Anomalous duality symmetry

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Hermann Nicolai's 60$^{\mathrm{th}}$ anniversary

$\mathcal{N} = 8$ supergravity

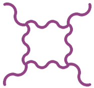
Is $\mathcal{N} = 8$ supergravity a consistent quantum field theory?

$\hookrightarrow$ Free of ambiguities associated to logarithmic divergences?

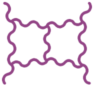
Explanation for the excellent ultra-violet behaviour of the 4-graviton amplitudes

- ⋆ Supersymmetry and $E_{7(7)}$ duality symmetry
- ⋆ Some more hidden symmetry of the quantum theory?

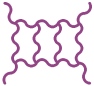
$\mathcal{N} = 8$ supergravity



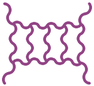
$$\sim \int \frac{d^4k}{(2\pi)^4}\frac{k^8}{k^8} \sim p^8 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^8} \qquad \mathcal{L}^{(3)} = ef_{-42}(\phi)C^2\bar{C}^2 + \ldots$$



$$\sim \int \frac{d^8k}{(2\pi)^8}\frac{k^{12}}{k^{14}} \sim p^{12} \int \frac{d^8k}{(2\pi)^4}\frac{1}{k^{14}} \qquad \mathcal{L}^{(5)} = ef_{-60}(\phi)\nabla^2C^2\nabla^2\bar{C}^2 + \ldots$$



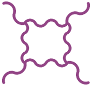
$$\sim \int \frac{d^{12}k}{(2\pi)^{12}}\frac{k^{16}}{k^{20}} \sim p^{14} \int \frac{d^{12}k}{(2\pi)^{12}}\frac{1}{k^{18}} \qquad \mathcal{L}^{(6)} = ef_{-60}(\phi)\nabla^3C^2\nabla^3\bar{C}^2 + \ldots$$



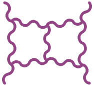
$$\sim \int \frac{d^{16}k}{(2\pi)^{16}}\frac{k^{20}}{k^{26}} \sim p^{16} \int \frac{d^{16}k}{(2\pi)^4}\frac{1}{k^{22}} \qquad \mathcal{L}^{(7)} = e\nabla^4C^2\nabla^4\bar{C}^2 + \ldots$$

[Bern, Carrasco, Dixon, Johansson, Kosower and Roiban]

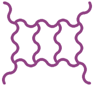
Maximal supergravity in higher dimensions



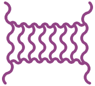
$$\sim \int \frac{d^8 k}{(2\pi)^4} \frac{k^8}{k^8} \sim p^8 \ln[\Lambda] \qquad \mathcal{L}_8^{(1)} = eR^4 + \ldots$$



$$\sim \int \frac{d^{2\times 7} k}{(2\pi)^{14}} \frac{k^{12}}{k^{14}} \sim p^{12} \ln[\Lambda] \qquad \mathcal{L}_7^{(2)} = e\nabla^4 R^4 + \ldots$$

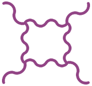


$$\sim \int \frac{d^{3\times 6} k}{(2\pi)^{18}} \frac{k^{16}}{k^{20}} \sim p^{14} \ln[\Lambda] \qquad \mathcal{L}_6^{(3)} = e\nabla^6 R^4 + \ldots$$

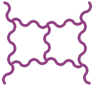


$$\sim \int \frac{d^{7\times 4} k}{(2\pi)^{28}} \frac{k^{32}}{k^{44}} \sim p^{16} \int \frac{d^{28} k}{(2\pi)^{28}} \frac{1}{k^{28}} \qquad \mathcal{L}_4^{(7)} = e\nabla^4 C^2 \nabla^4 \bar{C}^2 + \ldots$$

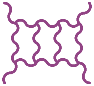
Half-maximal supergravity in higher dimensions



$$\sim \int \frac{d^8 k}{(2\pi)^4} \frac{k^8}{k^8} \sim p^8 \ln[\Lambda] \qquad \mathcal{L}_8^{(1)} = eR^4 + \dots$$



$$\sim \int \frac{d^{2\times 5} k}{(2\pi)^{10}} \frac{k^{12}}{k^{14}} \sim p^8 \int \frac{d^{10} k}{(2\pi)^{10}} \frac{1}{k^{10}} \qquad \mathcal{L}_5^{(2)} = eR^4 + \dots$$



$$\sim \int \frac{d^{3\times 4} k}{(2\pi)^{12}} \frac{k^{16}}{k^{20}} \sim p^8 \int \frac{d^{12} k}{(2\pi)^{12}} \frac{1}{k^{12}} \qquad \mathcal{L}_4^{(3)} = eC^2 \bar{C}^2 + \dots$$

But the explicit computations suggest they are finite.

[Bern, Davies, Dennen, Huang]

[Vanhove, Tourkine]

Outline

- The $SL(2)$ anomaly
- R^4 type invariants in $\mathcal{N} = 4$ supergravity
- Five dimensions and vector multiplets
- Toward algebraic renormalisation in harmonic superspace
- Conclusion

[G. Bossard, P. S. Howe and K. S. Stelle, to appear]

[G. Bossard, C. Hillmann and H. Nicolai, 1007.5472]
[G. Bossard, P. S. Howe, K. S. Stelle and P. Vanhove, 1105.6087]

$\mathcal{N} = 4$ supergravity

$\mathcal{N} = 4$ supergravity includes

- 1 complex scalar field τ parametrizing $SL(2)/SO(2)$
- 2×4 Dirac fermions χ^i_α
- 2×6 vectors $F_{\alpha\beta ij}$
- 2×4 gravitinos $\rho_{\alpha\beta\gamma i}$
- 2 graviton $C_{\alpha\beta\gamma\delta}$

of $SL(2, \mathbb{C}) \times U(4)$.

The same letters are used for the corresponding superfields.

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$\mathcal{N} = 4$ supergravity

$\mathcal{N} = 4$ supergravity includes

⋆ 1 complex scalar field τ parametrizing $SL(2)/SO(2)$

⋆ 2×4 Dirac fermions χ^i_α	$-4 \times 3 \times 1/24$
⋆ 2×6 vectors $F_{\alpha\beta ij}$	$-6 \times 2 \times 1/6$
⋆ 2×4 gravitinos $\rho_{\alpha\beta\gamma i}$	$+4 \times 1 \times 7/8$
	$= 1$
⋆ 2 graviton $C_{\alpha\beta\gamma\delta}$	[Marcus]

of $SL(2, \mathbb{C}) \times U(4)$.

The same letters are used for the corresponding superfields.

$\mathcal{N} = 4$ supergravity's 1-loop anomaly

The rigid $SL(2)$ symmetry is anomalous, and is broken to its parabolic subgroup.

$$\mathbf{f}\,\Gamma^{(1)} = \frac{2+n}{32\pi^2}\int \left(e^{-2\phi}R^{ab}\wedge R_{ab} + \ldots\right)$$

where $\mathbf{f}\tau = -\tau^2$ with $\tau = a + ie^{-2\phi}$.

Correspondingly at 1-loop

$$\langle g_{\mu\nu}(p_1)g_{\sigma\rho}(p_2)a(-p_1-p_2)\rangle = \frac{2+n}{16\pi^2}\varepsilon_{\sigma)(\mu}{}^{\kappa\lambda}\left(\eta_{\nu)(\rho}p_1\cdot p_2 - p_{2\nu)}p_{1(\rho}\right)p_{1\kappa}p_{2\lambda}$$

Consistency with supersymmetry

The supersymmetric extension is

$$\mathcal{A}_{\mathbf{f}} = \int \left(e^{-2\phi} R^{ab} \wedge R_{ab} + \frac{1}{2} \varepsilon_{abcd}\, a R^{ab} \wedge R^{cd} + \dots \right)$$

and the Wess–Zumino condition

$$\mathbf{e}\mathcal{A}_{\mathbf{f}} + A_{\mathbf{h}} = 0$$

then implies that

$$\mathcal{A}_{\mathbf{h}} = -\int \frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd}$$

so the rigid symmetry is preserved, but

$$dJ_{\mathbf{h}} = -\frac{2+n}{64\pi^2} \varepsilon_{abcd} R^{ab} \wedge R^{cd}$$

Consistency with supersymmetry

The supersymmetry invariant $\mathcal{A}_{\mathbf{f}}$ is not defined as a superspace integral, but only as a d-closed super 4-form. [Gates]

$$d\mathcal{L}_4 = \frac{1}{24} E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A \big(D_A \mathcal{L}_{BCDE} + 2\, T_{AB}{}^F \mathcal{L}_{FCDE} \big)$$

Focusing on the lowest dimensional components

($[r_1, r_2, r_3]$ such that $r_1 + 2r_2 + r_3 = 5$)

$$\begin{aligned}
D^p_\eta \mathcal{L}^{ijkl}_{\alpha\beta\gamma\delta} + 2\, T^{pi}_{\eta\,\alpha}{}^{\dot\varsigma q} \mathcal{L}^{jkl}_{\beta\gamma\delta\dot\varsigma q} + \circlearrowleft \quad &= \quad 0 \\
D_{\dot\eta p} \mathcal{L}^{ijkl}_{\alpha\beta\gamma\delta} + 4 D^i_\alpha \mathcal{L}^{jkl}_{\beta\gamma\delta\dot\eta p} + \circlearrowleft \quad &\approx \quad 0 \\
2 D_{\dot\eta p} \mathcal{L}^{ijk}_{\alpha\beta\gamma\dot\delta l} + T_{\dot\eta p \dot\delta l}{}^\varsigma_q \mathcal{L}^{ijkq}_{\alpha\beta\gamma\varsigma} + \circlearrowleft \quad &\approx \quad 0 \\
T_{\dot\eta p \dot\gamma l}{}^\varsigma_q \mathcal{L}^{ijq}_{\alpha\beta\varsigma\dot\delta l} + \circlearrowleft \quad &\approx \quad 0
\end{aligned}$$

Consistency with supersymmetry

The supersymmetry invariant $\mathcal{A}_{\mathbf{f}}$ is not defined as a superspace integral, but only as a d-closed super 4-form.

$$d\mathcal{L}_4 = \frac{1}{24} E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A \bigl(D_A \mathcal{L}_{BCDE} + 2 T_{AB}{}^F \mathcal{L}_{FCDE} \bigr)$$

For an arbitrary anti-holomorphic function $\mathcal{F}[\bar{T}]$

with $U\bar{U}(1 - T\bar{T}) = 1$, $\begin{pmatrix} U & UT \\ \bar{U}\bar{T} & \bar{U} \end{pmatrix} \in SU(1,1)$

$$\begin{aligned}\mathcal{L}^{ijkl}_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\Bigl(&\mathcal{F}[\bar{T}] F^{ij}_{\dot\alpha\dot\beta} F^{\dot\alpha\dot\beta kl} - \bar{U}^{-2} \bar\partial \mathcal{F}[\bar{T}] \varepsilon^{ijpq} \chi_{\dot\alpha p} \chi_{\dot\beta q} F^{\dot\alpha\dot\beta kl} \\ &+ \frac{1}{6} \bar{U}^{-4} \Bigl(\bar\partial - \frac{2T}{1 - T\bar{T}} \Bigr) \bar\partial \mathcal{F}[\bar{T}] \varepsilon^{ijpq} \varepsilon^{klrs} \chi_{\dot\alpha p} \chi_{\dot\beta q} \chi^{\dot\alpha}_r \chi^{\dot\beta}_s \Bigr) + \circlearrowleft\end{aligned}$$

and

$$\mathcal{L}^{ijk}_{\alpha\beta\gamma\dot\delta l} = -\varepsilon_{\alpha\beta} \varepsilon^{\dot\eta\dot\varsigma} \chi^k_\gamma \chi_{\dot\eta l} \Bigl(\mathcal{F}[\bar{T}] F^{ij}_{\dot\delta\dot\varsigma} - \frac{1}{3} \bar{U}^{-2} \bar\partial \mathcal{F}[\bar{T}] \varepsilon^{ijpq} \chi_{\dot\delta p} \chi_{\dot\varsigma q} \Bigr) + \circlearrowleft$$

Consistency with supersymmetry

For $\mathcal{F}[T] = 1$

$$\mathcal{L}_4[1] = \frac{1}{2} R^{ab} \wedge R_{ab} - \frac{i}{4} \varepsilon_{abcd} R^{ab} \wedge R^{cd}$$

therefore for $\mathcal{F}[T] = \tau[T] \equiv i\frac{1-T}{1+T}$

$$i\mathcal{L}[\bar{\tau}] - i\bar{\mathcal{L}}[\tau] = \mathrm{Im}[\tau] R^{ab} \wedge R_{ab} + \mathrm{Re}[\tau] \frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge R^{cd} + \dots$$

defines the anomaly $\mathcal{A}_{\mathbf{f}}$.

Anomalous Ward identity

At higher order, the anomaly is renormalised consistently

$$\mathbf{f}\,\Gamma_\epsilon = \frac{1}{16\pi^2}[\mathcal{A}\cdot\Gamma]_\epsilon$$

If there would be a logarithmic divergence at 3-loop

$$\mathbf{f}\left(\Gamma^R - \frac{\beta_3}{\epsilon}\kappa^4 S^{(3)}\right) \approx \frac{1}{16\pi^2}\left([\mathcal{A}\cdot\Gamma]^R - \frac{\gamma_2}{\epsilon}\kappa^4\mathcal{A}^{(2)}\right)$$

so that

$$16\pi^2\beta_3\,\mathbf{f}\,S^{(3)} = \gamma_2\mathcal{A}^{(2)}\ , \qquad \beta_3\,\mathbf{h}S^{(3)} = 0\ , \quad \beta_3\,\mathbf{e}S^{(3)} = 0$$

Anomalous Ward identity

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$$\mathbf{f}\,\Gamma = \frac{1}{16\pi^2}[\mathcal{A}\cdot\Gamma]$$

If there would be a logarithmic divergence at 3-loop

$$\left(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\right)\Gamma \approx -\beta_3\kappa^4[S^{(3)}\cdot\Gamma] + \mathcal{O}(\kappa^6)$$

and

$$\left(\frac{\partial}{\partial\mu} - \kappa\frac{\partial}{\partial\kappa}\right)[\mathcal{A}\cdot\Gamma] \approx -\gamma_2\kappa^4[\mathcal{A}^{(2)}\cdot\Gamma] + \mathcal{O}(\kappa^6) \approx -16\pi^2\beta_3\kappa^4\mathbf{f}[S^{(3)}\cdot\Gamma] + \mathcal{O}(\kappa^6)$$

so that

$$16\pi^2\beta_3\,\mathbf{f}S^{(3)} = \gamma_2\mathcal{A}^{(2)}\ , \qquad \beta_3\,\mathbf{h}S^{(3)} = 0\ , \quad \beta_3\,\mathbf{e}S^{(3)} = 0$$

Constraints on the 3-loop counter-term

According to the symmetries, it must be supersymmetric, and invariant with respect to the parabolic subgroup of $SL(2)$.

Therefore we need to study generic invariants of the form

$$\int d^4x d^{16}\theta \, \mathrm{Ber}[E] K[T, \bar{T}]$$

or possible generalisations thereof which cannot be written as full-superspace integrals.

Normal coordinate expansion

To study the integral

$$\int d^4x d^{16}\theta \, \mathrm{Ber}[E] K[T, \bar{T}]$$

it is convenient to integrate over four θ's.

$\hookrightarrow$ Requires vector fields in involution.

Only exist in harmonic superspace

$$u^I{}_i \in \big(U(1) \times U(2) \times U(1)\big) \backslash U(4)$$

Then the following vectors are in involution

$$u^1{}_i \tilde{E}^i_\alpha \ , \quad u^i{}_4 \tilde{E}_{\dot{\alpha} i} \ , \quad D^1{}_r \ , \quad D^r{}_4 \ , \quad D^1{}_4$$

where $\tilde{E}_A = E_A - u^I{}_i \Omega_A{}^i{}_j u^j{}_J D^J{}_I$.

Normal coordinate expansion

$$u^1{}_i \tilde{E}^i_\alpha \ , \quad u^i{}_4 \tilde{E}_{\dot{\alpha} i} \ , \quad D^1{}_r \ , \quad D^r{}_4 \ , \quad D^1{}_4$$

The associated normal coordinate are complex.

$$\mathfrak{u}(4) \cong \overline{\mathbf{1}}^{(-2)} \oplus (2 \times \overline{\mathbf{2}})^{(-1)} \oplus \big(\mathfrak{u}(1) \oplus \mathfrak{u}(2) \oplus \mathfrak{u}(1)\big)^{(0)} \oplus (2 \times \mathbf{2})^{(1)} \oplus \mathbf{1}^{(2)}$$

$\hookrightarrow$ holomorphic normal coordinate expansion

⋆ Usual for Grassmann variables
[Hermann Nicolai, Nucl. Phys. B 140, 294 (1978)]

⋆ Not for ordinary commuting variables.

Normal coordinate expansion

However the normal coordinate expansion factorizes.

- ⋆ The expansion of du depends only on ${z^r}_4, {z^1}_r, {z^1}_4$
- ⋆ The expansion of $\mathrm{Ber}[E]K[T, \bar{T}]$ depends only on $\zeta_1^{\alpha}, \zeta^{\dot{\alpha}4}$

Therefore one can simply expand in the fermionic coordinates only.

Normal coordinate expansion

As a result

$$
\begin{aligned}
& \int d^4x d^{16}\theta \, \mathrm{Ber}[E] K[T, \bar{T}] \\
= \; & \frac{1}{4} \int d\mu_{(4,1,1)} \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \chi^1_\alpha \chi^1_\beta \chi_{\dot{\alpha}4} \chi_{\dot{\beta}4} \, (\Delta - 2)\Delta \, K[T, \bar{T}] \\
\sim \; & \int d^4x \Big(C^2 \bar{C}^2 (\Delta - 2)\Delta \, K[\tau, \bar{\tau}] + \dots \Big)
\end{aligned}
$$

It is zero if K is holomorphic, and it is duality invariant for

$$K = -\ln(1 - T\bar{T})$$

the Kähler potential.

Back to the anomaly

All 3-loop candidates can be written as

$$\begin{aligned} S^{(3)}[G] &= \frac{1}{4}\int d\mu_{(4,1,1)}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\chi^1_\alpha\chi^1_\beta\chi_{\dot\alpha 4}\chi_{\dot\beta 4}\,G \\ &\sim \int d^4x\Big(C^2\bar{C}^2 G + \dots\Big) \end{aligned}$$

So $\delta S^{(3)}[G] = S^{(3)}[\delta G]$ and

$$\mathbf{e}S^{(3)}[G] = \mathbf{h}S^{(3)}[G] = 0 \implies G = \mathrm{const}$$

The invariance with respect to the parabolic subgroup implicates duality invariance at this order.

Back to the anomaly

Therefore $\delta S^{(3)} = 0$ and the only available counter-term satisfying all required symmetries is the duality invariant

$$\begin{aligned} & \int d^4x d^{16}\theta \, \mathrm{Ber}[E] \ln(1 - T\bar{T}) \\ = \ & \frac{1}{2} \int d\mu_{(4,1,1)} \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \chi^1_\alpha \chi^1_\beta \chi_{\dot{\alpha}4} \chi_{\dot{\beta}4} \\ \sim \ & \int d^4x \Big(C^2 \bar{C}^2 + \ldots \Big) \end{aligned}$$

$\mathcal{N} = 4$ supergravity coupled to n vector multiplets

The available invariants are

- 1-loop :
  - A duality invariant $1/2$ BPS F^4 counter-term
  - Infinitely many non-duality invariant pure cocycle in R^2
- 2-loop :
  - A $1/4$ BPS $\partial^2 F^4$ counter-term
  - A complex non-duality invariant pure cocycle in $F^2 R^2$
- 3-loop :
  - A duality invariant $1/4$ BPS R^4 counter-term
  - A duality invariant full-superspace integral of $-\ln(1 - T\bar{T})$
  - Infinitely many non-duality invariant full-superspace integrals.

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- 2-loop :
  - A 1/4 BPS $\partial^2 F^4$ counter-term $SO(6,n)$ invariance?
  - A complex non-duality invariant pure cocycle in $F^2 R^2$
- 3-loop :
  - A duality invariant 1/4 BPS R^4 counter-term
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$\mathcal{N} = 4$ supergravity coupled to n vector multiplets

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  - ⋆ Infinitely many non-duality invariant full-superspace integrals.

$\mathcal{N} = 2$ supergravity in five dimensions

The available invariants are

- 2-loop :
  - A duality invariant full-superspace integral of Φ
  - Infinitely many non-duality invariant full-superspace integrals.

$$\begin{aligned}&\int d^5x d^{16}\theta \mathrm{Ber}[E] K[\Phi] \\ = \quad & \frac{1}{24}\int d\mu_{(4,1)} \varepsilon^{\alpha\beta\gamma\delta}\chi^1_\alpha\chi^1_\beta\chi^1_\gamma\chi^1_\delta(\partial+3)(\partial+\tfrac{3}{2})\partial(\partial-\tfrac{3}{2})K[\Phi]\end{aligned}$$

$\mathcal{N} = 2$ supergravity in five dimensions

The available invariants are

⋆ 2-loop :

- A duality invariant full-superspace integral of Φ
- ⋆ Infinitely many non-duality invariant full-superspace integrals.

$$\begin{aligned}
&\int d^5x d^{16}\theta \mathrm{Ber}[E] K[\Phi] \\
= \; &\frac{1}{24} \int d\mu_{(4,1)} \varepsilon^{\alpha\beta\gamma\delta} \chi^1_\alpha \chi^1_\beta \chi^1_\gamma \chi^1_\delta (\partial + 3)(\partial + \tfrac{3}{2})\partial(\partial - \tfrac{3}{2}) K[\Phi]
\end{aligned}$$

Off-shell formulation of $\mathcal{N} = 4$ supergravity

Counting arguments: can only exist with finitely many auxiliary fields for $\mathcal{N} = 4$ supergravity with 6 vector multiplets
[Rivelles and Taylor]

It does exist in ten dimensions for type I supergravity, with $\tilde{B}_6$
[Howe, Nicolai and Van Proeyen]

In four dimensions $SO(6,6)$ is broken to $GL(6,\mathbb{R})$ and 15 scalars are realised as 2-form gauge fields.

In four dimensions $SO(5,5)$ is broken to $GL(5,\mathbb{R})$ and 10 scalars are realised as 3-form gauge fields, 5 vectors as 2-forms.

A duality invariant formulation with n vector multiplets can only exist in harmonic superspace

Off-shell formulation of $\mathcal{N} = 4$ supergravity

Strong assumption:

It exists a harmonic superspace formulation of $\mathcal{N} = 4$ supergravity coupled to n vector multiplets with all 16 supercharges realised linearly, and for which the action is duality invariant *à la* Henneaux–Teitelboim.

Can we then understand the absence of divergence at 3-loop (respectively 2-loop in five dimensions)?

$\hookrightarrow$ algebraic renormalisation in harmonic superspace

Algebraic renormalisation

The n-loop β_n function associated to an invariant counter-term $\int \mathcal{L}_{(n)}$ is related to the n-loop anomalous dimension γ_n of the classical Lagrange density $\mathcal{L}$ as a local operator for mixing with the local operator $\mathcal{L}_{(n)}$.

For a first divergence, one proves that

$$(n-1)\beta_n = \gamma_n$$

Algebraic renormalisation

If the Lagrange density is not invariant with respect to a symmetry of the action

$$
\begin{aligned}
\delta \mathcal{L} &= -D_M \mathcal{L}^M \\
\delta \mathcal{L}^M &= -D_N \mathcal{L}^{NM} \\
\delta \mathcal{L}^{NM} &= -D_P \mathcal{L}^{PNM} \\
&\dots
\end{aligned}
$$

The sources transforms as forms $\delta u = 0$, $\delta u_M = -D_M u$, $\delta u_{MN} = -2 D_{[M} u_{N]}$, ...

and they all must renormalise consistently

$$
\begin{aligned}
\mathcal{L}_\flat &= \mathcal{L} + \gamma_n \ln(\tfrac{\Lambda}{\mu}) \mathcal{L}_{(n)} + \dots \\
\mathcal{L}_\flat^M &= \mathcal{L}^M + \gamma_n \ln(\tfrac{\Lambda}{\mu}) \mathcal{L}_{(n)}^M + \dots \\
\mathcal{L}_\flat^{NM} &= \mathcal{L}^{NM} + \gamma_n \ln(\tfrac{\Lambda}{\mu}) \mathcal{L}_{(n)}^{NM} + \dots \\
&\dots
\end{aligned}
$$

$\mathcal{N} = 4$ counter-term density

We have seen that the only available duality invariant full-superspace integral at 3-loop is the integral of the Kähler potential.

Under duality

$$\delta\Big(\mathrm{Ber}[E](-1)\ln\big(\tfrac{-i}{2}(\tau-\bar{\tau})\big)\Big) = -2h\mathrm{Ber}[E] + f\mathrm{Ber}[E](\tau+\bar{\tau})$$

Within an off-shell formulation,

$$\mathrm{Ber}[E] = D_M\psi^M[E,V]$$

and respectively for $\mathrm{Ber}[E](\tau+\bar{\tau})$, where V is some prepotential for the supergravity fields.

$\hookrightarrow$ Whereas explicit prepotential are forbidden within the background field method.

Conclusion

Can supersymmetry and duality invariance explain the absence of logarithm divergence at 3-loop in $\mathcal{N} = 4$ $d = 4$ supergravity, respectively 2-loop in 5 dimensions?

With the rather strong assumption that an off-shell harmonic formulation with a duality invariant action exists

$\hookrightarrow$ the answer might be yes
(although more work is required).

If yes, what about $\mathcal{N} = 8$ supergravity?

$$\int \partial^8 R^4 + \cdots \sim \int d^4x d^{28}\theta \mathcal{E} \chi^4 \overset{?}{\sim} \int d^4x d^{32}\theta \mathrm{Ber}[E] K[\phi]$$

But asking for an off-shell formulation with all 32 supercharges does not seem reasonable then.

Happy Birthday Hermann !!