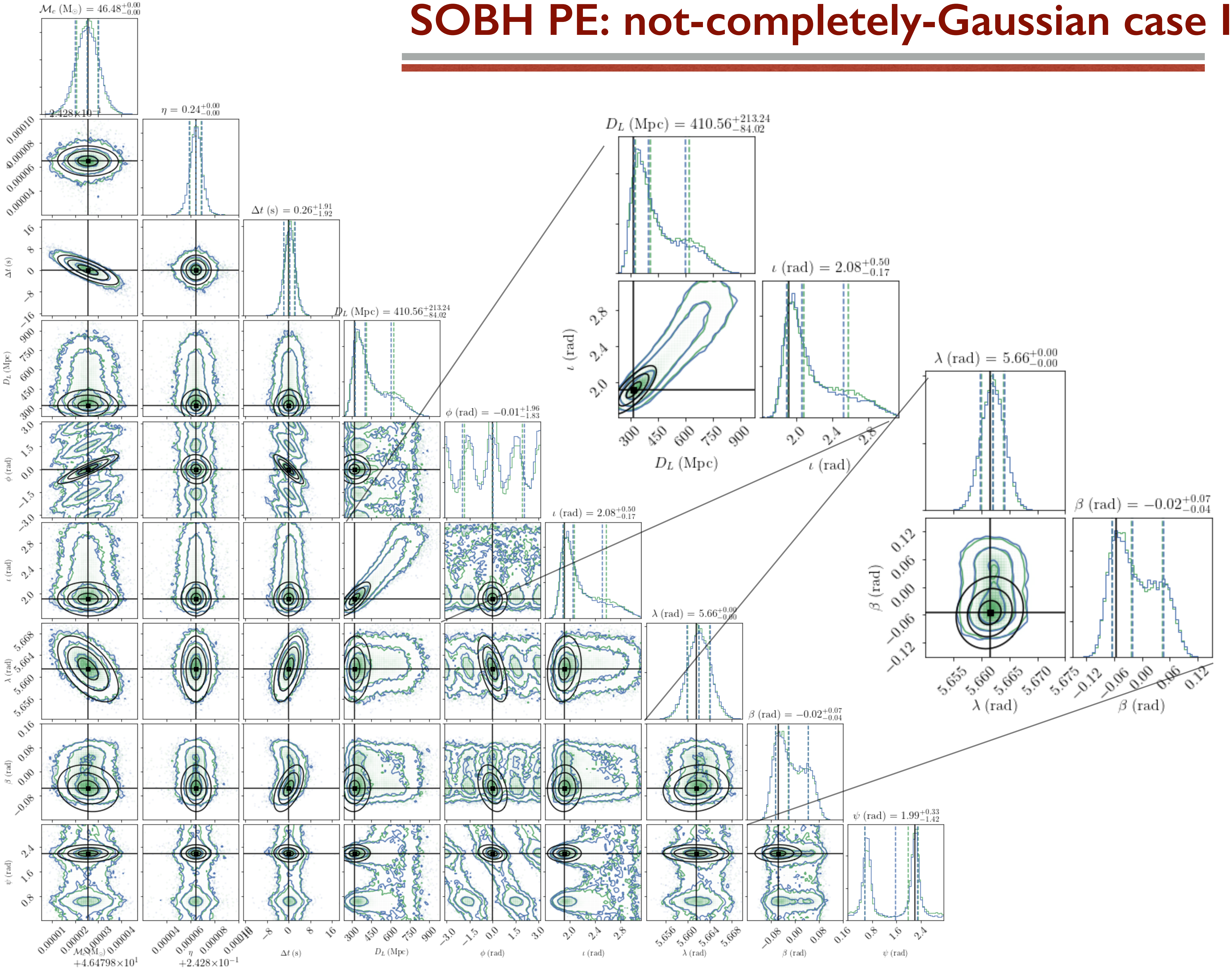
SNR

- SNR 27
- SNR 12

Evaluations

Multinest: $2 - 5 \cdot 10^6$
PTMCMC: $60 \cdot 10^6$

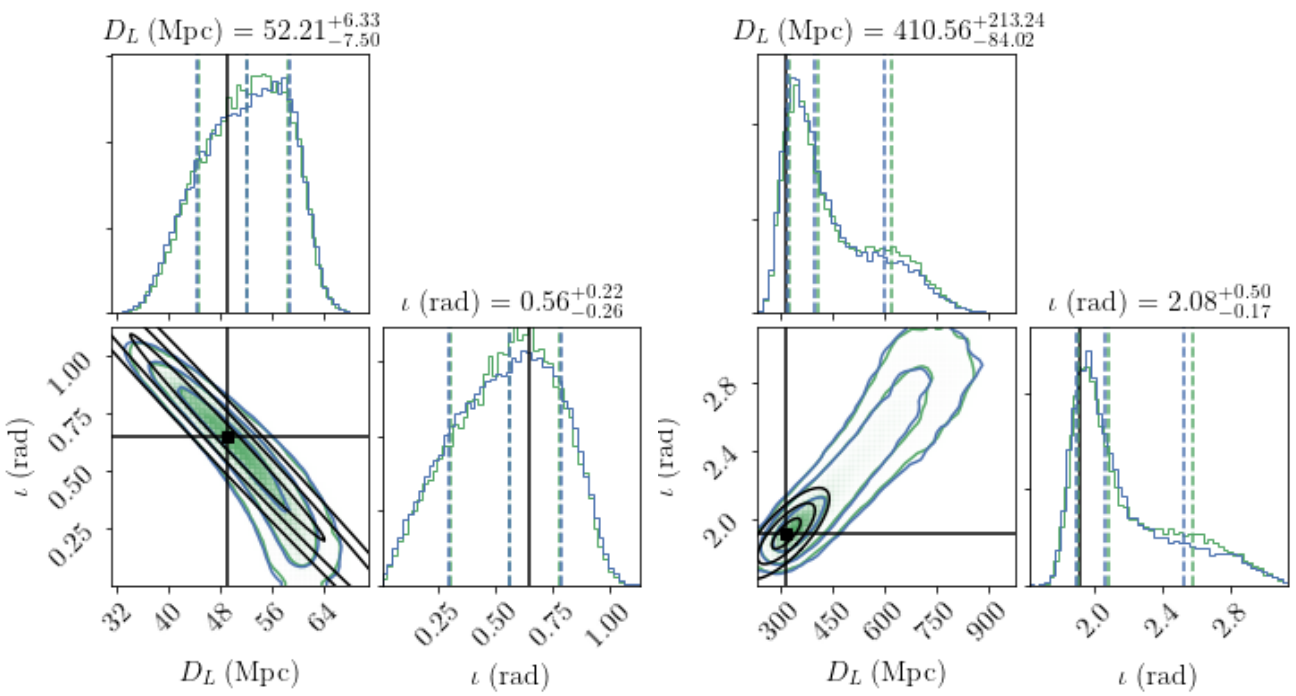
SOBH PE: not-completely-Gaussian case II



SOBH PE: highlights

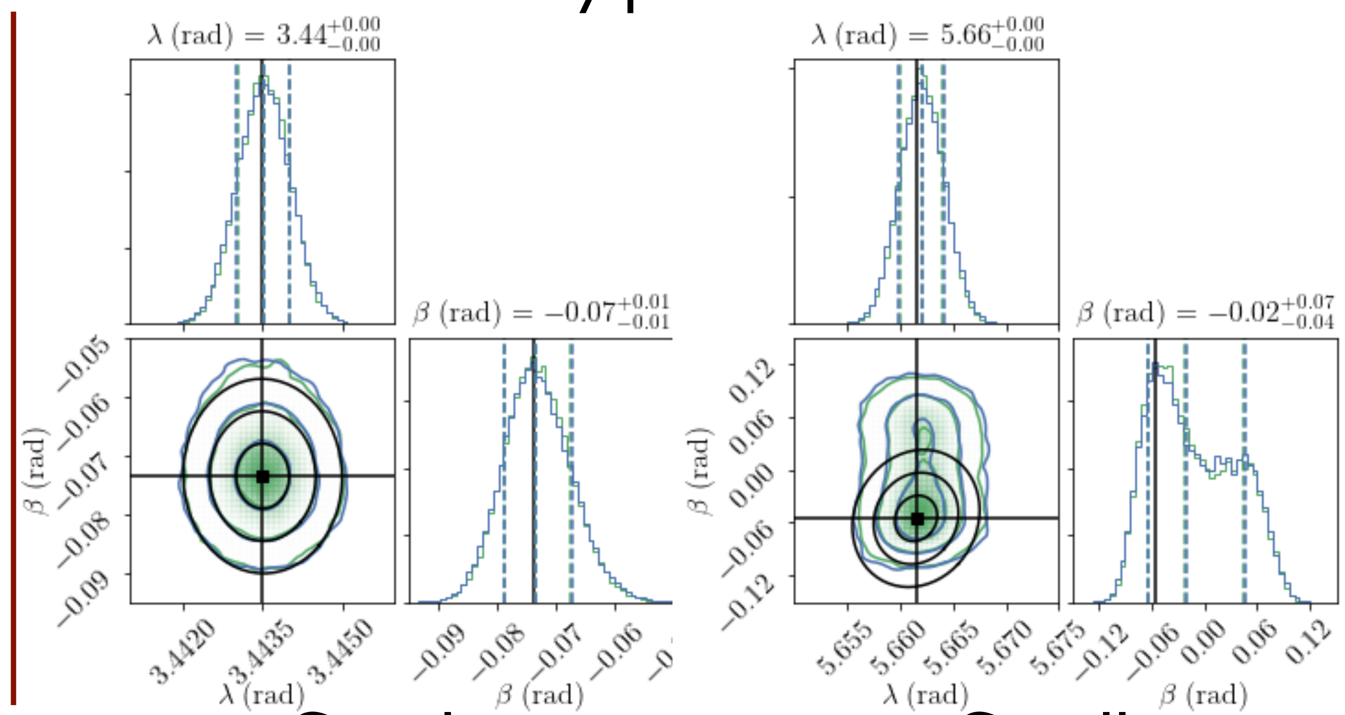
Mostly Gaussian posteriors, but…

Distance-inclination



Case I Case II

Sky position



Case I Case II

Highlights and limitations

- Mostly Gaussian posteriors, the two samplers agree very well
- Very accurate extraction of masses (but will be affected by spin)
- Good sky localization, even for these low SNRs

- We assumed we solved the search problem
- Narrow priors used in masses
- Low-SNR zero-noise analysis less reliable
- Single-source assumption ignores the population of other SOBHs
- Including spins will introduce degeneracies with the masses
- To be extended to narrow-band signals

Understanding degeneracies in the likelihood

The face-on / face-off limit

- Two branches: close to face-on or face-off
- Effective amplitude and phase degenerate in distance/inclination and in phase/polarization

$$\mathcal{A}(D_L, \iota) \sim \cos^4(\iota/2)/D_L$$
$$\xi(\varphi_L, \psi_L) = -\varphi_L - \psi_L$$

For example for $\sin^4 \frac{\iota}{2} \ll 1$

$$s_a \simeq i\mathcal{A}e^{2i\xi}\left(F_a^+ + iF_a^\times\right) \, ,$$
$$s_e \simeq i\mathcal{A}e^{2i\xi}\left(F_e^+ + iF_e^\times\right) \, ,$$

Explicit solution for the degeneracy

Reproduce s_a, s_e of injection if condition on sky position is met:

$$r = \frac{s_a^{\text{inj}}}{s_e^{\text{inj}}} = \frac{F_a^+ + iF_a^\times}{F_e^+ + iF_e^\times}(\lambda_L, \beta_L)$$

Then **line degeneracy** for both (φ_L, ψ_L) and (D_L, ι)

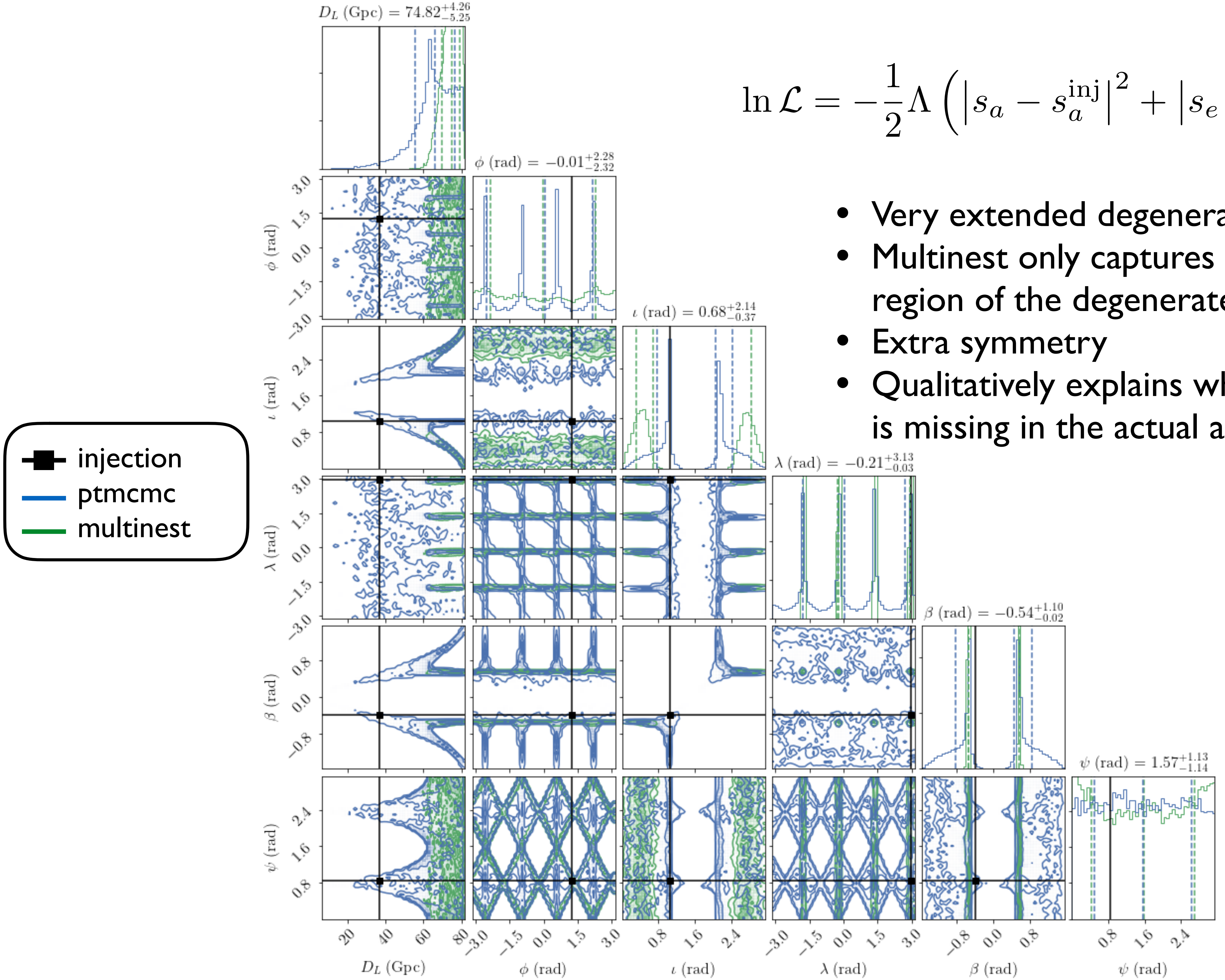
Solution : $\rho = \sqrt{\left|\frac{1+ir}{1-ir}\right|}$

$$\sin\beta_L^* = \frac{\rho - 1}{\rho + 1}$$
$$\lambda_L^* = -\frac{\pi}{12} + \frac{1}{4}\text{Arg}\frac{1+ir}{1-ir} + \frac{k\pi}{2} \, .$$

+ approximate symmetry

$$(\lambda_L, \beta_L) \leftrightarrow (\varphi_L, \iota)$$

Exploring the analytic simplified extrinsic likelihood



$$\ln \mathcal{L} = -\frac{1}{2}\Lambda\left(\left|s_a - s_a^{\mathrm{inj}}\right|^2 + \left|s_e - s_e^{\mathrm{inj}}\right|^2\right)$$

- Very extended degeneracies
- Multinest only captures a small region of the degenerate likelihood
- Extra symmetry
- Qualitatively explains what multinest is missing in the actual analysis