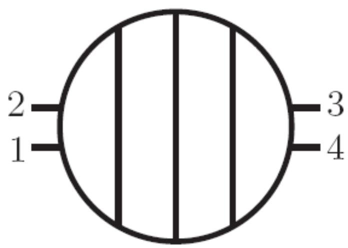
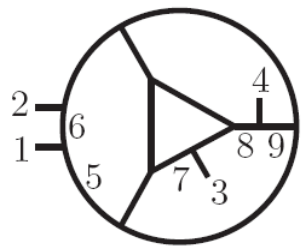
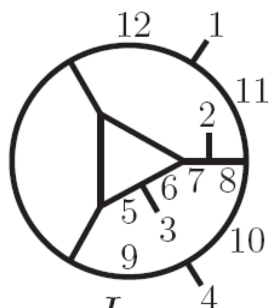
N=8 Supergravity: Scattering and Ultraviolet Behavior through Four Loops



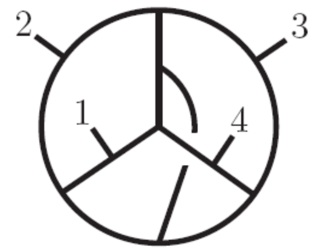
I_1



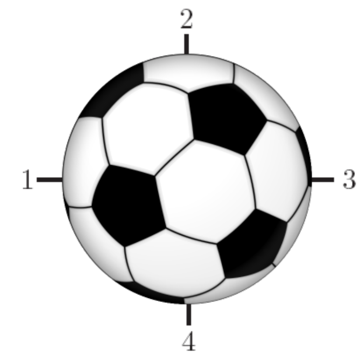
I_{25}



I_{32}



I_{50}



Lance Dixon (SLAC)

for the C.G.C.:

Z. Bern, J.J. Carrasco, LD, H. Johansson & R. Roiban

Symmetries and Quantum Gravity

Hermann Nicolai-Fest – MPI Potsdam

September 7, 2012

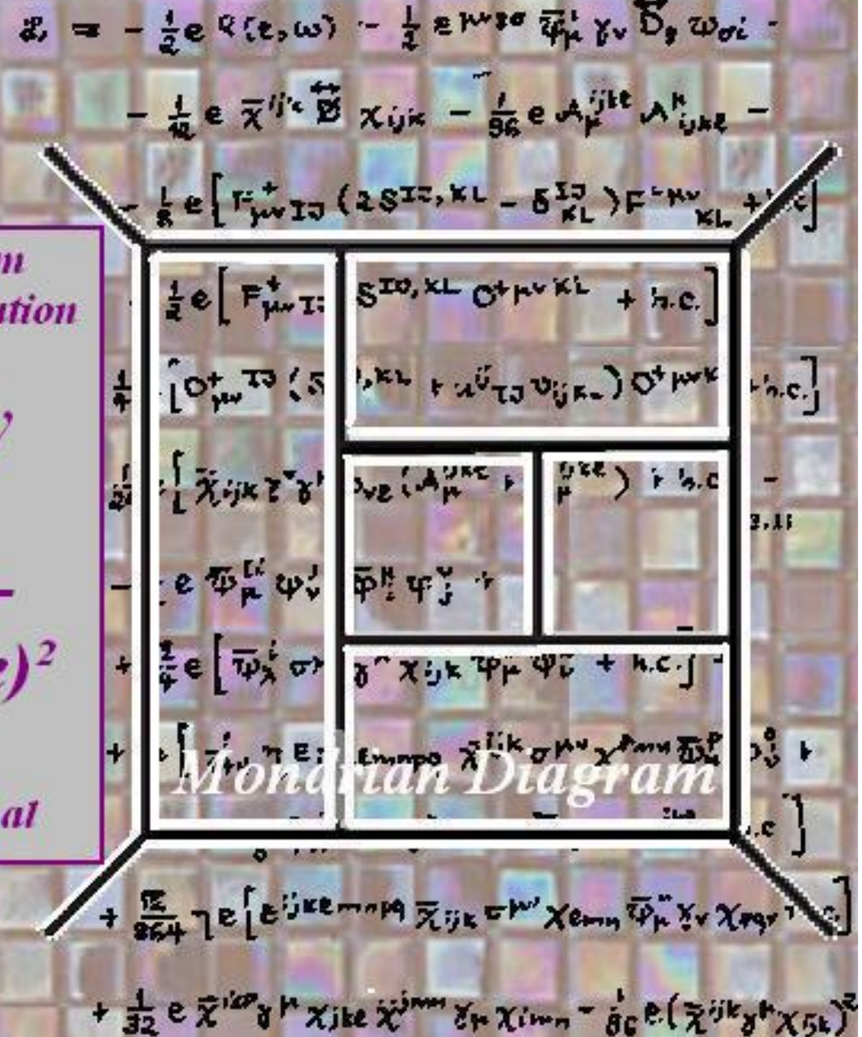
Colors of Invincibility

1982 Lagrangian $\mathcal{L}$ of the Unified Field (Formula) =>

Results from String-Unification

Gravity ~ (Vector-Gauge)²

Zvi Bern et al

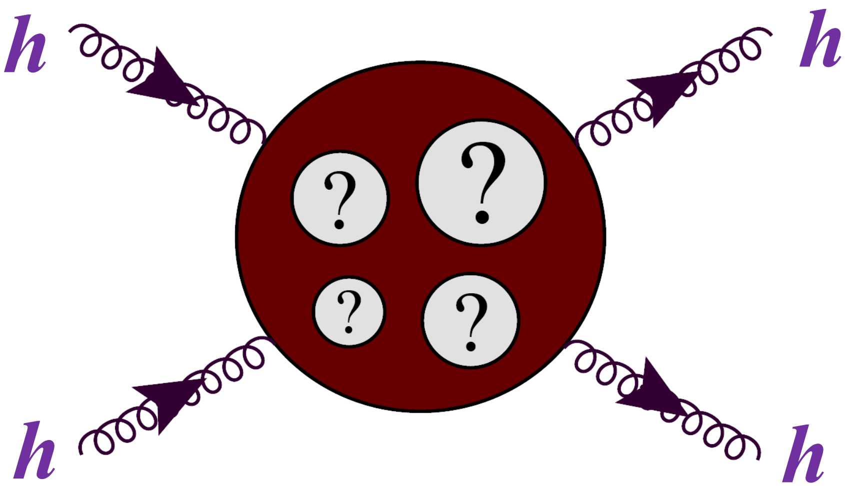


2009 Vanquishing Infinity (Diagrams) =>

Graviton Scattering: a Gedanken Experiment

"Mathematics is the part of physics where experiments are cheap"

– V.I. Arnold



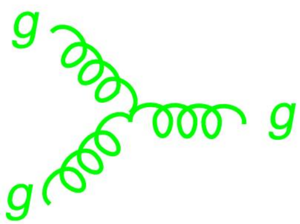
Introduction

- Quantum gravity nonrenormalizable by power counting: Newton's constant, $G_N = 1/M_{\rm Pl}^2$ is dimensionful
- String theory cures divergences of quantum gravity – but particles are no longer pointlike.
- Is this necessary? Or could enough symmetry, e.g. **N=8** supersymmetry, allow a point particle theory of quantum gravity to be perturbatively ultraviolet finite?
- N=8 supergravity (ungauged)

DeWit, Freedman (1977); Cremmer, Julia, Scherk (1978); Cremmer, Julia (1978,1979)

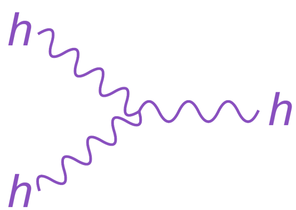
Why gravity should behave badly

gauge theory (spin 1)   renormalizable



$$\supset \ \ell^{\mu}\,\eta^{\nu\rho} \ + \ \cdots$$

gravity (spin 2)   nonrenormalizable



$$\supset \ \ell^{\mu_1}\,\ell^{\mu_2}\,\eta^{\nu_1\rho_1}\,\eta^{\nu_2\rho_2} \ + \ \cdots$$

Extra $\dfrac{\ell^2}{M_{\mathrm{Pl}}^2}$ per loop

Counterterm Basics

- Divergences associated with local counterterms
- On-shell counterterms are generally covariant,
built out of products of Riemann tensor $R_{\mu\nu\sigma\rho}$ (& derivatives $\mathcal{D}_\mu$)
- Terms containing Ricci tensor $R_{\mu\nu}$ and scalar R
removable by nonlinear field redefinition in Einstein action

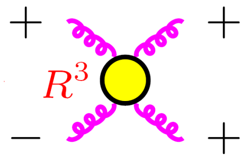
$R^{\mu}_{\nu\sigma\rho} \sim \partial_\rho \Gamma^{\mu}_{\nu\sigma} \sim g^{\mu\kappa}\partial_\rho\partial_\nu g_{\kappa\sigma}$ has mass dimension 2

$G_N = 1/M_{Pl}^2$ has mass dimension -2

Each additional $R_{\mu\nu\sigma\rho}$ or $\mathcal{D}^2$ ←→ 1 more loop (in D=4)

One-loop → $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$
However, $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$
is Gauss-Bonnet term, total derivative in four dimensions.
So pure gravity is UV finite at one loop (but not with matter)
't Hooft, Veltman (1974)

Pure supergravity ($\mathcal{N} \geq 1$):
Divergences deferred to at least three loops



$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$ cannot be supersymmetrized

produces helicity amplitude (−+++) incompatible with SUSY Ward identitites

Grisaru (1977); Deser, Kay, Stelle (1977); Tomboulis (1977)

However, at three loops, there is an N=8 supersymmetric counterterm, abbreviated R^4 ,
plus (many) other terms containing other fields in N=8 multiplet.

Deser, Kay, Stelle (1977); Howe, Lindström (1981); Kallosh (1981);
Howe, Stelle, Townsend (1981)

R^4 produces first subleading term in low-energy limit of 4-graviton scattering in type II string theory:

$$\alpha'^3 R^4 \;\Rightarrow\; \alpha'^3 stu \underbrace{M_4^{\text{tree}}(1,2,3,4)}_{\text{4-graviton amplitude in (super)gravity}}$$

Gross, Witten (1986)

$E_{7(7)}$ Constraints on Counterterms

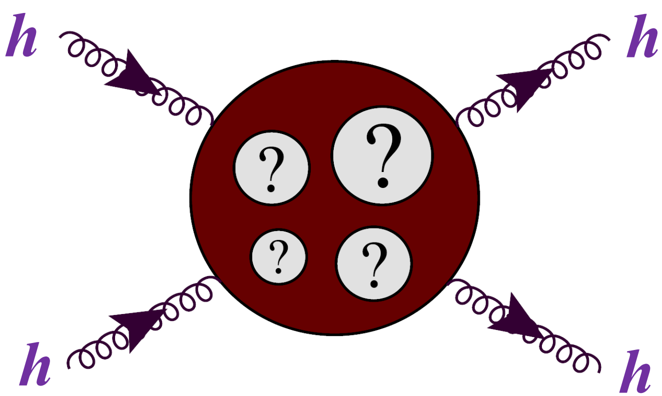
Talk by DeWit

- N=8 SUGRA has continuous symmetries: noncompact form of E_7.
- 70 scalars → coset $E_{7(7)}/\mathrm{SU}(8)$. Non-$\mathrm{SU}(8)$ part realized nonlinearly.

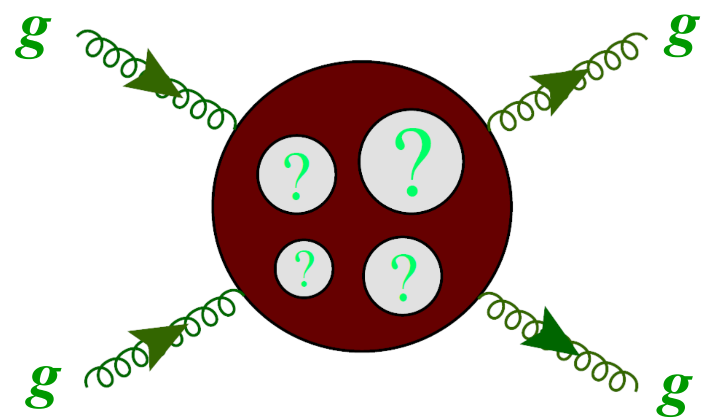
Cremmer, Julia (1978,1979) quantum level: Bossard, Hillmann, Nicolai, 1007.5472

- $E_{7(7)}$ implies amplitude Ward identities, associated with limits as one or two scalars become soft Bianchi, Elvang, Freedman, 0805.0757; Arkani-Hamed, Cachazo, Kaplan, 0808.1446; Kallosh, Kugo, 0811.3414
- Soft limit of NMHV 6-point matrix element of R^4 doesn't vanish; violates $E_{7(7)}$ Elvang, Kiermaier, 1007.4813
- Similar arguments also rule out $\mathcal{D}^4R^4$ and $\mathcal{D}^6R^4$
- However, $\mathcal{D}^8R^4$ is allowed (L=7 for D=4) Beisert et al., 1009.1643
- Same conclusions reached by other methods Bossard, Howe, Stelle, 1009.0743; next talk by Bossard?
- Volume of full N=8 superspace is same dimension as $\mathcal{D}^8R^4$ – but it vanishes! Invariant candidate $\mathcal{D}^8R^4$ counterterm exists, but not full superspace integral Bossard, Howe, Stelle, Vanhove, 1105.6087
- Could it be that on-shell counterterms are inconsistent with duality and/or local SUSY when continued off-shell?? Talk by Kallosh

Strategy for Assessing UV Behavior of N=8 Supergravity



vs.



N=8 SUGRA

N=4 Super-Yang-Mills

How does N=8 SUGRA compare to N=4 SYM? What is the critical dimension $D_c(L)$ in which it first diverges?

A "mere" gauge theory. UV finite in $D = 4$.
Strong evidence that it's also finite at L loops for

$$D < 4 + \frac{6}{L}$$

Full color N=4 SYM and N=8 SUGRA

- Compute N=4 SYM amplitudes for two reasons:

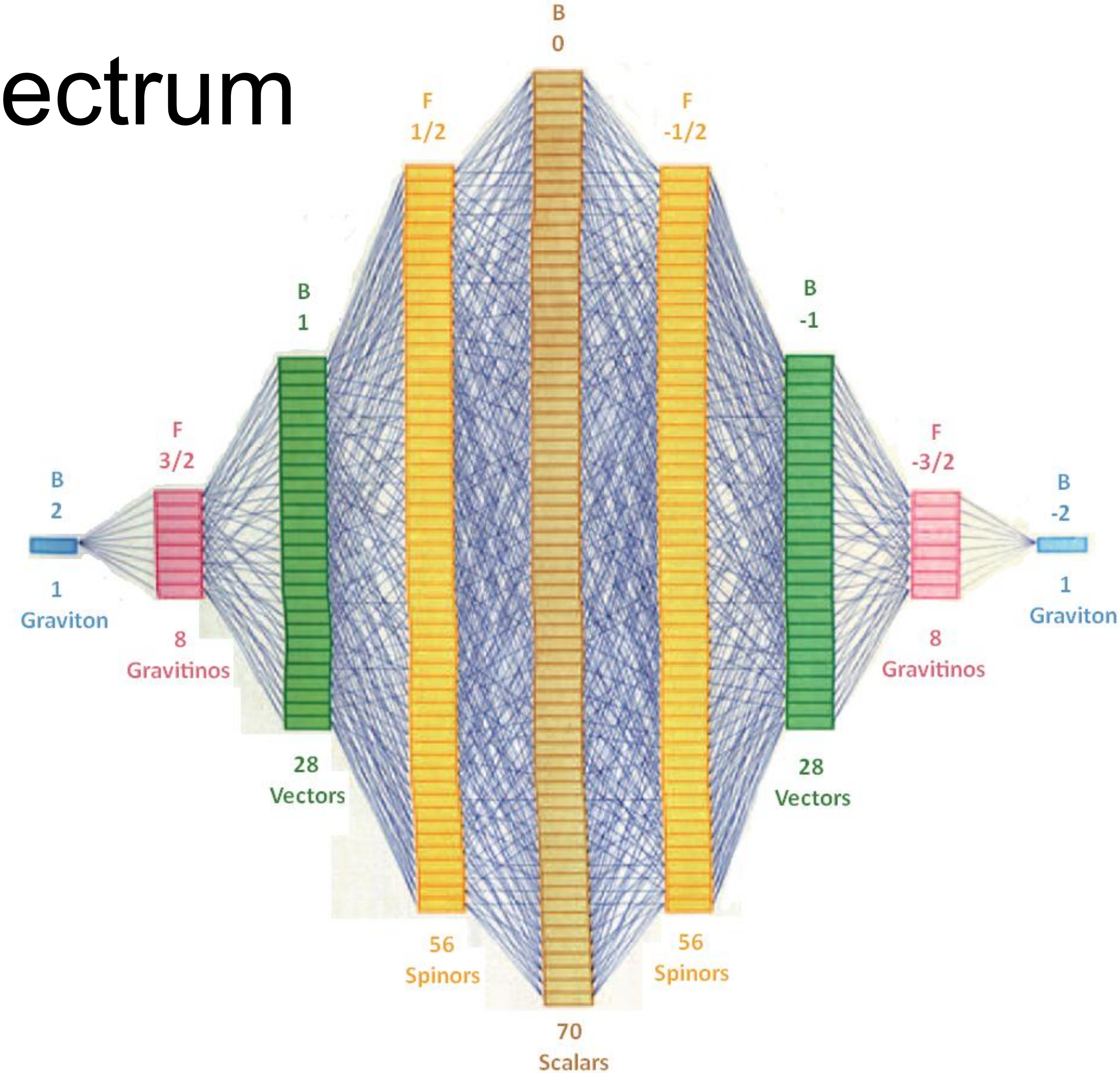
1. Relations between gauge theory and gravity
   (KLT → BCJ/color kinematic duality + double copy) a huge help in constructing gravity amplitudes
   Kawai, Lewellen, Tye (1986); Bern, Carrasco, Johansson, 0805.3993

2. Assess how N=8 SUGRA is doing by comparing UV behavior in D > 4 to N=4 SYM critical dimension,

$$D_c = 4 + \frac{6}{L}$$

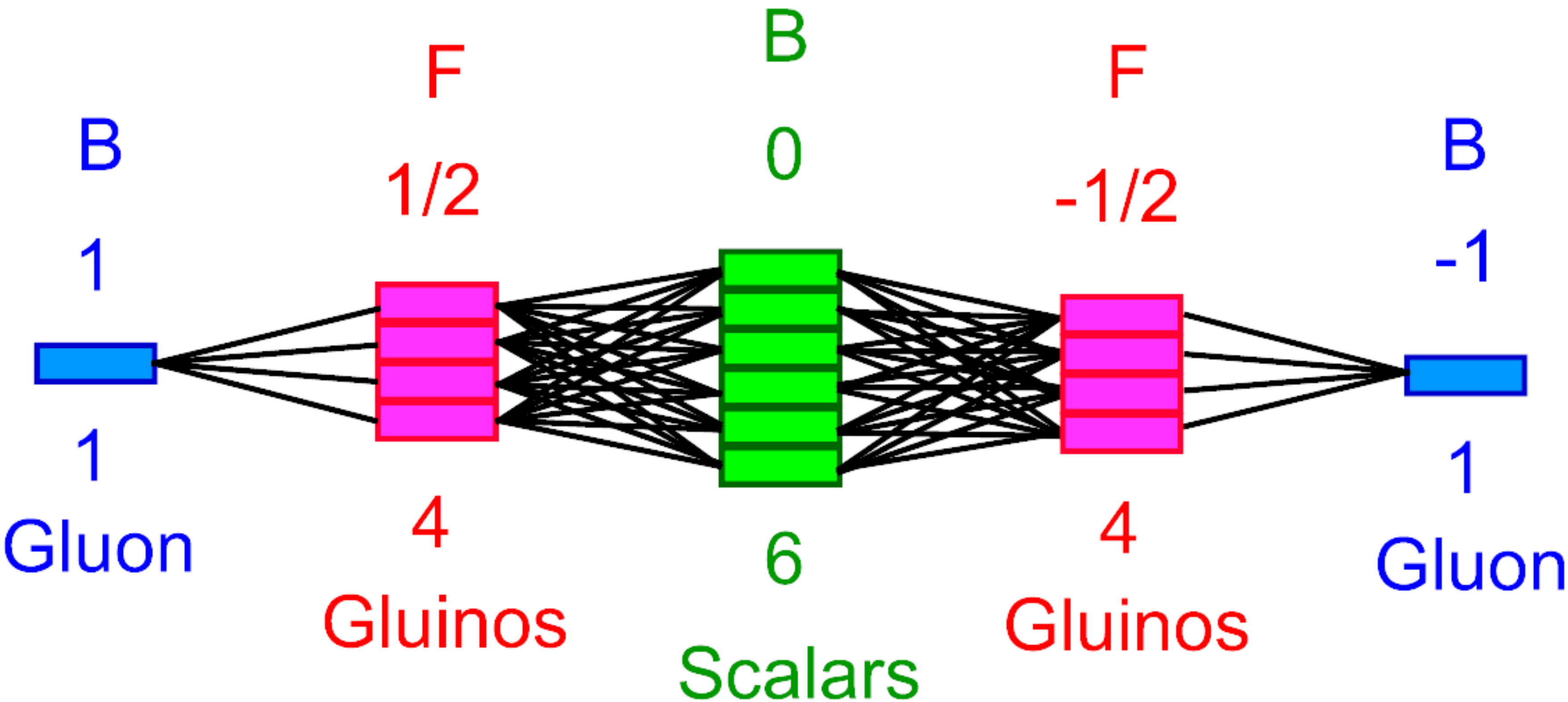
- Need full color N=4 SYM for task 1, but it also provides interesting information for task 2.

N=8 spectrum



B
0
F
1/2
F
-1/2
B
1
B
-1
F
3/2
F
-3/2
B
2
B
-2
1
Graviton
1
Graviton
8
Gravitinos
8
Gravitinos
28
Vectors
28
Vectors
56
Spinors
56
Spinors
70
Scalars

N=4 spectrum much simpler



Color-Kinematic Duality

- First realized for 4-point non-Abelian gauge theory amplitudes by Zhu (1980), Goebel, Halzen, Leveille (1981)
- Massless adjoint gauge theory, color factors $C \sim f^{abe} f^{cde}$:

$$\mathcal{A}_4^{\text{tree}} = \frac{n_s C_s}{s} + \frac{n_t C_t}{t} + \frac{n_u C_u}{u}$$

- Group theory → 3 terms are not independent (Jacobi identity):

$$C_t - C_u = C_s$$

- In suitable gauge, kinematic numerators obey: $\boxed{n_t - n_u = n_s}$
  Same structure can be extended to an arbitrary number of legs and provides a new "KLT-like" relation to gravity ($n_i = \tilde{n}_i$):

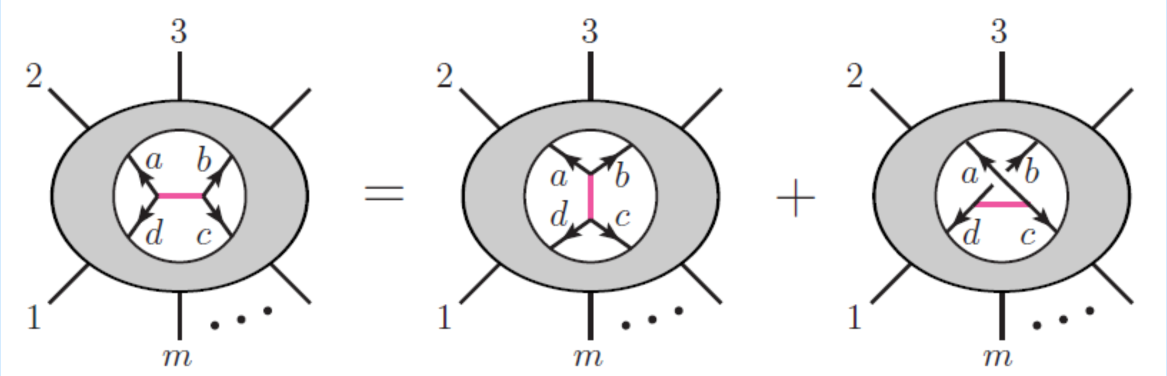
$$M_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

Bern, Carrasco, Johansson, 0805.3993

Color-Kinematic Duality at loop level

BCJ, 1004.0476

- Consider any 3 graphs connected by a Jacobi identity



- Color factors obey

$$C_s = C_t - C_u$$

- Duality requires

$$n_s = n_t - n_u$$

- Very strong constraint on structure of integrands; only a handful of independent integral numerators left after imposing it.

Double-copy formula for gravity

BCJ, 1004.0476; Bern, Dennen, Huang, Kiermaier, 1004.0476

- If an all-adjoint gauge-theory amplitude is represented in terms of cubic graphs Γ:

$$\mathcal{A}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

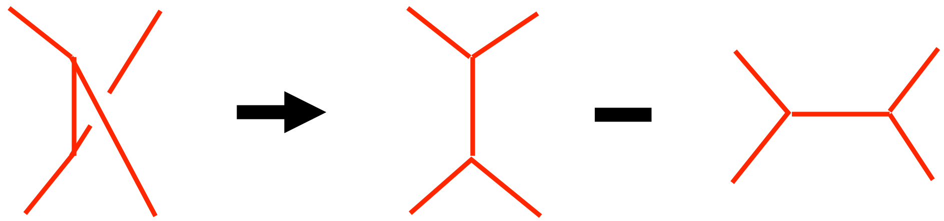
- And numerator factors n_i obey color-kinematics duality
- Then corresponding gravity amplitude is given by

$$\mathcal{M}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

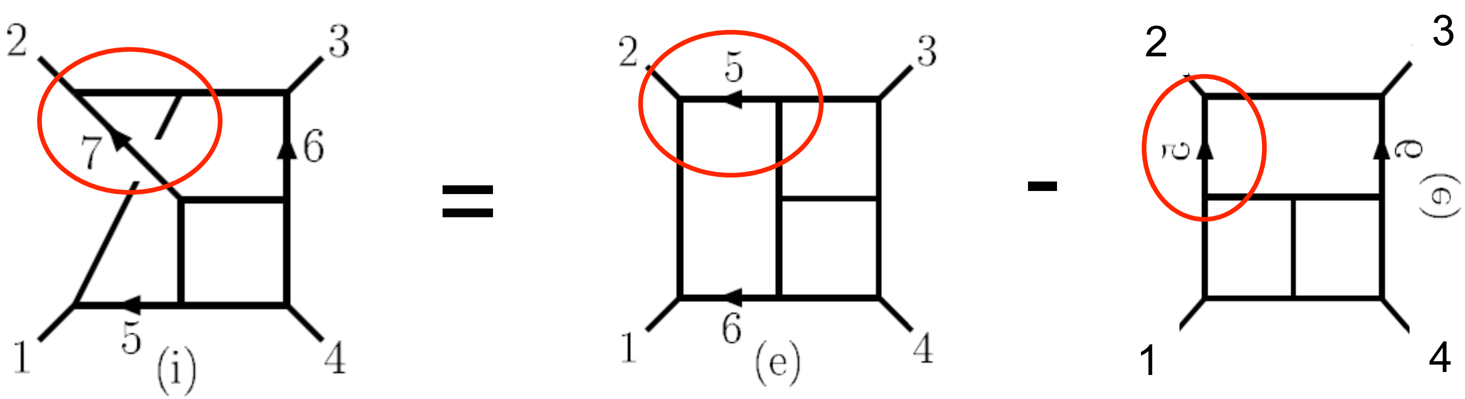
- Argument based on a recursion relation on the integrand.

Simple 3 loop example

Using



we can relate non-planar topologies to planar ones



$$n^{(i)} = n^{(e)} - n^{(e)}|_{1 \leftrightarrow 3}$$

In fact all N=4 SYM 3 loop topologies related to (e) (master graph)

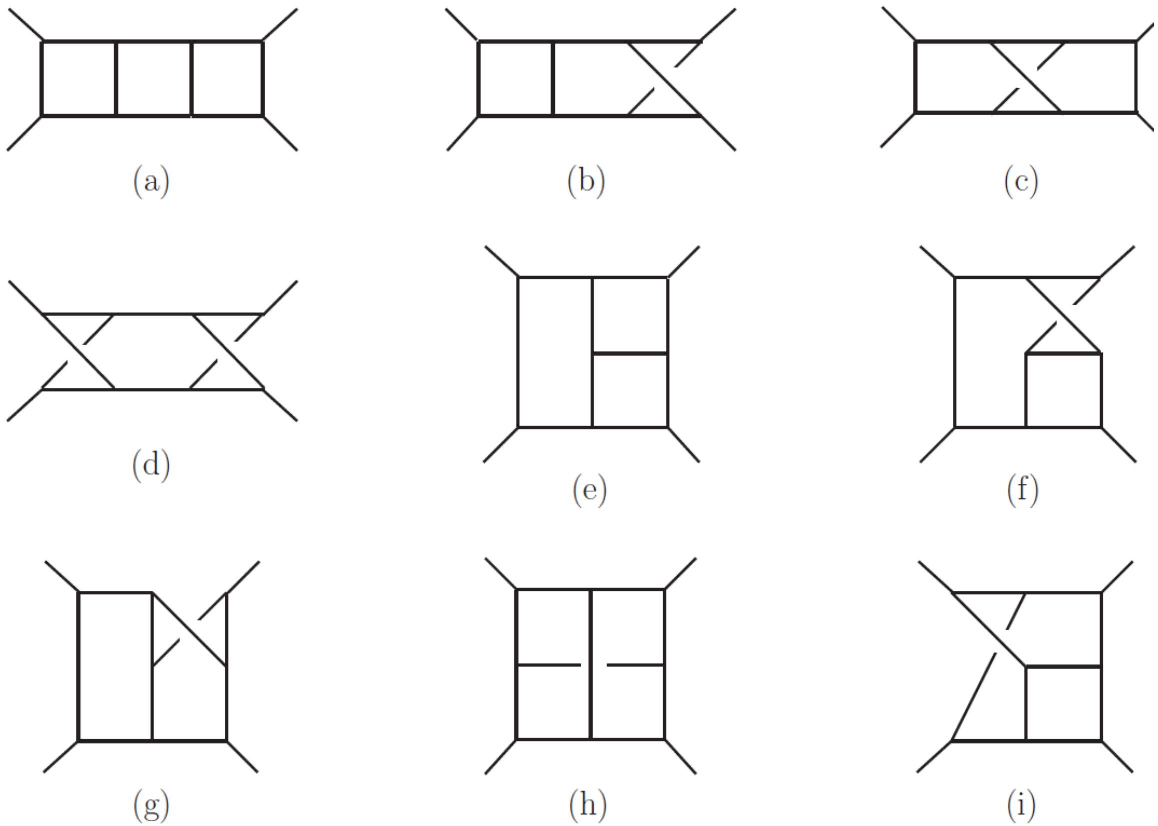
Carrasco, Johansson, 1103.3298

3 loop amplitude before color-kinematics duality

BCDJKR hep-th/0702112;
BCDJR, 0808.4112

Nine basic integral topologies:

- Cubic 1PI graphs only, no triangle subgraphs

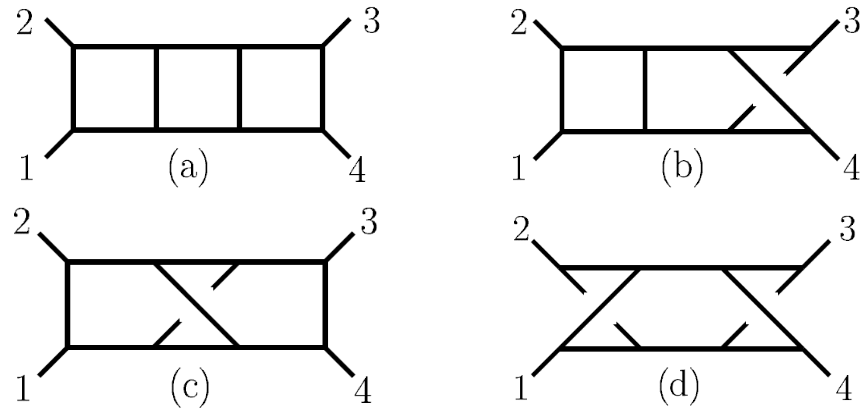


Old N=4 numerators at 3 loops

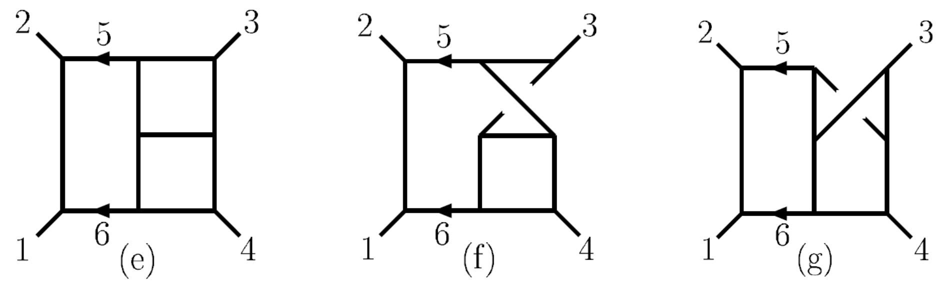
Overall $st\, A_4^{\text{tree}}$

$$s_{iM} = (k_i + \ell_M)^2$$
$$\tau_{iM} = 2k_i \cdot \ell_M$$

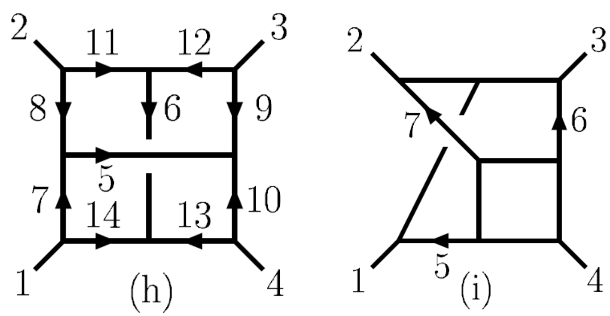
s_{12}^2



$s_{12}\, s_{46}$



$$s_{12}(\tau_{26} + \tau_{36}) + s_{14}(\tau_{15} + \tau_{25}) + s_{12}s_{14}$$



$$- \; s_{12}\, s_{45} \; - \; s_{14}\, s_{46} \; - \; \frac{1}{3}(s_{12} - s_{14})\, \ell_7^2$$

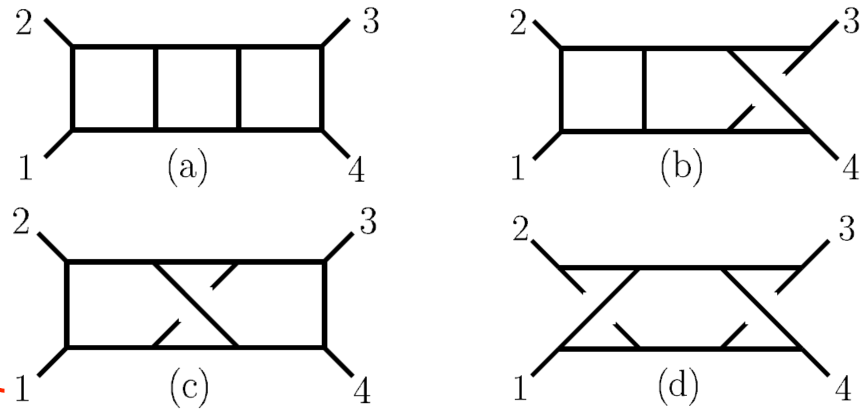
manifestly quadratic in loop momentum ℓ_M

Old N=8 numerators at 3 loops

Overall

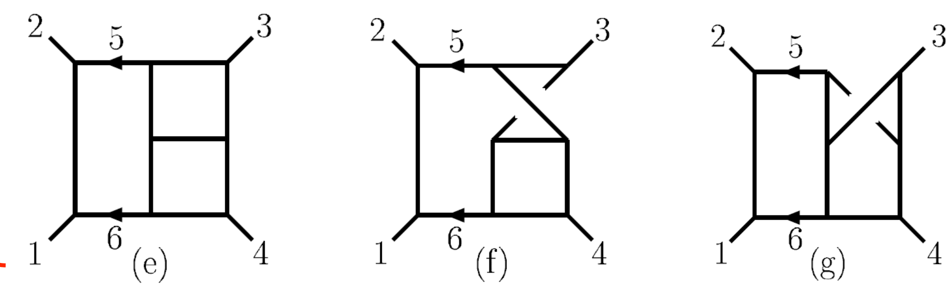
$(stA_4^{\text{tree}})^2 = stu\, M_4^{\text{tree}}$

$$s_{12}^4$$

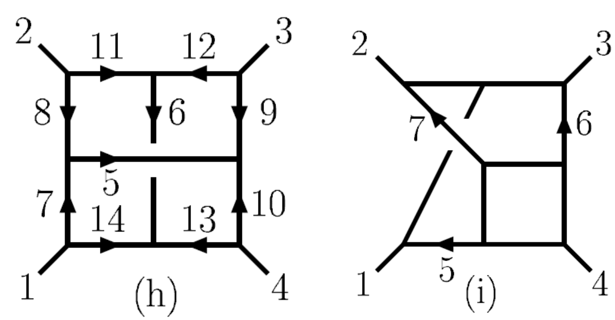


$$\begin{aligned} s_{iM} &= (k_i + \ell_M)^2 \\ \tau_{iM} &= 2k_i \cdot \ell_M \end{aligned}$$

$$s_{12}^2\, \tau_{35}\, \tau_{46}$$



$$\begin{aligned} &(s_{12}(\tau_{26}+\tau_{36}) + s_{14}(\tau_{15}+\tau_{25}) + s_{12}s_{14})^2 \\ &+ (s_{12}^2(\tau_{26}+\tau_{36}) - s_{14}^2(\tau_{15}+\tau_{25})) \\ &\times (\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10}) \\ &+ s_{12}^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) \\ &+ s_{14}^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10}) \\ &+ s_{13}^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10}) \end{aligned}$$



$$\begin{aligned} &(s_{12}\tau_{45} - s_{14}\tau_{46})^2 \\ &- \tau_{27}(s_{12}^2\tau_{45} + s_{14}^2\tau_{46}) \\ &- \tau_{15}(s_{12}^2\tau_{47} + s_{13}^2\tau_{46}) \\ &- \tau_{36}(s_{14}^2\tau_{47} + s_{13}^2\tau_{45}) \\ &+ l_5^2\, s_{12}^2\, s_{14} + l_6^2\, s_{12}\, s_{14}^2 \\ &- \frac{1}{3} l_7^2\, s_{12}\, s_{13}\, s_{14} \end{aligned}$$

Had to work hard to make manifestly quadratic in ℓ_M

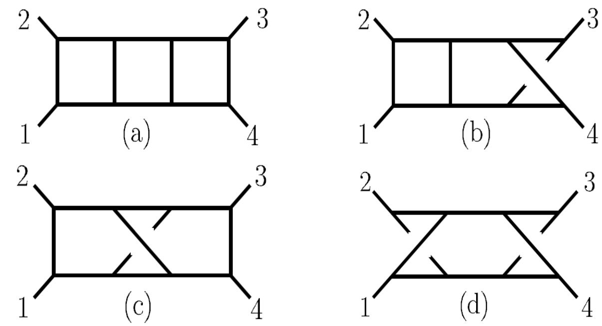
BCDJR (2008)

3 loop amplitude after color-kinematics duality

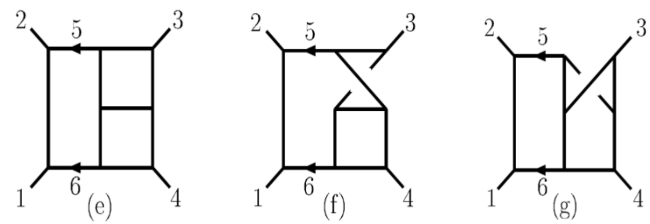
BCJ, 1004.0476

N=8 SUGRA

$$[\, s^2 \,]^2$$

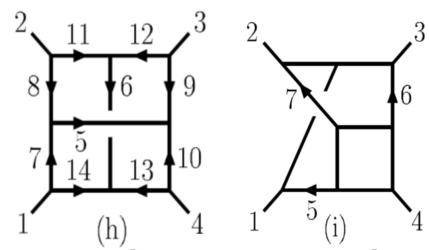


$$\left[\ \tfrac{1}{3}\ \ [s(t-\tau_{36}-\tau_{46}) - t(\tau_{26}+\tau_{46}) + u(\tau_{26}+\tau_{36}) - s^2]\ \right]^2$$



Linear in ℓ_M

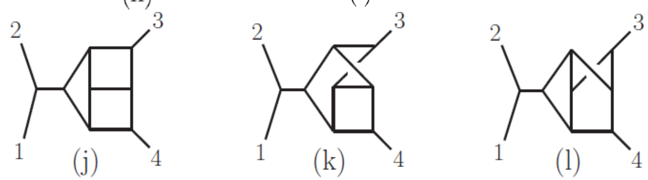
$$[N^{(h)}(\tau_{ij})\,]^2$$



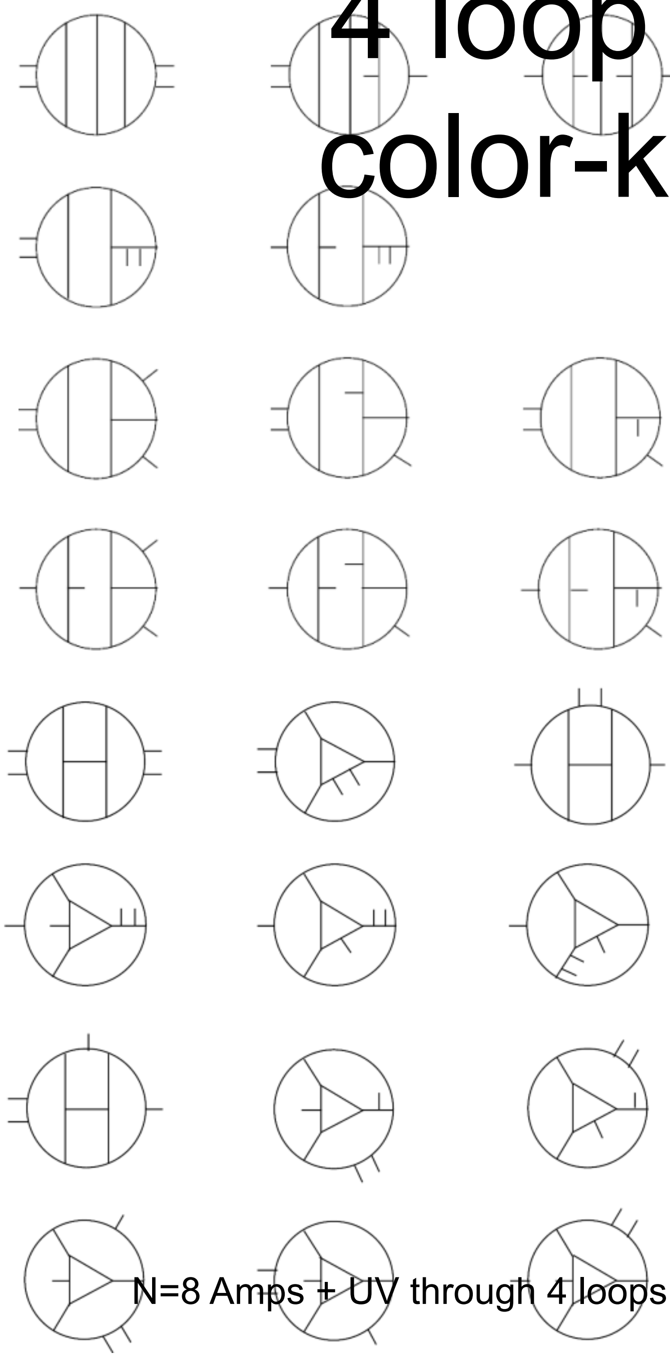
$$[N^{(i)}(\tau_{ij})\,]^2$$

Add 3 1PR graphs:

$$\left[\frac{1}{3}s(t-u)\right]^2$$



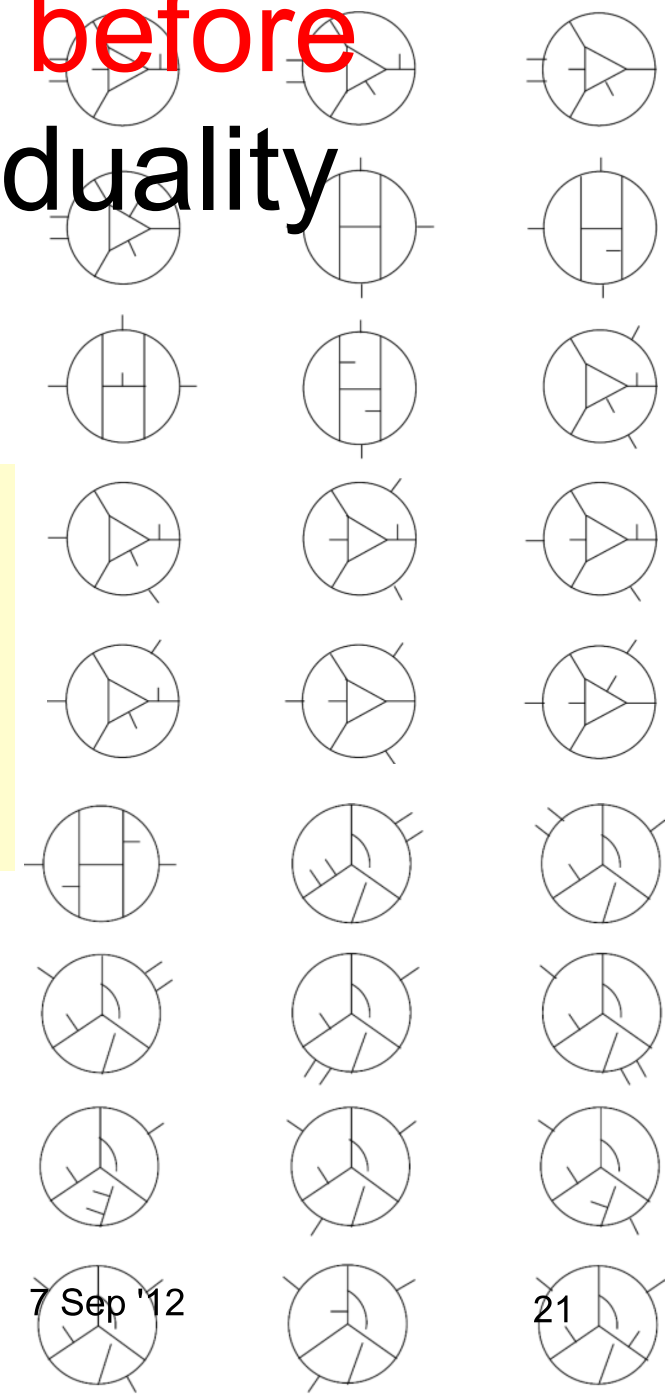
4 loop amplitude before color-kinematics duality



BCDJR, 0905.2326, 1008.3327

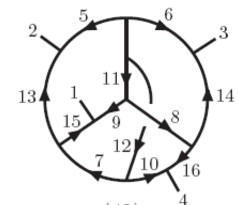
50 nonvanishing 4-point graphs

- Cubic 1PI graphs only, no triangle or bubble subgraphs



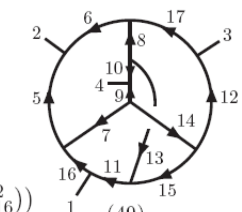
N=4 SYM numerators for most complex graphs [N=8 SUGRA numerators much larger]

$$
\begin{aligned}
&s_{12}(s_{2,10}s_{39} - s_{47}s_{18} + s_{2,10}s_{59} + s_{39}s_{6,10} + s_{23}s_{6,11}) - s_{23}s_{57}s_{68} - s_{13}s_{59}s_{6,10} \\
&+l_6^2(s_{12}s_{35} + s_{12}s_{4,\overline{12}} - s_{23}s_{59}) + l_5^2(s_{12}s_{26} + s_{12}s_{1,\overline{11}} - s_{23}s_{6,10}) \\
&+l_9^2(s_{12}s_{12,\overline{13}} - s_{13}s_{10,11}) + l_{10}^2(s_{12}s_{11,\overline{14}} - s_{13}s_{9,12}) \\
&-l_{13}^2 s_{12}s_{11,\overline{14}} - l_{14}^2 s_{12}s_{12,\overline{13}} + (s_{13} - 2s_{12})l_9^2 l_{10}^2 \\
&+s_{23}(l_5^2l_6^2 - l_7^2l_8^2 + l_6^2l_7^2 + l_5^2l_8^2) + s_{12}l_{13}^2l_{14}^2 + s_{12}l_5^2l_6^2 \\
&+s_{12}(-l_5^2l_8^2 + l_5^2l_9^2 - l_5^2l_{11}^2 - l_5^2l_{15}^2 - l_9^2l_{15}^2) \\
&+s_{12}(-l_6^2l_7^2 + l_6^2l_{10}^2 - l_6^2l_{12}^2 - l_6^2l_{16}^2 - l_{10}^2l_{16}^2) \\
&+s_{23}(l_9^2l_{12}^2 + l_{10}^2l_{11}^2 - l_7^2l_9^2 - l_8^2l_{10}^2) + s_{13}(l_9^2l_{11}^2 + l_{10}^2l_{12}^2)
\end{aligned}
$$



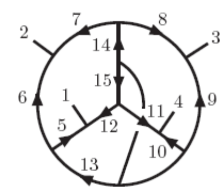
(48)

$$
\begin{aligned}
&s_{12}(s_{47}s_{5,12} - s_{19}s_{36} - s_{48}s_{36}) + s_{23}(s_{48}s_{6,11} - s_{15}s_{3,10} - s_{15}s_{47}) - s_{12}s_{23}s_{11,12} \\
&+l_5^2(s_{23}s_{7,12} - s_{23}s_{4,15} - s_{13}s_{10,11}) + l_6^2(s_{12}s_{8,11} - s_{12}s_{4,\overline{15}} - s_{13}s_{9,12}) \\
&+l_9^2(s_{23}s_{3,15} - s_{12}s_{3\overline{8}} + s_{23}s_{6,10}) + l_{10}^2(s_{12}s_{1,\overline{15}} - s_{23}s_{1\overline{7}} + s_{12}s_{59}) \\
&+l_{13}^2(s_{12}s_{23} + s_{12}s_{38} - s_{23}s_{6,11}) + l_{14}^2(s_{23}s_{12} + s_{23}s_{17} - s_{12}s_{5,12}) \\
&+l_{11}^2 s_{23}(s_{4,12} - s_{6,10}) + l_{12}^2 s_{12}(s_{4,11} - s_{59}) \\
&+s_{13}(l_7^2l_8^2 + l_5^2l_8^2 + l_6^2l_7^2 + l_{11}^2l_{12}^2 + l_{10}^2l_{16}^2 + l_9^2l_{17}^2 - l_9^2l_{12}^2 - l_{10}^2l_{11}^2) \\
&+s_{12}(-l_5^2l_{10}^2 + l_6^2(l_{14}^2 + l_{13}^2 - l_{10}^2) + l_{12}^2(l_9^2 - l_5^2 - l_7^2 + l_{14}^2) + l_8^2(l_9^2 + l_{16}^2)) \\
&+s_{23}(-l_6^2l_9^2 + l_5^2(l_{13}^2 + l_{14}^2 - l_9^2) + l_{11}^2(l_{10}^2 - l_6^2 - l_8^2 + l_{13}^2) + l_7^2(l_{10}^2 + l_{17}^2)) \\
&+s_{12}(l_{12}^2l_{13}^2 - l_8^2l_{13}^2 - l_{10}^2l_{13}^2 - l_{10}^2l_{14}^2 - l_{13}^2l_{17}^2) + s_{23}(l_{11}^2l_{14}^2 - l_7^2l_{14}^2 - l_9^2l_{14}^2 - l_9^2l_{13}^2 - l_{14}^2l_{16}^2)
\end{aligned}
$$



(49)

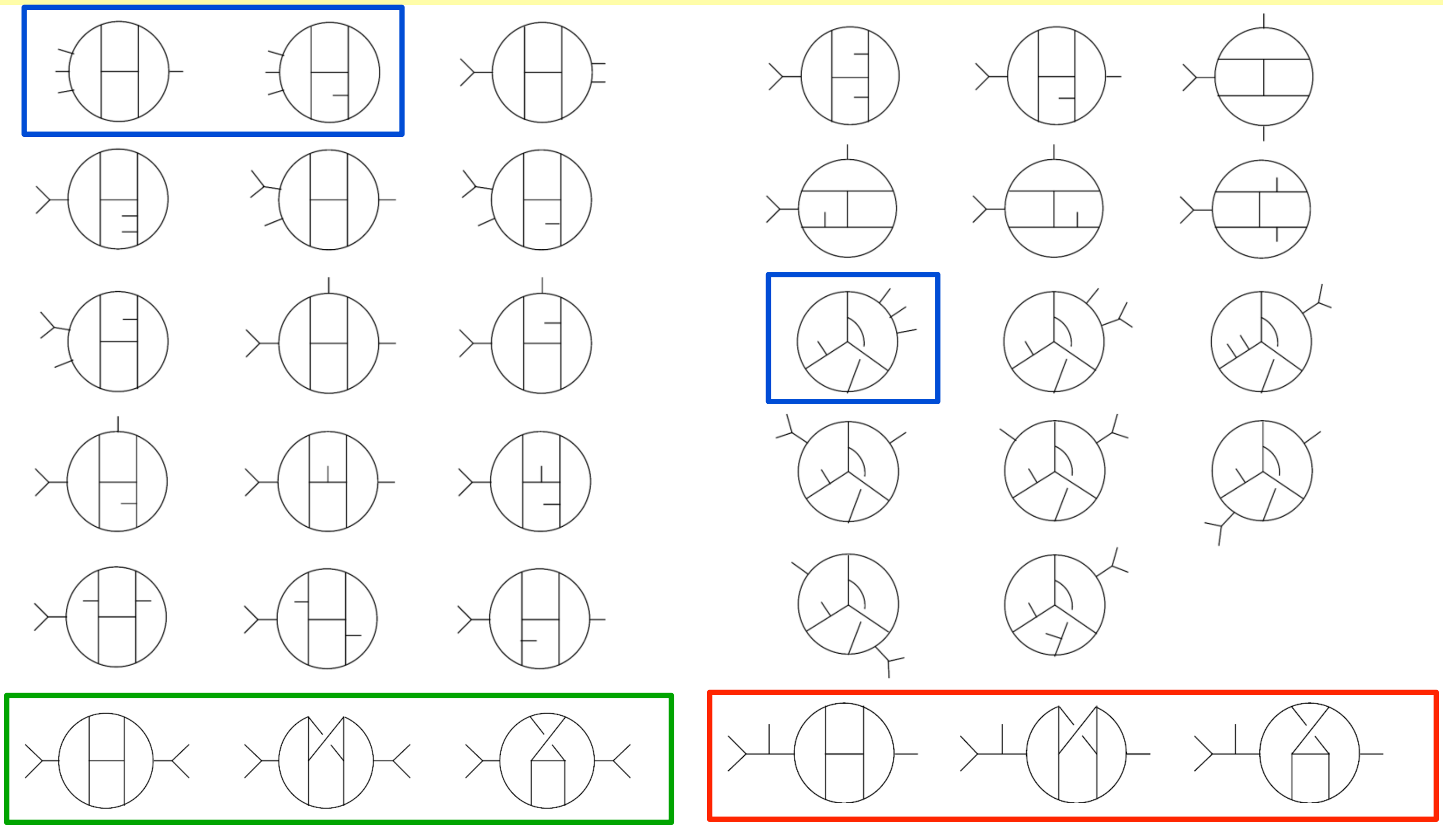
$$
\begin{aligned}
&s_{12}s_{28}s_{4,12} - s_{12}s_{37}s_{1,11} - s_{23}s_{16}s_{3,10} \\
&+s_{23}s_{25}s_{49} + \tfrac{1}{2}s_{12}s_{23}(s_{13,15} - s_{13,14}) \\
&+s_{12}(l_6^2l_{10}^2 - l_5^2l_9^2) + s_{23}(l_7^2l_{11}^2 - l_8^2l_{12}^2)
\end{aligned}
$$



(50)

4 loop amplitude after color-kinematics duality

50 nonvanishing 1PI cubic 4-point graphs BCDJR, 1201.5366
+ 3 more 1PI graphs (0 in previous representation)
+ 32 1PR graphs (6 of which are 2PR) → 85 in all



Minimal set of (simplified) duality relations

$$n_i \equiv st A_4^{\text{tree}} N_i$$

2 term relations, due to generation of vanishing triangle subgraphs:

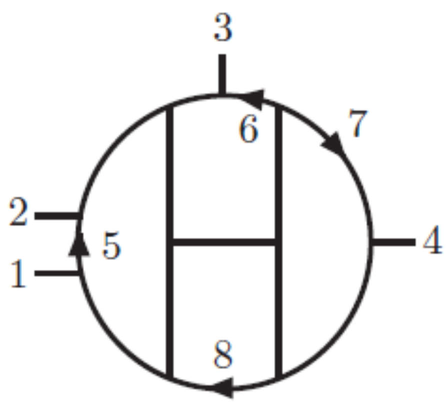
$$\begin{aligned}
N_5 &= N_4 = N_3 = N_2 = N_1\,,\\
N_{11} &= N_{10} = N_9 = N_8 = N_7 = N_6\,,\\
N_{40} &= N_{13} = -N_{12}\,,\\
N_{41} &= -N_{17} = -N_{16} = -N_{15} = N_{14}\,,\\
N_{42} &= N_{20} = -N_{19} = N_{18}\,,\\
N_{43} &= -N_{23} = -N_{22} = -N_{21}\,,\\
N_{25} &= N_{24}\,,\\
N_{44} &= -N_{26}\,,\\
N_{31} &= -N_{30} = N_{29} = N_{28}\,,\\
N_{46} &= N_{34} = N_{32}\,,\\
N_{36} &= -N_{35} = -N_{33}\,,\\
N_{47} &= N_{38}\,,\\
N_{72} &= N_{52} = N_{51}\,,\\
N_{74} &= -N_{54} = -N_{53}\,,\\
N_{73} &= N_{57} = N_{56} = N_{55}\,,\\
N_{76} &= -N_{62} = -N_{61} = -N_{60} = -N_{59} = -N_{58}\\
N_{77} &= -N_{65} = N_{64} = N_{63}\,,\\
N_{78} &= -N_{67} = N_{66}\,,\\
N_{75} &= N_{71} = N_{70} = N_{69} = N_{68}\,,\\
N_{82} &= N_{81} = N_{80}\,,\\
N_{85} &= N_{84} = N_{83}\,.
\end{aligned}$$

3 term relations:

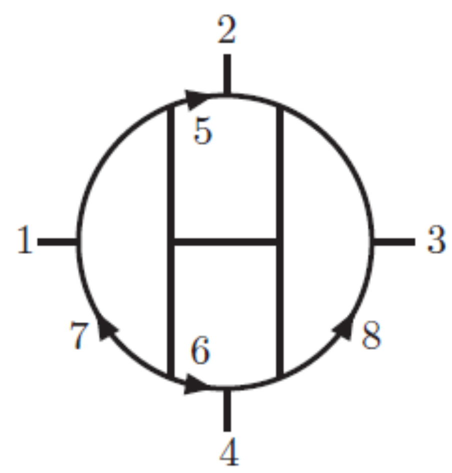
$$\begin{aligned}
N_{58} &= N_{18}(k_1,k_2,k_3,k_2-l_6,l_5,l_7,l_8) - N_{18}(k_2,k_1,k_3,k_1-l_6,l_5,l_7,l_8)\,,\\
N_{33} &= N_{28}(k_4,k_3,k_2,k_3-l_5,k_2-l_6+l_7,l_7,l_8) - N_{18}(k_1,k_2,k_3,k_2-l_6,k_3-l_5,l_7,l_8)\,,\\
N_{50} &= N_{28}(k_2,k_1,k_4,l_5,k_3-l_7,l_7,l_8) - N_{28}(k_1,k_2,k_3,l_6,k_4-l_8,l_7,l_8)\,,\\
N_6 &= -N_{33}(k_1,k_2,k_4,l_7,l_5-l_6,k_1-l_6,l_8) - N_{33}(k_2,k_1,k_4,l_7,l_6,k_2-l_5+l_6,l_8)\,,\\
N_{14} &= -N_{33}(k_3,k_2,k_1,l_5,-l_5-l_7,k_3-l_7+l_8,l_6) - N_{33}(k_3,k_2,k_1,l_5,k_2+l_7,l_7-l_8,l_8)\,,\\
N_{24} &= -N_{28}(k_1,k_2,k_3,l_5-l_7,-l_6,l_7,l_8) - N_{33}(k_1,k_2,k_4,-l_6,-l_7,-l_5,l_8)\,,\\
N_{32} &= -N_{28}(k_4,k_2,k_1,l_7,k_3-l_5,l_7,l_8) - N_{33}(k_2,k_1,k_3,l_5,l_6,k_2+l_5+l_6-l_7,l_8)\,,\\
N_{48} &= N_{28}(k_3,k_4,k_1,l_8,k_2-l_5,l_7,l_8) - N_{33}(k_1,k_2,k_3,k_3-l_6,k_2-l_5,l_7,l_8)\,,\\
N_{49} &= -N_{33}(k_1,k_2,k_3,k_3-l_8,k_2-l_5,-l_7,l_8) - N_{33}(k_4,k_1,k_2,l_5,-l_7,l_6,l_8)\,,\\
N_{66} &= N_{58}(k_1,k_2,k_4,l_5-k_3-l_6,l_6,l_7,l_8) - N_{58}(k_1,k_2,k_3,k_3+l_6,l_6,l_7,l_8)\,,\\
N_1 &= -N_6(k_1,k_2,k_3,l_6,l_5,l_7,l_8) - N_6(k_1,k_2,k_4,l_6,l_5,l_7,l_8)\,,\\
N_{68} &= N_{14}(k_1,k_2,k_3,k_1-l_5,-l_6,-l_7,-l_8) - N_{14}(k_1,k_2,k_4,l_5-k_2,-l_7,-l_6,l_8)\,,\\
N_{21} &= -N_{14}(k_2,k_1,k_3,l_5,l_6,l_7,l_8) - N_{18}(k_2,k_1,k_3,-l_5,k_1+k_3+l_5-l_6,l_7,l_8)\,,\\
N_{26} &= N_{24}(k_2,k_1,k_3,-l_5,-k_4-l_6-l_7,l_8,l_6) - N_{24}(k_2,k_1,k_4,-l_5,l_7-k_3,l_6-k_1-l_5-l_8,l_6)\,,\\
N_{27} &= -N_{18}(k_2,k_1,k_4,-l_5,l_7,l_7,l_8) - N_{24}(k_1,k_2,k_4,l_5,-k_3-l_7-l_8,k_3-l_6+l_7+l_8,l_8)\,,\\
N_{37} &= -N_{28}(k_2,k_1,k_3,k_1-l_5,k_4+l_8,l_7,l_6) - N_{49}(k_2,k_1,k_3,k_1-l_5,-l_8,l_7-k_2,l_6)\,,\\
N_{39} &= N_{28}(k_2,k_1,k_3,-l_5-l_7,k_4+l_6+l_8,l_5,l_6) - N_{48}(k_1,k_2,k_3,l_7,l_8,-l_5-l_7,-l_6-l_8)\,,\\
N_{45} &= N_{49}(k_1,k_2,k_3,l_5-l_6-l_7-l_8,k_4-l_6,l_5,l_7)\\
&\quad + N_{49}(k_1,k_2,k_4,k_2+l_6+l_7+l_8,l_7,l_5,k_4-l_6)\,,\\
N_{38} &= N_{49}(k_2,k_1,k_4,l_6,k_3+l_5+l_7,-l_5+l_6,k_4-l_8)\\
&\quad - N_{49}(k_1,k_2,k_4,l_5-l_6,k_3+l_5+l_7,-l_6,l_7+l_8)\,,\\
N_{53} &= N_{58}(k_1,k_2,k_3,k_3-l_8,l_6,l_7,l_8) + N_{66}(k_1,k_2,k_4,l_8,-k_4-l_5,l_7,l_8)\,,\\
N_{12} &= N_{18}(k_4,k_3,k_2,l_6,k_2+l_8,l_5,l_7) + N_{26}(k_3,k_4,k_1,-l_6,l_8,-l_5,l_8)\,,\\
N_{51} &= N_{18}(k_3,k_2,k_1,k_1+k_2-l_5,-l_6,l_7,l_8) - N_{21}(k_2,k_3,k_1,l_5-k_1-k_2,-l_6,l_7,l_8)\,,\\
N_{63} &= N_{21}(k_1,k_2,k_3,k_2-l_5,k_1+k_2-l_5-l_6,l_7,l_8)\\
&\quad - N_{21}(k_2,k_1,k_3,k_1-l_5,k_1+k_2-l_5-l_6,l_7,l_8)\,,\\
N_{79} &= N_{45}(k_1,k_2,k_3,k_2-l_5,k_4-l_7,l_6,-l_6-l_8)\\
&\quad - N_{45}(k_1,k_2,k_3,l_5-k_1,l_7,k_3-l_6,k_4+l_5-l_7-l_8)\,,\\
N_{80} &= N_{53}(k_1,k_2,k_3,k_3-l_7,l_6,l_7,l_8) + N_{53}(k_1,k_2,k_3,l_7-k_4,l_5,l_6,l_8)\,,\\
N_{55} &= N_{51}(k_1,k_2,k_3,k_1+l_5,l_6,l_7,l_8) - N_{51}(k_1,k_3,k_2,k_1+l_5,l_6,l_7,l_8)\,,\\
N_{83} &= -N_{55}(k_3,k_1,k_2,k_1+k_2-l_5,l_8,l_6,l_7) - N_{55}(k_3,k_1,k_2,l_5-k_3,l_6,l_7,l_8)\,.
\end{aligned}$$

Duality relations and master graphs

- Solve system of linear relations between numerators in terms of numerators for a few “master” integrals
- Convenient to choose 2 planar integrals as masters, (18) and (28).
- Generalized unitarity determines N_{18} and N_{28}

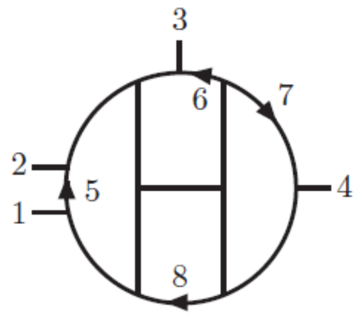


(18)

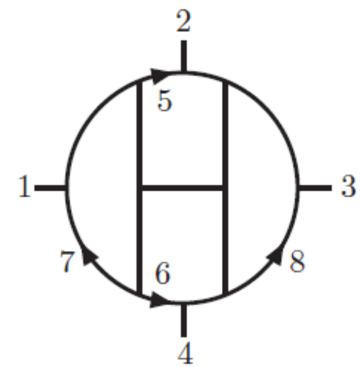


(28)

Full integrand specified by



(18)



(28)

$$
\begin{aligned}
N_{18} &= \tfrac{1}{4}(6u^2\tau_{25} + u(2s(5\tau_{25}+2\tau_{26}) - \tau_{15}(7\tau_{16}+6t)) \\
&\quad + t(\tau_{15}\tau_{26} - \tau_{25}(\tau_{16}+7\tau_{26})) + s(4\tau_{15}(t-\tau_{26}) + 6\tau_{36}(\tau_{35}-\tau_{45}) \\
&\quad - \tau_{16}(4t+5\tau_{25}) - \tau_{46}(5\tau_{35}+\tau_{45})) + 2s^2(t+\tau_{26}-\tau_{35}+\tau_{36}+\tau_{56})) \\
N_{28} &= \tfrac{1}{4}(s(2\tau_{15}t + \tau_{16}(2t-5\tau_{25}+\tau_{35}) + 5\tau_{35}(\tau_{26}+\tau_{36}) + 2t(2\tau_{46}-\tau_{56}) - 10u\tau_{25}) \\
&\quad - 4s^2\tau_{25} - 6u(\tau_{46}(t-\tau_{25}+\tau_{45}) + \tau_{25}\tau_{26}) - t(\tau_{15}(4\tau_{36}+5\tau_{46}) + 5\tau_{25}\tau_{36}))
\end{aligned}
$$

plus duality relations for the rest!

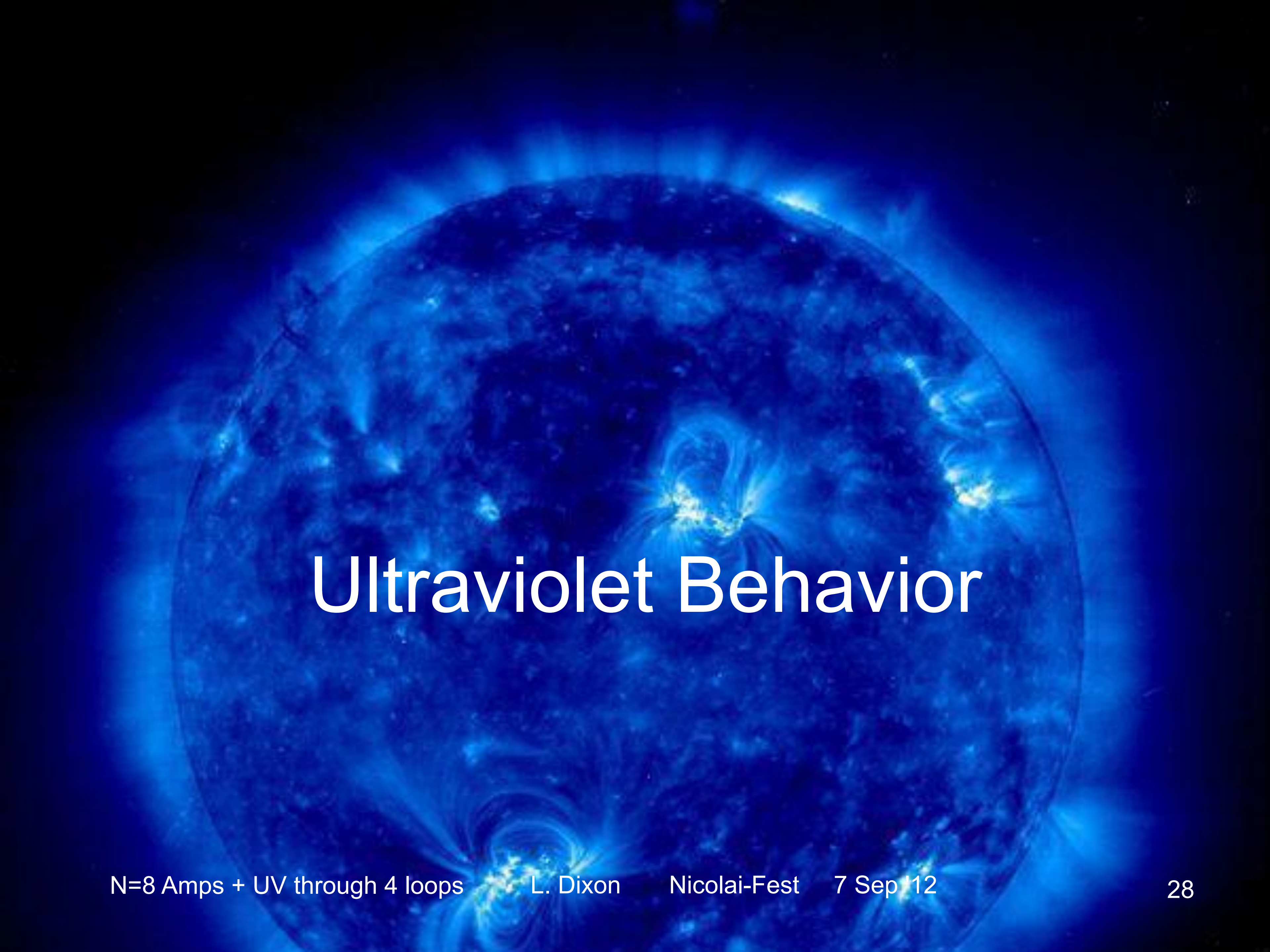
Checks and gravity amplitude

- Unitarity cuts of new N=4 SYM integrand agree with those of an old form computed without BCJ [1008.3327].
- To get N=8 SUGRA, we use double copy formula:

$$\mathcal{A}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

$$\mathcal{M}_4^{(L)} = \sum_{i\in\Gamma} \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i^2}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

- Cuts of new N=8 supergravity amplitude also agree with a previous (KLT driven) construction [0905.2326]



Ultraviolet Behavior

UV divergences at 3 loops in $D_c = 4 + 6/3 = 6$

- N=4 SYM: 1PI graphs (x) = (a), (b), … , (i) all have 10 propagators, and numerators $N^{(x)}(l_i)$ that are at most linear in loop momenta l_i.

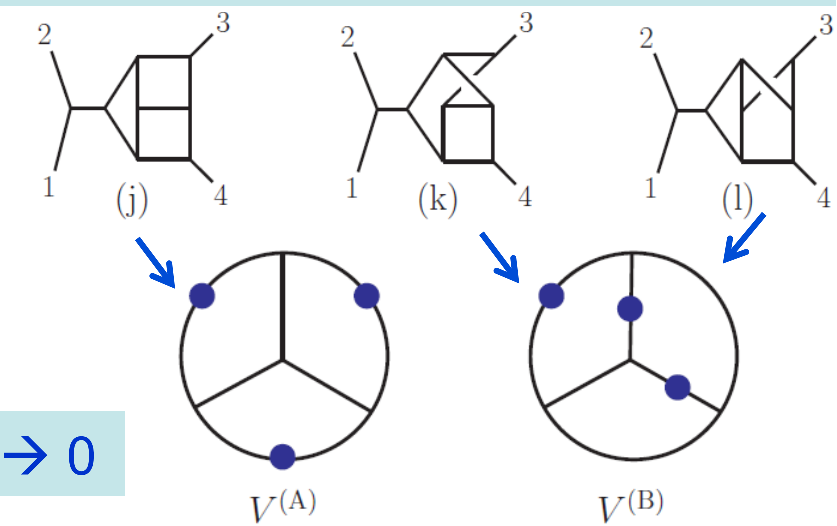
$$\Rightarrow \quad I^{(x)} \sim \int \frac{(d^6 l_i)^3 \; l_i^\mu}{[(l_i)^2]^{10}}$$

finite in $D = 6$

Only divergences come from 1PR 9 propagator graphs (y) = (j), (k), (l)

$$N^{(y)} = \frac{1}{3} s(t-u)$$

$$\Rightarrow \quad I^{(y)} \sim \int \frac{(d^{6-2\epsilon} l_i)^3}{[(l_i)^2]^9}$$

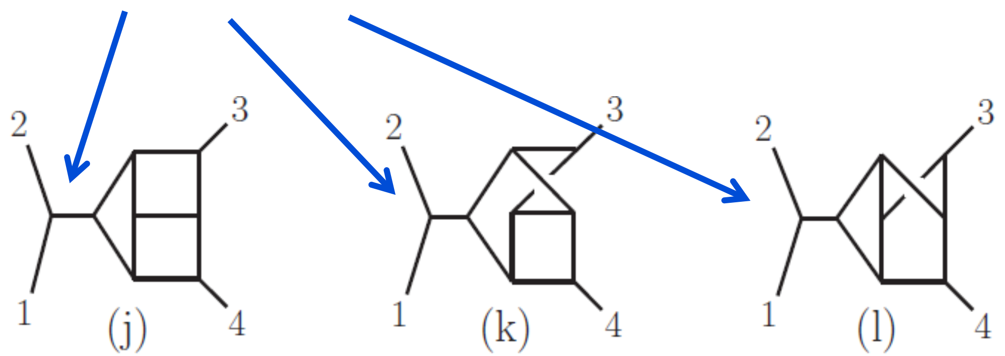


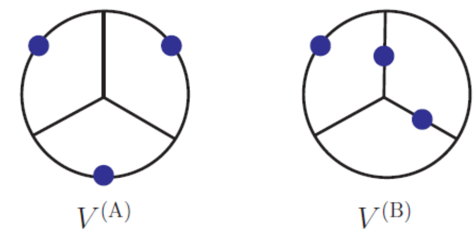
Log divergence → just set external $k_i \to 0$

3 loop N=4 SYM UV color structure

- BCJ form manifestly has no double trace terms in $D_c = 6$:
- Color factors for divergent graphs contain explicit

$$f^{a_1 a_2 b} f^{b a_3 a_4} = \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \pm \cdots$$





$$\mathrm{Tr}_{ijkl} \equiv \mathrm{Tr}(T^{a_i} T^{a_j} T^{a_k} T^{a_l})$$

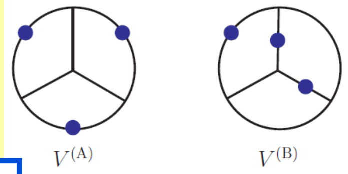
$$\mathcal{A}_4^{(3)}(1,2,3,4)\Big|_{\text{pole}}^{SU(N_c)} = 2\, g^8\, \mathcal{K} \left(N_c^3\, V^{(\mathrm{A})} + 12\, N_c \,(V^{(\mathrm{A})} + 3\, V^{(\mathrm{B})}) \right)$$
$$\times \left(s\,(\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + t\,(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u\,(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432}) \right)$$

LD @ Amps 2009, BCDJR, 1008.3327

String-theory argument for double-trace absence via collision of 2 vertex operators: Berkovits, Green, Russo, Vanhove, 0908.1923

3 loop N=8 SUGRA UV structure

- 1PI graphs (x) = (a) – (d) have loop-momentum independent (scalar) numerators, also after squaring
→ finite in $D = 6$.
- 1PI graphs (x) = (e) – (i) were linear in l_i in SYM, become quadratic in SUGRA, so they do contribute to the UV pole
- As do 1PR scalar graphs (y) = (i), (j), (k).
- Total:



$$\mathcal{M}_4^{(3)}\Big|_{\rm pole} = -\left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\rm tree}\left[10\,(V^{(A)}+3\,V^{(B)})\right]$$

Curiously, **same linear combination** of $V^{(A)}$ and $V^{(B)}$ as in subleading-color part of N=4 SYM divergence! Understandable for (y) graphs, but why for 1PI ones?

3 loop summary: N=8 no worse than N=4 SYM in UV

Manifest quadratic representation at 3 loops – same as N=4 SYM – implies same critical dimension (as for $L = 2$):

$$I_3^{\text{quad.}} \sim \int \frac{(d^6 l_i)^3 \; l_i^2}{[(l_i)^2]^{10}} \sim \ln \Lambda$$

$$D_c = 4 + \frac{6}{L} = 6$$

$$M_4^{(3),D=6-2\epsilon}\Big|_{\text{pole}} = \frac{1}{\epsilon}\,\frac{5\zeta_3}{(4\pi)^9}\left(\frac{\kappa}{2}\right)^8 (s_{12}s_{13}s_{14})^2 M_4^{\text{tree}}$$

$\mathcal{D}^6 R^4$ counterterm

Also recovered via string theory argument (up to factor of 9?) Green, Russo, Vanhove,1002.3805; talk by Green?

UV divergences at 4 loops in $D = 4 + 6/4 = 11/2 = 5.5$

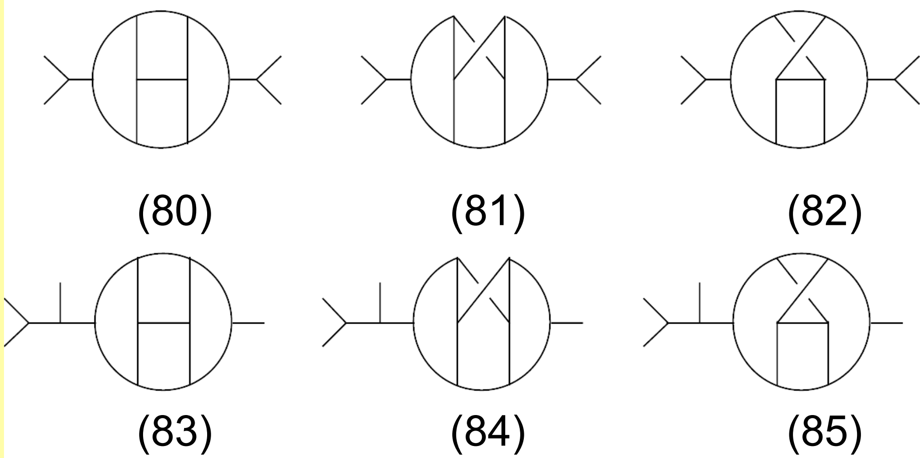
- N=4 SYM: Master numerators N_{18} and N_{28} quadratic in l_i. Duality relations preserve quadratic in l_i for all numerators → 1PI, 13-propagator graphs (1)-(52) and (72)

$$\Rightarrow \quad I \ \sim \ \int \frac{(d^{11/2} l_i)^4 \ \ l_i^2}{[(l_i)^2]^{13}}$$

are finite in $D = 11/2$

- 1PR but 2PI 12-propagator graphs actually linear → also finite in $D = 11/2$
- SYM divergences again only from most reducible graphs: scalar 2PR 11-propagators

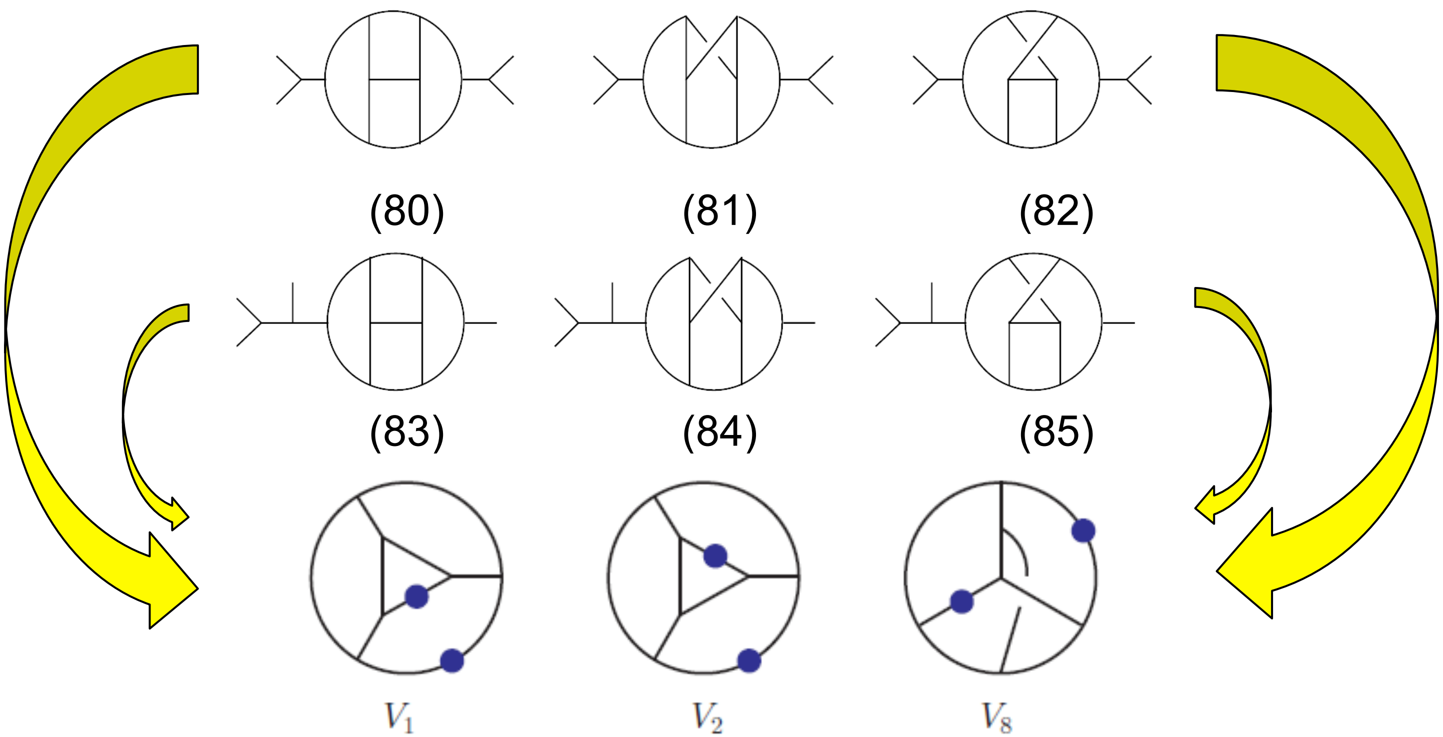
$$I \ \sim \ \int \frac{(d^{11/2-2\epsilon} l_i)^4}{[(l_i)^2]^{11}}$$



4 loop N=4 SYM UV color structure

- BCJ form again manifestly has no double trace terms in critical dimension $D_c = 11/2$:
- Color factors for only divergent graphs contain explicit

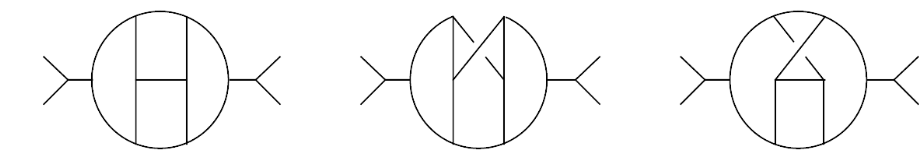
$$f^{a_1 a_2 b} f^{b a_3 a_4} = \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) \pm \cdots$$



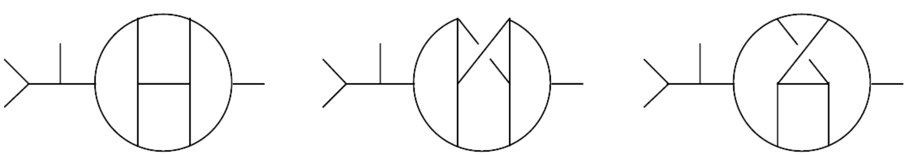
$k_i \rightarrow 0$

4 loop N=4 SYM UV pole

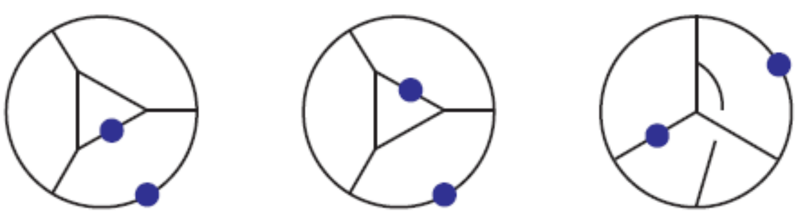
$$\mathcal{A}_4^{(4)}(1,2,3,4)\Big|_{\text{pole}}^{SU(N_c)} = -6\,g^{10}\,\mathcal{K}\,N_c^2\Big(N_c^2\,V_1 + 12\,\underline{(V_1 + 2\,V_2 + V_8)}\Big)$$
$$\times\Big(s\,(\mathrm{Tr}_{1324} + \mathrm{Tr}_{1423}) + t\,(\mathrm{Tr}_{1243} + \mathrm{Tr}_{1342}) + u\,(\mathrm{Tr}_{1234} + \mathrm{Tr}_{1432})\Big)$$



(80) (81) (82)



(83) (84) (85)



V_1 V_2 V_8

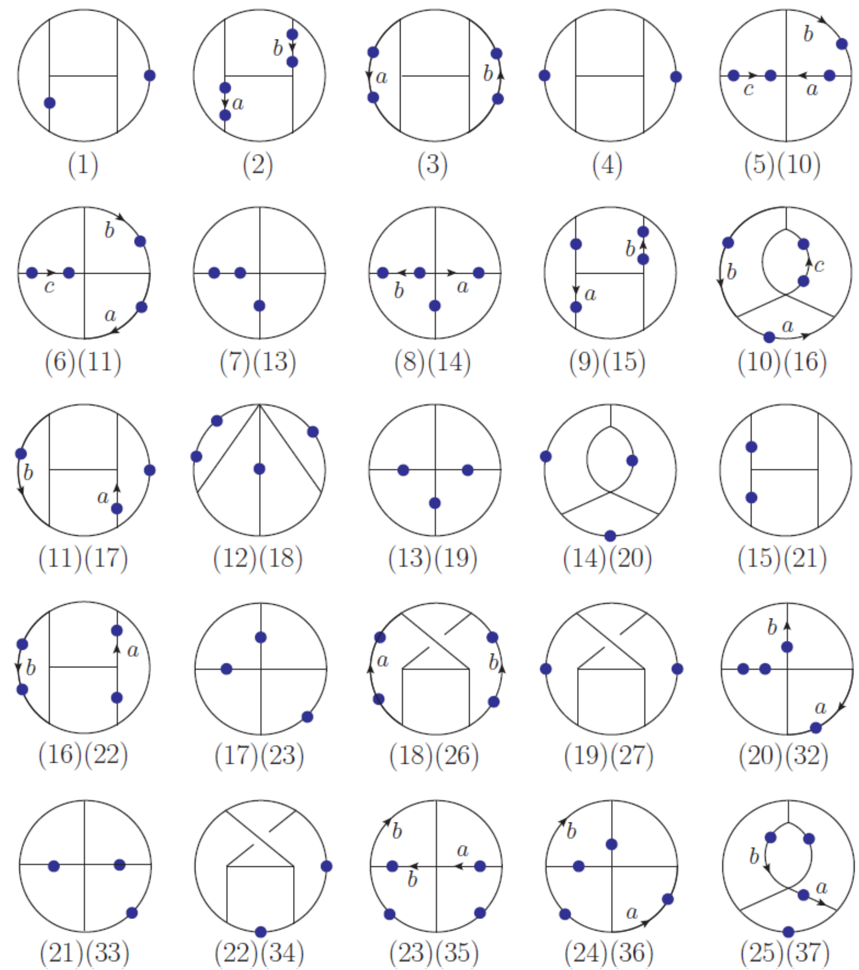
- No $N_c^{\,0}$ term
- As at 3 loops, relative factors in $N_c^{\,2}$ term are purely from graph symmetry factors S_i

4 loop N=8 SUGRA UV pole

- In BCJ form of amplitude, all integrals are at worst log-divergent in $D = 11/2$.
- After standard tensor reductions like

$$l_i^{\mu_i} l_j^{\mu_j} \mapsto \frac{1}{D}\,\eta^{\mu_i \mu_j}\, l_i \cdot l_j$$

we set $k_i \rightarrow 0$ inside integrals → 69 different 4 loop vacuum integrals (25 shown here).

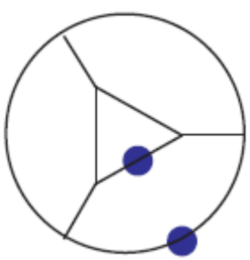


4 loop UV pole in $D = 11/2$

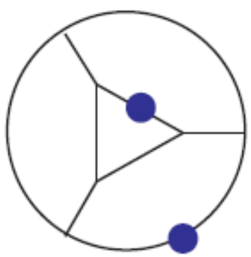
- Reduce integrals to basis $\{V_1, V_2, V_8\}$
- Final answer is remarkably simple:

$$\mathcal{M}_4^{(4)}\Big|_{\text{pole}} = -\frac{23}{8}\left(\frac{\kappa}{2}\right)^{10} stu\,(s^2+t^2+u^2)^2\, M_4^{\text{tree}}\,\underline{(V_1+2V_2+V_8)}$$

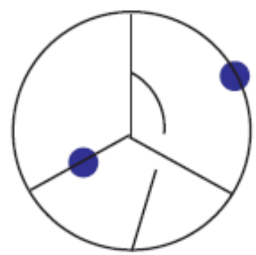
- Again, same linear combination as in $N_c^{\,2}$ part of N=4 SYM pole!



V_1



V_2



V_8

I^v	Effective numerator	V_1	V_2	V_8
I_1^v	$-\frac{117674}{1485}$	0	$-\frac{117674}{1485}$	0
I_2^v	$\frac{19112}{1485}\ \tau_{a,b}^2$	$\frac{8798687}{5346000}$	$\frac{212621}{27000}$	0
I_3^v	$\frac{9556}{1485}\ \tau_{a,b}^2$	$\frac{15937019}{1782000}$	$-\frac{140951}{33000}$	0
I_4^v	$-\frac{16427}{495}$	$-\frac{16427}{495}$	0	0
I_5^v	$\frac{19112}{1485}\ \tau_{a,c} - \frac{19112}{1485}\ \tau_{b,c}$	$-\frac{2389}{2970}$	$-\frac{2389}{1485}$	0
I_6^v	$-\frac{4778}{495}\ \tau_{a,c} + \frac{4778}{1485}\ \tau_{b,c}$	$\frac{16723}{2970}$	$-\frac{4778}{1485}$	0
I_7^v	$-\frac{9556}{1485}$	$\frac{109894}{7425}$	$\frac{90782}{2475}$	0
I_8^v	$\frac{38224}{1485}\ \tau_{a,b}$	$-\frac{2389}{675}$	$-\frac{19112}{2475}$	0
I_9^v	$\frac{38224}{1485}\ \tau_{a,b}^2$	$-\frac{1617353}{148500}$	$\frac{2606399}{74250}$	0
I_{10}^v	$-\frac{19112}{1485}\ \tau_{a,c} - \frac{19112}{1485}\ \tau_{b,c}$	$-\frac{2389}{2970}$	$-\frac{2389}{1485}$	0
I_{11}^v	$-\frac{38224}{1485}\ \tau_{a,b}$	$\frac{90782}{22275}$	$-\frac{9556}{825}$	0
I_{12}^v	$-\frac{19112}{1485}$	$\frac{31057}{990}$	$\frac{38224}{495}$	0
I_{13}^v	$\frac{10048}{99}$	$\frac{2512}{99}$	$\frac{10048}{99}$	0
I_{14}^v	$-\frac{19112}{1485}$	$-\frac{4778}{275}$	$-\frac{324904}{7425}$	0
I_{15}^v	$\frac{19112}{1485}$	$\frac{66892}{4455}$	$\frac{19112}{495}$	0
I_{16}^v	$\frac{19112}{1485}\ \tau_{a,b}^2$	$\frac{977101}{267300}$	$\frac{88393}{14850}$	0
I_{17}^v	$\frac{39676}{1485}$	$\frac{9919}{495}$	$\frac{19838}{1485}$	0
I_{18}^v	$\frac{9556}{1485}\ \tau_{a,b}^2$	$-\frac{1478791}{297000}$	$\frac{661753}{148500}$	$\frac{2389}{396}$
I_{19}^v	$-\frac{64441}{1485}$	0	0	$-\frac{64441}{1485}$
I_{20}^v	$\frac{38224}{1485}\ \tau_{a,b}$	$-\frac{102727}{14850}$	$-\frac{74059}{7425}$	0
I_{21}^v	$\frac{5284}{1485}$	$\frac{18494}{7425}$	$\frac{34346}{7425}$	0
I_{22}^v	$\frac{934}{165}$	$\frac{467}{165}$	$\frac{1868}{165}$	$-\frac{934}{165}$
I_{23}^v	$\frac{526}{135}\ \tau_{a,b} - \frac{91}{1485}\ \tau_{a,c}$	$\frac{279199}{297000}$	$\frac{72052}{37125}$	0
I_{24}^v	$\frac{3736}{495}\ \tau_{a,b}$	$\frac{26152}{12375}$	$-\frac{91532}{12375}$	0
I_{25}^v	$-\frac{9556}{1485}\ \tau_{a,b}$	$-\frac{2389}{2475}$	$-\frac{16723}{7425}$	0
I_{26}^v	$-\frac{11048}{1485}$	$-\frac{17953}{2475}$	$-\frac{44192}{2475}$	0
I_{27}^v	$-\frac{1228}{135}$	$-\frac{307}{50}$	$-\frac{10438}{675}$	0
I_{28}^v	$-\frac{3736}{495}$	$\frac{934}{825}$	$-\frac{14944}{825}$	0
I_{29}^v	$\frac{3736}{495}\ \tau_{a,b}$	$-\frac{48568}{4125}$	$\frac{76588}{4125}$	0
I_{30}^v	$\frac{934}{495}$	$\frac{90131}{24750}$	$\frac{119552}{12375}$	0
I_{31}^v	$\frac{4778}{1485}\ \tau_{a,b} + \frac{9556}{1485}\ \tau_{a,c}$	$-\frac{45391}{19800}$	$-\frac{112283}{14850}$	0
I_{32}^v	$\frac{19112}{1485}\ \tau_{a,b}^2$	$\frac{2721071}{148500}$	$-\frac{327293}{74250}$	$-\frac{2389}{495}$
I_{33}^v	$-\frac{3736}{495}\ \tau_{a,b}$	$-\frac{1868}{2475}$	$-\frac{1868}{275}$	$\frac{1868}{165}$
I_{34}^v	$\frac{4778}{297}$	$-\frac{155285}{2376}$	$-\frac{47780}{297}$	0
I_{35}^v	$-\frac{7904}{1485}$	$-\frac{3952}{495}$	$-\frac{27664}{1485}$	0
		...		
Total	–	$\frac{23}{2}$	23	$\frac{23}{2}$

What about $L = 5$?

N=4 SYM: Bern, Carrasco, Johansson, Roiban, 1207.6666

- Motivation: Various arguments point to 7 loops as the possible first divergence for N=8 SUGRA in D=4, associated with a D^8R^4 counterterm:

Howe, Lindstrom, NPB181, 487 (1981); Bossard, Howe, Stelle, 0908.3883; Kallosh, 0903.4630; Green, Russo, Vanhove, 1002.3805; Bjornsson, Green, 1004.2692; Bossard, Howe, Stelle, 1009.0743; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger, 1009.1643

- Same D^8R^4 counterterm shows up at $L = 4$ in $D = 5.5$
- Does 5 loops → $D^{10}R^4$ (same UV as N=4 SYM)?
  or → D^8R^4 (worse UV as N=4 SYM)?
- 5 loops would be a very strong indicator for 7 loops
- Now 100s of nonvanishing cubic 4-point graphs!

What if it's finite?

• Then we should determine the finite values of N=8 scattering amplitudes near D=4.

• These all have IR divergences, but fortunately they exponentiate, much more simply than in Yang-Mills theory

Weinberg (1965); Naculich, Schnitzer, 1101.1524; White, 1103.2981; Akhoury, Saotome, Sterman, 1109.0270

$$\ln \frac{\mathcal{M}_4}{M_4^{\text{tree}}} = \left(\frac{\kappa}{8\pi}\right)^2 \frac{M_4^{1-\text{loop}}}{M_4^{\text{tree}}} + \mathcal{F}_4$$

divergent   finite

$$\mathcal{F}_4 = \left(\frac{\kappa}{8\pi}\right)^4 F_4^{(2)} + \cdots$$

Two-loop finite remainder

- Known for N = 4,5,6,8 four-graviton scattering

Boucher-Veronneau, LD, 1110.1132

- Of course N = 8 is the simplest ☺

[see also Naculich, Nastase, Schnitzer, 0805.2347;
Brandhuber, Heslon, Nasti, Spence, Travaglini, 0805.2763]

$$F_4^{(2),\mathcal{N}=8}\Big|_{s-\text{channel}} = 8\left\{t\,u\left[f_1\!\left(\frac{-t}{s}\right)+f_1\!\left(\frac{-u}{s}\right)\right]+s\,u\left[f_2\!\left(\frac{-t}{s}\right)+f_3\!\left(\frac{-t}{s}\right)\right]\right.$$
$$\left.+s\,t\left[f_2\!\left(\frac{-u}{s}\right)+f_3\!\left(\frac{-u}{s}\right)\right]\right\},$$

- f_2 and f_3 are related to f_1 by crossing symmetry (analytic continuation).

Two-loop finite remainder (cont.)

- f_1 is a very simple, maximal (weight 4) transcendental function of $x = -t/s$:

$$f_1(x) \;=\; \zeta_4 + i\pi\,\zeta_3 - \int_x^1 \frac{dt}{t(1-t)}\left[\frac{1}{6}\ln^3 t + \frac{i\pi}{2}\ln^2 t\right]$$

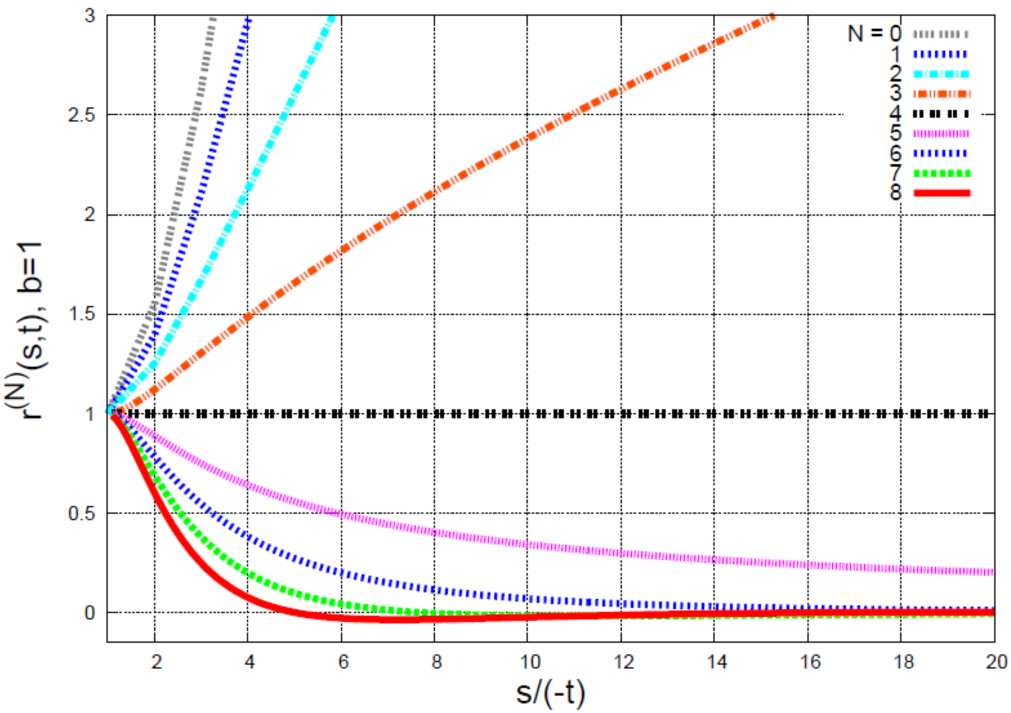
- Would be quite interesting to determine $F_4^{(3)}$, $F_4^{(4)}$, etc.

- Possible starting point: $x \to 0$ limit

→ Forward scattering ~ Regge limit.

Leading Regge Double Logs

- $[t/s\ \ln^2(-t/s)]^L$ recently resummed to all orders for any number of supersymmetries Bartels, Lipatov, Sabio Vera, 1208.3423
- Note that these terms are heavily power-suppressed, by $(t/s)^L$, with respect to the leading eikonal behavior.
- N=8 Regge terms most heavily damped in HE limit
- N=4 Regge terms are totally boring…

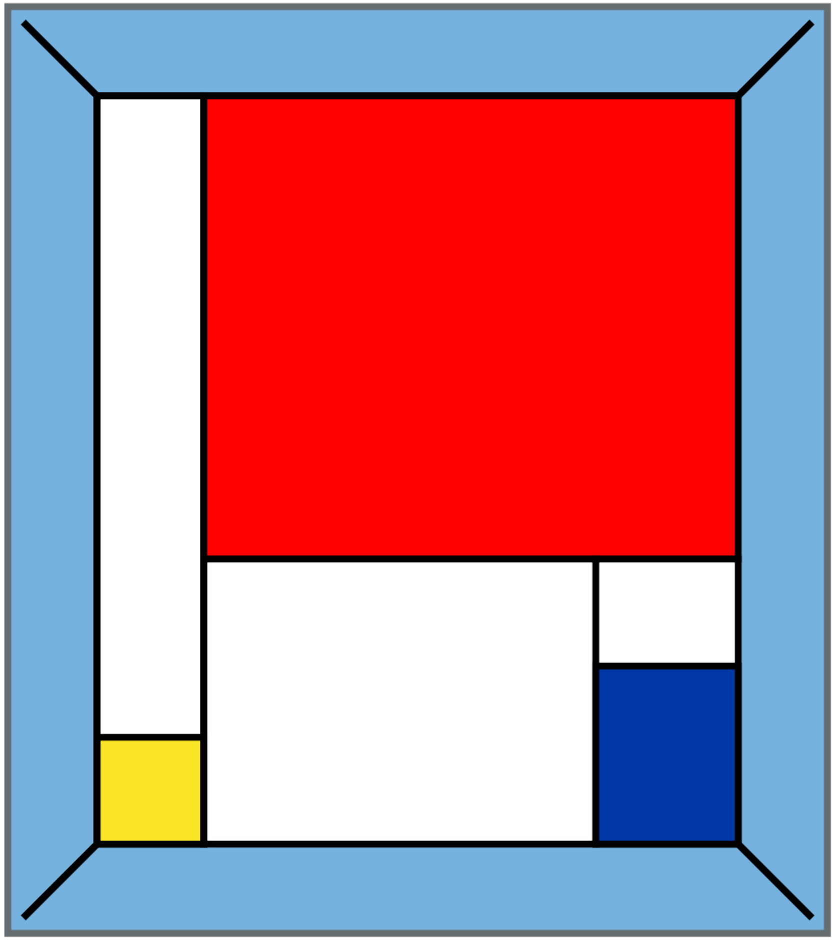


Outlook

- Through 4 loops, the 4-graviton scattering amplitude of N=8 supergravity has UV behavior no worse than the corresponding 4-gluon amplitude of N=4 SYM.
- Finite remainder also remarkably simple (at 2 loops).
- Precise pole for N=8 supergravity bears a remarkable relation with subleading-color single trace pole in N=4 SYM in the same critical dimension, at 2, 3 and 4 loops.
- Is this an accident, or could it foreshadow equal critical dimensions $D_c = 26/5$ also at 5 loops? Which in turn would suggest that 7 loops is not where N=8 supergravity first diverges… If not there, where? L = 8? L = ∞ ?

Last word goes to Hermann

(with apologies)



- *DW-TV: Well, what about the "man on the street"? Are there any achievements or concrete benefits that just your average person will be able to see from this research?*
- Dr. Hermann Nicolai: Well, whatever comes out of this … will be extremely hard to communicate. But …one should never forget that if it was not for human curiosity and the desire to explore the world around us, we would still be sitting in the caves and painting the walls.