Applications of vertex operator constructions and character theory to branching problems of affine Kac-Moody algebras

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Fixed points $\mathfrak{k} = \mathfrak{g}^\sigma$ form a subalgebra of type G_2, $dim(\mathfrak{k}) = 14$, and $\mathfrak{g} = \mathfrak{k} \oplus W_7 \oplus W_7$, where W_7 is the 7-dimensional irrep of G_2.

Example 2: $\mathfrak{g}$ is of type E_6, $dim(\mathfrak{g}) = 78$, τ is the order 2 Dynkin diagram automorphism, the fixed point subalgebra $\mathfrak{k} = \mathfrak{g}^\tau$ with $dim(\mathfrak{k}) = 52$ is of type F_4, and $V = \mathfrak{g} = \mathfrak{k} \oplus W_{26}$, where W_{26} is the 26-dimensional irrep of F_4.

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Computer programs do such computations when $\mathfrak{g}$ is finite dimensional simple. More challenging to solve the branching rule problem for an infinite dimensional module $\hat{V}$ of the infinite dimensional affine Kac-Moody Lie algebras $\hat{\mathfrak{k}} \subset \hat{\mathfrak{g}}$.

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Main tool: Goddard-Kent-Olive coset construction of Virasoro operators which commute with $\hat{\mathfrak{k}}$ and give the space of $\hat{\mathfrak{k}}$ highest weight vectors in $\hat{V}$ the structure of a Virasoro module.

We present here the results of two such projects:

(1) The dissertation research of Quincy Loney on branching of the four level-1 irreps of $\hat{\mathfrak{g}}$ of type $D_4^{(1)}$ w.r.t. its subalgebra $\hat{\mathfrak{k}}$ of type $G_2^{(1)}$, using the fermionic spinor construction,

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(2) The dissertation research of Christopher Mauriello on branching of the three level-1 irreps of $\hat{\mathfrak{g}}$ of type $E_6^{(1)}$ w.r.t. its subalgebra $\hat{\mathfrak{k}}$ of type $F_4^{(1)}$, using the bosonic lattice construction.

Affine Algebra Background

For $\mathfrak{g}$ finite dimensional simple of type X_ℓ with normalized Killing form $\langle \cdot, \cdot \rangle$, the affinization of $\mathfrak{g}$ of type $X_\ell^{(1)}$ is

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

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and writing $x(m) = x \otimes t^m$ for $x \in \mathfrak{g}$, $m \in \mathbb{Z}$, the brackets are

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The central element c acts on an irreducible $\hat{\mathfrak{g}}$-module by a scalar called the level of that module.

Loney Project Background

In 1991, Feingold, Frenkel and Ries gave a spinor construction of the vertex operator para-algebra

$$\hat{V} = \hat{V}^0 \oplus \hat{V}^1 \oplus \hat{V}^2 \oplus \hat{V}^3,$$

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The triality group $S_3 = \langle \sigma, \tau \mid \sigma^3 = 1 = \tau^2, \tau\sigma\tau = \sigma^{-1} \rangle$ in $Aut(\hat{V})$ was constructed, preserving $\hat{V}^0$ and permuting $\hat{V}^i$, $i = 1, 2, 3$.

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$\hat{V}$ is $\frac{1}{2}\mathbb{Z}$-graded; $\hat{V}^i_n$ denotes the n-graded subspace of $\hat{V}^i$.
Vertex operators $Y(v, z)$ for $v \in \hat{V}^0_1$ represent $D_4^{(1)}$ on $\hat{V}$, while those for which $\sigma(v) = v$ represent $G_2^{(1)}$.

$\hat{V}$ decomposes into a direct sum of $G_2^{(1)}$ irreps by a two-step process, first decomposing with respect to the intermediate subalgebra $B_3^{(1)}$ represented by $Y(v,z)$ for $\tau(v) = v$.

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each representing the Virasoro algebra given by the Sugawara constructions from the three algebras.
These give two commuting coset Virasoro constructions,

$$Y(\omega_{D_4}-\omega_{B_3},z)\quad \text{and} \quad Y(\omega_{B_3}-\omega_{G_2},z),$$

with central charges $1/2$ and $7/10$, resp., the first commuting with $B_3^{(1)}$, the second commuting with $G_2^{(1)}$.

This gives the space of highest weight vectors for $G_2^{(1)}$ in $\hat{V}$ as sums of tensor products of irreducible Virasoro modules $L(1/2, h_1) \otimes L(7/10, h_2)$.

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The dissertation research of Quincy Loney explicitly constructs these coset Virasoro operators, and uses them to find the decomposition of $\hat{V}$ with respect to $G_2^{(1)}$ by finding 12 highest weight vectors w.r.t. $Vir^{1/2} \times Vir^{7/10} \times G_2^{(1)}$.

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This work also provides a spinor construction of the $7/10$ Virasoro modules inside $\hat{V}$, and of a vertex operator algebra naturally associated with the basic representation of $G_2^{(1)}$.

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The dissertation research of Christopher Mauriello uses the bosonic lattice construction of the vertex operator para-algebra

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The vertex operators $Y(\omega_{E_6} - \omega_{F_4}, z)$ provide a coset Virasoro representation with central charge $4/5$, giving the decomposition of each $\hat{V}^i$ as a sum of tensor products $L(4/5, h) \otimes W(\Omega_j)$, where $W(\Omega_j)$, $j = 0, 4$, are the two level-1 $F_4^{(1)}$-modules.

Finite Spinor Constructions

The spinor construction of four irreps of D_ℓ starts with $A \cong \mathbb{C}^{2\ell}$ equipped with a nondegenerate symmetric bilinear form, $\langle \cdot, \cdot \rangle$, and a polarization of $A = A^+ \oplus A^-$ into isotropic subspaces, $A^\pm \cong \mathbb{C}^\ell$.

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The Clifford algebra $Cliff_\ell(A)$ is the $2^{2\ell}$-dimensional associative algebra with unit 1 generated by A with relations $ab + ba = \langle a, b \rangle 1$, $\forall a, b \in A$.

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Define the “normal ordered" elements, $:ab: = \frac{1}{2}(ab - ba)$, and the subspace $\mathfrak{g}$ spanned by them. Then the commutator

$$[:ab:, :cd:] = \langle a, d\rangle :bc: - \langle a, c\rangle :bd: + \langle b, c\rangle :ad: - \langle b, d\rangle :ac:$$

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shows that $\mathfrak{g}$ is a Lie algebra inside $Cliff_\ell(A)$. It is not hard to show that it is isomorphic to $so(2\ell, \mathbb{C})$ of type D_ℓ.

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$$CM_\ell(A) = Cliff_\ell(A) \cdot vac = \bigwedge(A^-) \cdot vac$$

is an irreducible left $Cliff_\ell(A)$-module of dimension 2^ℓ.

Under the action of $\mathfrak{g}$ we have the decomposition into two $\mathfrak{g}$-modules

$$CM_\ell(A) = CM_\ell(A)^0 \oplus CM_\ell(A)^1 \ ,$$

where the decomposition is according to the parity of the number of "creation operators" applied to vac,

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All of this generalizes to the affine algebra $\hat{\mathfrak{g}}$ of type $D_\ell^{(1)}$, providing four level 1 irreps.

Affine Spinor Constructions

Let $Z = \mathbb{Z} + \frac{1}{2}$ or $Z = \mathbb{Z}$ and let

$$A(Z) = \bigoplus_{m \in Z} A \otimes t^m$$

spanned by the elements $a(m) = a \otimes t^m$, $a \in A$, $m \in Z$.

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and polarize $A(Z) = A(Z)^+ \oplus A(Z)^-$ so that $a(\pm m) \in A(Z)^\pm$ when $m > 0$ for all $a \in A$, but $a(0) \in A(\mathbb{Z})^\pm$ for $a \in A^\pm$.

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$Cliff_\ell(A(Z))$ is the associative algebra with unit $\mathbf{1}$ generated by $A(Z)$ with relations

$$a(m)b(n) + b(n)a(m) = \langle a(m), b(n) \rangle 1.$$

Let $\mathcal{I}(Z)^+$ be the left ideal of $Cliff_\ell(A(Z))$ generated by $A(Z)^+$ and define the left $Cliff_\ell(A(Z))$-module

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For convenience, let $vac = vac(\mathbb{Z} + \frac{1}{2})$ and $vac' = vac(\mathbb{Z})$.

Let

$$\begin{aligned} CM_\ell(A(Z))^0 &= \bigwedge^{even}(A(Z)^-) \cdot vac(Z) \\ CM_\ell(A(Z))^1 &= \bigwedge^{odd}(A(Z)^-) \cdot vac(Z). \end{aligned}$$

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Define the "fermionic normal ordering"

$$: a(m)b(n) : = \begin{cases} a(m)b(n) & \text{for } m < n \\ 1/2\Big(a(m)b(n) - b(n)a(m)\Big) & \text{for } m = n \\ -b(n)a(m) & \text{for } m > n. \end{cases}$$

Then the coefficients of the generating functions

$$: a(w)b(w) := \sum_{k\in\mathbb{Z}} \left(\sum_{m\in Z} : a(k-m)b(m) : \right) w^{-k-1}.$$

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$$\begin{aligned} Y(a(-1/2)vac, w) &= a(w), \quad \text{and} \\ Y(a(-1/2)b(-1/2)vac, w) &= : a(w)b(w) : . \end{aligned}$$

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$$\begin{aligned} \hat{V}^0 &= CM_4(A(\mathbb{Z} + \tfrac{1}{2}))^0, & \hat{V}^2 &= CM_4(A(\mathbb{Z}))^0, \\ \hat{V}^1 &= CM_4(A(\mathbb{Z} + \tfrac{1}{2}))^1, & \hat{V}^3 &= CM_4(A(\mathbb{Z}))^1, \end{aligned}$$

$$\hat{V} = \hat{V}^0 \oplus \hat{V}^1 \oplus \hat{V}^2 \oplus \hat{V}^3.$$

Triality When $n = 4$

When $\ell = 4$ a special S_3 symmetry occurs, called "triality", which plays a vital role. Let

$$V^1 = A, \qquad V^2 = CM_4(A)^0, \qquad V^3 = CM_4(A)^1,$$

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$$\begin{aligned} \hat{V}^0 &= CM_4(A(\mathbb{Z} + \tfrac{1}{2}))^0, & \hat{V}^2 &= CM_4(A(\mathbb{Z}))^0, \\ \hat{V}^1 &= CM_4(A(\mathbb{Z} + \tfrac{1}{2}))^1, & \hat{V}^3 &= CM_4(A(\mathbb{Z}))^1, \end{aligned}$$

$$\hat{V} = \hat{V}^0 \oplus \hat{V}^1 \oplus \hat{V}^2 \oplus \hat{V}^3.$$

C has a commutative, nonassociative operation, $\circ$, so that the $\circ$ action of V^1 on $V^2 \oplus V^3$ is the Clifford module action. This "Chevalley algebra", has an automorphism σ of order 3, cyclically permuting V^1, V^2, V^3.

The symmetry σ and the bilinear form on C allows each V^k to generate a Clifford algebra $Cliff_4^{(k)}$, and lets us identify $V^{k'} \oplus V^{k''}$ as its module $CM_4^{(k)}$, where (k, k', k'') is a cyclic permutation of $(1, 2, 3)$.

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But then the three constructions of $\mathfrak{g}$ of type D_4 given by $: ab :$ for $a, b \in V^k$ can be shown to coincide as operators on C, related by the Lie algebra automorphism $\sigma(: ab :) = \, : (\sigma a)(\sigma b) :$.

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The eigenspace decomposition of $\mathfrak{g}$ under σ then yields a 14-dimensional Lie subalgebra of fixed points, $\mathfrak{g}_0$, and two 7-dimensional subspaces, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which are irreducible $\mathfrak{g}_0$-modules. In fact, $\mathfrak{g}_0$ is of type G_2.

Spinor Construction of $G_2^{(1)}$

We may lift σ to $\hat{\sigma} : \hat{V} \longrightarrow \hat{V}$ so that $\hat{\sigma}(\hat{V}^0) = \hat{V}^0$ and $\hat{\sigma}$ permutes $\hat{V}^1$, $\hat{V}^2$ and $\hat{V}^3$ cyclically. For $v \in \hat{V}^0$ we have

$$\hat{\sigma} Y(v, z) \hat{\sigma}^{-1} = Y(\hat{\sigma}(v), z).$$

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For $v \in \hat{V}_1^0$ fixed by $\hat{\sigma}$, the coefficients of the vertex operator $Y(v, z)$ represent the affine algebra $\hat{\mathfrak{g}}_0$ of type $G_2^{(1)}$ on $\hat{V}$. The subspace of all fixed points of $\hat{\sigma}$ in $\hat{V}^0$ is a sub-VOA of $\hat{V}^0$ whose relationship with the basic level 1 module of $G_2^{(1)}$ we may study.

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For $v \in \hat V_1^0$ fixed by $\hat\sigma$, the coefficients of the vertex operator $Y(v,z)$ represent the affine algebra $\hat{\mathfrak{g}}_0$ of type $G_2^{(1)}$ on $\hat V$. The subspace of all fixed points of $\hat\sigma$ in $\hat V^0$ is a sub-VOA of $\hat V^0$ whose relationship with the basic level 1 module of $G_2^{(1)}$ we may study.

The main point is to find the decomposition of each $\hat V^i$ into a direct sum of $G_2^{(1)}$-modules.

Representations of Virasoro Algebras

In $\hat{V}_2^0$ there is a special element ω_{D_4} such that the coefficients of

$$Y(\omega_{D_4}, z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$$

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$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C_{D_4}$$

where $C_{D_4} = 4$ is the central charge, and for any $x(k)$ in $D_4^{(1)}$, we have $[L_m, x(k)] = -kx(k+m)$.

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Let bases of A^+ and A^- be $\{a_1, a_2, a_3, a_4\}$ and $\{a_1^*, a_2^*, a_3^*, a_4^*\}$, respectively, such that

$$\langle a_i, a_j^*\rangle = \delta_{i,j}, \quad \langle a_i, a_j\rangle = 0 = \langle a_i^*, a_j^*\rangle$$

.

Use the following notations for elements in $\hat{V}_2^0$, where $i^{\circledast} = a_i^{\circledast}$ and each appearance of $^{\circledast}$ may be * or a blank:

$$
\begin{aligned}
i^{\circledast} j^{\circledast} &= a_i^{\circledast}(-3/2) a_j^{\circledast}(-1/2) vac, \\
i i^* j j^* &= a_i(-1/2) a_i^*(-1/2) a_j(-1/2) a_j^*(-1/2) vac, \\
1^{\circledast} 2^{\circledast} 3^{\circledast} 4^{\circledast} &= a_1^{\circledast}(-1/2) a_2^{\circledast}(-1/2) a_3^{\circledast}(-1/2) a_4^{\circledast}(-1/2) vac, \\
J &= 11^*22^* + 11^*33^* - 22^*33^* \\
&\quad +1^*234 + 12^*3^*4^* - 1^*234^* - 12^*3^*4.
\end{aligned}
$$

We can then write

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$$\begin{aligned}\omega_{G_2} &= \omega_{B_3} - \Big[\tfrac{1}{10} \textstyle\sum_{i=1}^{3} (ii^* + i^*i) \\ &\quad + \tfrac{1}{20} \left(44^* + 4^*4 - 44 - 4^*4^*\right) - \tfrac{1}{5} J \Big]\end{aligned}$$

Then the coefficients of the corresponding vertex operators

$$
\begin{aligned}
Y(\omega_{B_3}, z) &= \sum_{m\in\mathbb{Z}} L_m^{B_3} z^{-m-2} \\
Y(\omega_{G_2}, z) &= \sum_{m\in\mathbb{Z}} L_m^{G_2} z^{-m-2}
\end{aligned}
$$

represent the Virasoro algebra on $\hat{V}$ with central charges

$$C_{B_3} = \frac{7}{2} \quad \text{and} \quad C_{G_2} = \frac{14}{5}, \ \text{resp.}$$

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These operators only satisfy bracket relations $[L_m^{T_r}, x(k)] = -kx(k+m)$ for $x(k)$ where x is in the corresponding subalgebra $T_r = B_3$ or $T_r = G_2$.

Then the coset Virasoro construction (Goddard, Kent, Olive) says that the differences

$$
\begin{aligned}
Y(\omega_{D_4} - \omega_{B_3}, z) &= \sum_{m \in \mathbb{Z}} L_m^{1/2} z^{-m-2} \\
Y(\omega_{B_3} - \omega_{G_2}, z) &= \sum_{m \in \mathbb{Z}} L_m^{7/10} z^{-m-2}
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represent the Virasoro algebra on $\hat{V}$ with central charges

$$C_{D_4} - C_{B_3} = \frac{1}{2} \quad \text{and} \quad C_{B_3} - C_{G_2} = \frac{7}{10}, \text{ resp,}$$

and have the commutation relations

$$[L_m^{1/2}, L_n^{7/10}] = 0, \ [L_m^{1/2}, B_3^{(1)}] = 0, \ [L_m^{7/10}, G_2^{(1)}] = 0.$$

This means that the Virasoro module, $L(1/2, h_1)$, generated by the operators $L_m^{1/2}$, $m < 0$, applied to any highest weight vector for $D_4^{(1)}$, is a space of highest weight vectors for $B_3^{(1)}$.

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$$L(1/2, h_1) \otimes L(7/10, h_2) \otimes W(\Omega_i)$$

for some irreducible level-1 $G_2^{(1)}$-module $W(\Omega_i)$.

Loney's Main Theorem

Theorem: $\hat{V} = \hat{V}^0 \oplus \hat{V}^1 \oplus \hat{V}^1 \oplus \hat{V}^3$ decomposes w.r.t. $G_2^{(1)}$ into twelve $Vir^{1/2} \times Vir^{7/10} \times G_2^{(1)}$-modules as follows:

$$\begin{aligned}
\hat{V} \;&= \left(L(\tfrac{1}{2},0) \oplus L(\tfrac{1}{2},\tfrac{1}{2})\right) \otimes \left(L(\tfrac{7}{10},0) \oplus L(\tfrac{7}{10},\tfrac{3}{2})\right) \otimes W(\Omega_0) \\
&\oplus \left(L(\tfrac{1}{2},0) \oplus L(\tfrac{1}{2},\tfrac{1}{2})\right) \otimes \left(L(\tfrac{7}{10},\tfrac{1}{10}) \oplus L(\tfrac{7}{10},\tfrac{3}{5})\right) \otimes W(\Omega_2) \\
&\oplus \left(L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{3}{80}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{3}{80})\right) \otimes W(\Omega_2) \\
&\oplus \left(L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{7}{16}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{7}{16})\right) \otimes W(\Omega_0).
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&\oplus \Big(L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{3}{80}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{3}{80})\Big) \otimes W(\Omega_2) \\
&\oplus \Big(L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{7}{16}) \oplus L(\tfrac{1}{2},\tfrac{1}{16}) \otimes L(\tfrac{7}{10},\tfrac{7}{16})\Big) \otimes W(\Omega_0).
\end{aligned}$$

Each summand is determined by $\phi(\hat{V}_n^i, h^{1/2}, h^{7/10}, \Omega_j)$ a HWV w.r.t. both coset Virasoro algebras and $G_2^{(1)}$, which has L_0^c eigenvalue h^c and $G_2^{(1)}$ weight Ω_j. The explicit form of these highest weight vectors is given below.

Highest Weight Vectors

The following are the highest weight vectors $\phi(\hat{V}_n^i, h^{1/2}, h^{7/10}, \Omega_j)$ corresponding to the summands in the branching rules for the level-1 $D_4^{(1)}$-modules:

$$
\begin{aligned}
\phi(\hat{V}_0^0, 0, 0, \Omega_0) &= \mathbf{1} = vac \\
\phi(\hat{V}_1^0, \tfrac{1}{2}, \tfrac{1}{10}, \Omega_2) &= 1(-\tfrac{1}{2})4(-\tfrac{1}{2})\mathbf{1} + 1(-\tfrac{1}{2})4^*(-\tfrac{1}{2})\mathbf{1} \\
\phi(\hat{V}_1^0, 0, \tfrac{3}{5}, \Omega_2) &= 2\Big(2(-\tfrac{1}{2})3(-\tfrac{1}{2})\mathbf{1}\Big) - 1(-\tfrac{1}{2})4(-\tfrac{1}{2})\mathbf{1} + 1(-\tfrac{1}{2})4^*(-\tfrac{1}{2})\mathbf{1} \\
\phi(\hat{V}_2^0, \tfrac{1}{2}, \tfrac{3}{2}, \Omega_0) &= 11^*44^* - 22^*44^* - 33^*44^* + 1^*234 + 1^*234^* + 12^*3^*4 + 12^*3^*4^* \\
\phi(\hat{V}_{1/2}^1, \tfrac{1}{2}, 0, \Omega_0) &= 4(-\tfrac{1}{2})\mathbf{1} + 4^*(-\tfrac{1}{2})\mathbf{1} \\
\phi(\hat{V}_{1/2}^1, 0, \tfrac{1}{10}, \Omega_2) &= 1(-\tfrac{1}{2})\mathbf{1} \\
\phi(\hat{V}_{3/2}^1, \tfrac{1}{2}, \tfrac{3}{5}, \Omega_2) &= 234 + 234^* - 144^* \\
\phi(\hat{V}_{3/2}^1, 0, \tfrac{3}{2}, \Omega_0) &= 11^*4 - 11^*4^* - 22^*4 + 22^*4^* - 33^*4 + 33^*4^* + 2\Big(1^*23 + 12^*3^*\Big) \\
\phi(\hat{V}_{1/2}^2, \tfrac{1}{16}, \tfrac{3}{80}, \Omega_2) &= \mathbf{1}' = vac' \\
\phi(\hat{V}_{1/2}^2, \tfrac{1}{16}, \tfrac{7}{16}, \Omega_0) &= 1^*(0)4^*(0)\mathbf{1}' - 2^*(0)3^*(0)\mathbf{1}' \\
\phi(\hat{V}_{1/2}^3, \tfrac{1}{16}, \tfrac{3}{80}, \Omega_2) &= 4^*(0)\mathbf{1}' \\
\phi(\hat{V}_{1/2}^3, \tfrac{1}{16}, \tfrac{7}{16}, \Omega_0) &= 1^*(0)\mathbf{1}' + 2^*(0)3^*(0)4^*(0)\mathbf{1}'
\end{aligned}
$$

Explicit Coset Virasoro Operators

Theorem: The following vertex operators provide two coset representations of the Virasoro algebra on the Neveu-Schwarz module, $CM_4(\mathbb{Z}+\frac{1}{2}) = \hat{V}^0 \oplus \hat{V}^1$, with central charges $\frac{1}{2}$ and $\frac{7}{10}$, respectively.

For all $k \in \mathbb{Z}$, we have

$$L_k^{1/2} = -\frac{1}{4} \sum_{r \in \mathbb{Z}+\frac{1}{2}} \left(r+\frac{1}{2}\right) \Big(:4(r)4(k-r): + :4^*(r)4^*(k-r): \\ + :4(r)4^*(k-r): + :4^*(r)4(k-r): \Big)$$

and for $r_1, r_2, r_3, r_4 \in \mathbb{Z} + \frac{1}{2}$, we have $L_k^{7/10} =$

$$-\frac{1}{10}\sum_{i=1}^{4}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (r+\tfrac{1}{2})\Big(: i^*(r)i(k-r) : + : i(r)i^*(k-r) : \Big)$$

$$+\frac{1}{20}\sum_{r\in\mathbb{Z}+\frac{1}{2}} (r+\tfrac{1}{2})\Big(: 4(r)4(k-r) : + : 4^*(r)4^*(k-r) :$$

$$+ : 4^*(r)4(k-r) : + : 4(r)4^*(k-r) : \Big)$$

$$-\frac{1}{5}\sum_{r_1+\cdots+r_4=k}\Big(:1(r_1)1^*(r_2)2(r_3)2^*(r_4): + :1(r_1)1^*(r_2)3(r_3)3^*(r_4):$$

$$- : 2(r_1)2^*(r_2)3(r_3)3^*(r_4) :$$

$$+ : 1^*(r_1)2(r_2)3(r_3)4(r_4) : + : 1(r_1)2^*(r_2)3^*(r_3)4^*(r_4) :$$

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Theorem: The following vertex operators provide two coset representations of the Virasoro algebra on the Ramond module, $CM_4(\mathbb{Z}) = \hat{V}^2 \oplus \hat{V}^3$, with central charges $\frac{1}{2}$ and $\frac{7}{10}$ respectively.

For all $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
L_k^{1/2} &= \tfrac{1}{16}\delta_{k,0} I \\
&\quad - \tfrac{1}{4}\sum_{r\in\mathbb{Z}} \left(r + \tfrac{1}{2}\right) \Big(:4(r)4(k-r): + :4^*(r)4^*(k-r): \\
&\qquad\qquad + :4(r)4^*(k-r): + :4^*(r)4(k-r): \Big)
\end{aligned}
$$

and for $r_1, r_2, r_3, r_4 \in \mathbb{Z}$, we have $L_k^{7/10} =$

$$\frac{7}{80}\delta_{k,0}I - \frac{1}{10}\sum_{i=1}^{4}\sum_{r\in\mathbb{Z}}(r+\tfrac{1}{2})\Big(:i^*(r)i(k-r): + :i(r)i^*(k-r):\Big)$$

$$+\frac{1}{20}\sum_{r\in\mathbb{Z}}(r+\tfrac{1}{2})\Big(:4(r)4(k-r): + :4^*(r)4^*(k-r):$$

$$+ :4^*(r)4(k-r): + :4(r)4^*(k-r):\Big)$$

$$-\frac{1}{5}\sum_{r_1+\cdots+r_4=k}\Big(:1(r_1)1^*(r_2)2(r_3)2^*(r_4): + :1(r_1)1^*(r_2)3(r_3)3^*(r_4):$$

$$- :2(r_1)2^*(r_2)3(r_3)3^*(r_4):$$

$$+ :1^*(r_1)2(r_2)3(r_3)4(r_4): + :1(r_1)2^*(r_2)3^*(r_3)4^*(r_4):$$

$$- :1^*(r_1)2(r_2)3(r_3)4^*(r_4): - :1(r_1)2^*(r_2)3^*(r_3)4(r_4):\Big).$$

Character Theory

For a $\mathfrak{g}$-module V^λ with weights Π^λ, the character

$$ch(V^\lambda) = \sum_{\mu \in \Pi^\lambda} dim(V^\lambda_\mu) e^\mu \in \mathbb{Z}[\mathfrak{h}^*]$$ satisfies

$$ch(V^\lambda \oplus V^\mu) = ch(V^\lambda) + ch(V^\mu)$$ and

$$ch(V^\lambda \otimes V^\mu) = ch(V^\lambda) \cdot ch(V^\mu).$$

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\end{aligned}
$$

For a $\hat{\mathfrak{g}}$-module, the graded dimension

$$gr(V^\Lambda) = e^{-\Lambda} \sum_{\mu\in\Pi^\Lambda} dim(V_\mu^\Lambda)e^\mu$$

captures the information about the infinite number of weight spaces V_μ^Λ in a formal power series of $\ell+1$ variables, $u_i = e^{-\alpha_i}$, $0 \le i \le \ell$.

The spinor construction of $\hat{V}$ gives a product form for its graded dimension:

$$\begin{aligned} gr(\hat{V}^0 \oplus \hat{V}^1) &= \prod_{i=1}^{4} \prod_{0<n\in\mathbb{Z}+\frac{1}{2}} (1+e^{\epsilon_i}q^n)(1+e^{-\epsilon_i}q^n) \\ gr(\hat{V}^2 \oplus \hat{V}^3) &= \left(\prod_{i=1}^{4}(1+e^{-\epsilon_i})\right) \prod_{i=1}^{4} \prod_{0<n\in\mathbb{Z}} (1+e^{\epsilon_i}q^n)(1+e^{-\epsilon_i}q^n) \end{aligned}$$

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gr(\hat{V}^2 \oplus \hat{V}^3) &= \left(\prod_{i=1}^{4}(1+e^{-\epsilon_i})\right) \prod_{i=1}^{4} \prod_{0<n\in\mathbb{Z}} (1+e^{\epsilon_i}q^n)(1+e^{-\epsilon_i}q^n)
\end{aligned}
$$

where $e^{\epsilon_i}q^n$ corresponds to the Clifford generator $a_i(-n)$ and $e^{-\epsilon_i}q^n$ corresponds to $a_i^*(-n)$.

Use the notations: $v_i = e^{\epsilon_i}$, $u_i = e^{-\alpha_i}$, $1 \leq i \leq 4$, $u_0 = e^{-\alpha_0} = e^{\theta - \delta}$ where $\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \epsilon_1 + \epsilon_2$ is the highest root of D_4.

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$$\begin{aligned} gr_{pr}(\hat{V}^0 \oplus \hat{V}^1) &= \prod_{0 \le m \in \mathbb{Z}} \prod_{i=1}^{4} (1 + u^{i-4} u^{6m+3})(1 + u^{4-i} q^{6m+3}) \\ &= 2 \prod_{1 \le n \in \mathbb{Z}} (1 + u^n)(1 + u^{3n}) = 2 \frac{\phi(u^2)\phi(u^6)}{\phi(u)\phi(u^3)} \end{aligned}$$

where

$$\phi(u) = \prod_{1 \le n \in \mathbb{Z}} (1 - u^n).$$

Similarly, for the Ramond modules, we get the same result:

$$
\begin{aligned}
& gr_{pr}(\hat{V}^2 \oplus \hat{V}^3) \\
&= \left(\prod_{i=1}^{4} (1 + u^{4-i}) \right) \prod_{i=1}^{4} \prod_{1 \le n \in \mathbb{Z}} (1 + u^{i-4} u^{6n})(1 + u^{4-i} u^{6n}) \\
&= 2 \prod_{1 \le n \in \mathbb{Z}} (1 + u^{n})(1 + u^{3n}) = 2 \frac{\phi(u^2)\phi(u^6)}{\phi(u)\phi(u^3)}.
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But triality symmetry gives $\hat{V}^1 \cong \hat{V}^2 \cong \hat{V}^3$ which implies $gr_{pr}(\hat{V}^1) = gr_{pr}(\hat{V}^2) = gr_{pr}(\hat{V}^3)$ and therefore,

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Theorem [Mandia]:

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where

$$\begin{aligned}
a(q) &= \prod_{n\geq 1} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})} = \frac{1}{\phi(q)} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k+3)/2} \\
b(q) &= \prod_{n\geq 1} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})} = \frac{1}{\phi(q)} \sum_{k\in\mathbb{Z}} (-1)^k q^{k(5k+1)/2} \\
\mathcal{F}(u) &= \frac{\phi(u^2)\phi(u^3)}{\phi(u)\phi(u^6)} = \prod_{n\geq 1} \frac{1}{(1-u^{6n-5})(1-u^{6n-1})}.
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Using this identity, the characters of Virasoro modules shown below, and the graded dimensions of the level-1 $D_4^{(1)}$ and $G_2^{(1)}$ modules, along with the twelve explicit HWVs, gives the equality in the above branching rule decomposition of $\hat{V}$.

The characters of the minimal model Virasoro modules we need are well-known (Feigin-Fuchs). For $c = \frac{1}{2}$:

$$
\begin{aligned}
\chi_{3,4}^{1,1}(q) &= \frac{q^{(-\frac{1}{48})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{12k^2}(q^{k} - q^{7k+1}) \\
\chi_{3,4}^{1,3}(q) &= \frac{q^{(\frac{1}{2}-\frac{1}{48})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{12k^2}(q^{-5k} - q^{13k+3}) \\
\chi_{3,4}^{1,2}(q) &= \frac{q^{(\frac{1}{16}-\frac{1}{48})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{12k^2}(q^{-2k} - q^{10k+2})
\end{aligned}
$$

and for $c = \frac{7}{10}$:

$$\begin{aligned}
\chi_{4,5}^{1,1}(q) &= \frac{q^{(-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{k} - q^{9k+1}) \\
\chi_{4,5}^{1,2}(q) &= \frac{q^{(\frac{1}{10}-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{3k} - q^{13k+2}) \\
\chi_{4,5}^{1,3}(q) &= \frac{q^{(\frac{3}{5}-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{7k} - q^{17k+3}) \\
\chi_{4,5}^{1,4}(q) &= \frac{q^{(\frac{3}{2}-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{11k} - q^{21k+4}) \\
\chi_{4,5}^{2,1}(q) &= \frac{q^{(\frac{7}{16}-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{6k} - q^{14k+2}) \\
\chi_{4,5}^{2,2}(q) &= \frac{q^{(\frac{3}{80}-\frac{7}{240})}}{\phi(q)} \cdot \sum_{k\in\mathbb{Z}} q^{20k^2}(q^{2k} - q^{18k+4}).
\end{aligned}$$

Mauriello's Main Theorem

Theorem: The decomposition of level-1 irreducible $E_6^{(1)}$-modules $\hat{V}^i$, $i = 0, 1, 6$, into $Vir^{4/5} \times F_4^{(1)}$-modules is:

$$\begin{aligned}
\hat{V}^0 &= \Big(L(\tfrac{4}{5}, 0) \oplus L(\tfrac{4}{5}, 3)\Big) \otimes W(\Omega_0) \oplus \Big(L(\tfrac{4}{5}, \tfrac{2}{5}) \oplus L(\tfrac{4}{5}, \tfrac{7}{5})\Big) \otimes W(\Omega_4) \\
\hat{V}^1 &= L(\tfrac{4}{5}, \tfrac{2}{3}) \otimes W(\Omega_0) \oplus L(\tfrac{4}{5}, \tfrac{1}{15}) \otimes W(\Omega_4) \\
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\end{aligned}$$

Each summand is determined by a HWV $\phi(\hat{V}_n^i, h^{4/5}, \Omega_j)$ w.r.t. $Vir^{4/5} \times F_4^{(1)}$ located in $\hat{V}_n^i$ with $L_0^{4/5}$ eigenvalue $h^{4/5}$ and $F_4^{(1)}$ weight Ω_j.

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The explicit HWVs found are listed on the next two slides.

Mauriello's Heighest Weight Vectors

Theorem: Let α_i and λ_i, $1 \leq i \leq 6$, be the simple roots and fundamental weights of E_6, resp. Then the highest weight vectors w.r.t. $Vir^{4/5} \times F_4^{(1)}$ are:

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$$\begin{aligned}
\phi(\hat{V}_3^0, 3, \Omega_0) &= \tfrac{1}{6}\big((-\lambda_1+\lambda_6)^3(-3) + (\lambda_3-\lambda_5)^3(-3) + (\lambda_1-\lambda_3+\lambda_5-\lambda_6)^3(-3)\big) \otimes e^0 \\
&+\tfrac{1}{2}((\lambda_3-\lambda_5)(-1) \otimes e^{\alpha_1-\alpha_6} + (\lambda_3-\lambda_5)(-1) \otimes e^{-\alpha_1+\alpha_6}) \\
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& + \tfrac{1}{2}((\lambda_3 - \lambda_5)(-1) \otimes e^{\alpha_1-\alpha_6} + (\lambda_3 - \lambda_5)(-1) \otimes e^{-\alpha_1+\alpha_6}) \\
& + \tfrac{1}{2}((-\lambda_1 + \lambda_6)(-1) \otimes e^{\alpha_3-\alpha_5} + (-\lambda_1 + \lambda_6)(-1) \otimes e^{-\alpha_3+\alpha_5}) \\
& + \tfrac{1}{2}(-\lambda_1 + \lambda_3 - \lambda_5 + \lambda_6)(-1) \otimes e^{\alpha_1+\alpha_3-\alpha_5-\alpha_6} \\
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\end{aligned}$$

$$\phi(\hat{V}_1^0, \tfrac{2}{5}, \Omega_4) = 1 \otimes e^{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} - 1 \otimes e^{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6}$$

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$$\begin{aligned}\phi(\hat{V}_2^0, \tfrac{7}{5}, \Omega_4) &= (-2\alpha_1(-1)-\alpha_3(-1)+\alpha_5(-1)+2\alpha_6(-1)) \otimes e^{\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6} \\ &+(2\alpha_1(-1)+\alpha_3(-1)-\alpha_5(-1)-2\alpha_6(-1)) \otimes e^{\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6} \\ &+3 \otimes e^{2\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5} + 3 \otimes e^{\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6}\end{aligned}$$

$$\phi(\hat{V}^{1}_{2/3}, \tfrac{2}{3}, \Omega_0) = 1 \otimes e^{-\lambda_1+\lambda_6} + 1 \otimes e^{\lambda_3-\lambda_5} - 1 \otimes e^{\lambda_1-\lambda_3+\lambda_5-\lambda_6}$$

$$\phi(\hat{V}^1_{2/3}, \tfrac{2}{3}, \Omega_0) = 1 \otimes e^{-\lambda_1+\lambda_6} + 1 \otimes e^{\lambda_3-\lambda_5} - 1 \otimes e^{\lambda_1-\lambda_3+\lambda_5-\lambda_6}$$

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$$\phi(\hat{V}^6_{2/3}, \tfrac{2}{3}, \Omega_0) = 1 \otimes e^{\lambda_1-\lambda_6} + 1 \otimes e^{-\lambda_3+\lambda_5} - 1 \otimes e^{-\lambda_1+\lambda_3-\lambda_5+\lambda_6}$$

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The proof of the main theorem uses another Ramanujan identity,

$$t^2 a(t^9)a(t) + b(t^9)b(t) = \frac{\phi(t^3)}{\phi(t)\phi(t^9)},$$

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the principal graded dimensions of each $E_6^{(1)}$-module, $\hat{V}^i$, $i = 0, 1, 6$, and of the two $F_4^{(1)}$-modules, $W(\Omega_0)$ and $W(\Omega_4)$, as well as the characters of six $Vir^{4/5}$-modules. The result for $\hat{V}^0$ seems to require a new identity, which is still being investigated.

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