Moduli space metrics for black holes

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Symmetries, unification and the search for quantum gravity

Nocolai Fest

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It is a a great honour to speak at this 60th birthday celebration for Hermann Nicolai. His role in the formulation of SUGRA theories is well known and has had an enormous influence on theoretical physics.

From the onset however, it seemed clear that if extended (ungauged) SUGRA theories were to make contact with real physics, it was by by non-perturbative effects, since otherwise, they are just theories of massless particles. In gravity theories the most important non-perturbative effects are connected with black holes, and in a SUSY theory, the most reliable calculations involve BPS states. It was natural therefore to investigate supersymmetric or extreme black holes. This has now become a vast subject, particularly with the incorporation of String Theory in the picture.

In this talk I want to review, and revisit, a small aspect of this topic : slow motion on the moduli spaces of extreme black holes, a subject which has many parallels and connections with with the slow motion on the moduli spaces of Yang-Mills monopoles. The subject is quite old, being originally developed in the 1980's and 1990's. However as as I began to think about it, I became aware that, in a certain sense, its origins and connections go back even further to the 1830's, and to the work of some of the greatest physicsts/mathematicians of all time. We shall see that SUGRA fits some of their boldest speculations as a hand fits a glove.

Since Hermann has often complemented me on my modest knowledge of history, and assuming that this was indeed a sincere form of flattery I hope he will enjoy the tale I have to tell. After all, it vindicates, a famous warning:

Those who cannot remember the past are condemned to repeat it.[1]

[1] George Santayana, The Life of Reason, Volume 1, 1905 US (Spanish-born) philosopher (1863 - 1952)

To acquaint myself with the history, I have consulted

- J.D. Jackson : *Classical Electrodynamics* 2nd Ed.
- E.T. Whittaker: *History of Theories of the Aether and Electricity*
- O. Darrigol: *Electrodynamics from Ampére to Einstein*

The Darwin Lagrangian

In 1920 [2] Charles Darwin (grandson of *the Charles Darwin*) wrote down an effective classical Lagrangian, valid to quadratic order in velocities, for n electromagnetically charged particles in which the light degree of freedom, i.e the Maxwell field, has been integrated out. The general form is

$$\begin{aligned} L &= \frac{1}{2}\sum m_a \mathbf{v}_a^2 + \sum_{1\leq a<b\leq n} L_{ab} \\ m_a \ddot{\mathbf{r}}_a &= \sum_{b\neq a} \mathbf{F}_{ab} \\ \mathbf{F}_{ab} &= \frac{\partial L_{ab}}{\partial \mathbf{r}_a} - \frac{d}{dt}\frac{\partial L_{ab}}{\partial \dot{\mathbf{r}}_a}. \end{aligned}$$

[2] C.G. Darwin M.A. (1920): The dynamical motions of charged particles , Philosophical Magazine Series 6, 39:233, 537-551

In Darwin's case

$$
\begin{aligned}
L_{ab} &= -\frac{q_a q_b}{r_{ab}}\Big\{1 - \frac{1}{2}\mathbf{v}_a \cdot \mathbf{v}_b - (\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_a)(\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_b)\Big\} \\
\mathbf{F}_{ab} &= q_a\Big(\mathbf{E}_b + \mathbf{v}_a \times \mathbf{B}_b\Big) \\
\mathbf{E}_b &= q_b\frac{\hat{\mathbf{r}}_{ab}}{r_{ab}^2}\Big\{1 + \frac{1}{2}\mathbf{v}_a^2 - \frac{3}{2}(\mathbf{v}_b \cdot \hat{\mathbf{r}}_{ab})^2 \ \Big\} \\
&- \frac{q_b}{2r_{ab}}\Big\{\mathbf{a}_b - \hat{\mathbf{r}}_{ab}(\hat{\mathbf{r}}_{ab} \cdot \mathbf{a}_b)\Big\}, \\
\mathbf{B}_b &= q_b\frac{\mathbf{v}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2} \\
\mathbf{F}_{ab}^{Darwin} &= q_a q_b\frac{\hat{\mathbf{r}}_{ab}}{r_{ab}^2}\Big\{1 + \frac{1}{2}\mathbf{v}_a^2 - \mathbf{v}_a \cdot \mathbf{v}_b - \frac{3}{2}(\mathbf{v}_b \cdot \hat{\mathbf{r}}_{ab})^2 \ \Big\} \\
&- \frac{q_a q_b}{2r_{ab}}\Big\{\mathbf{a}_b - \hat{\mathbf{r}}_{ab}(\hat{\mathbf{r}}_{ab} \cdot \mathbf{a}_b)\Big\}, \\
&+ \frac{q_a q_b}{r_{ab}^2}\mathbf{v}_b(\mathbf{v}_a \cdot \hat{\mathbf{r}}_{ab})
\end{aligned}
$$

Darwin's Lagrangian exhibits some puzzling features which have been the subject of much discussion for many years The force $\mathbf{F}_{ab}^{Darwin}$ is conservative but acceleration dependent . It is not central, and does not satisfy Newton's third law and hence the system is not Galilei invariant. In fact, after adding a (velocity)^4 term from special relativity the system is Poincaré invariant to order $\frac{1}{c^2}$ [3].

[3]S. Coleman and J. H. Van Vleck, Origin of 'Hidden Momentum Forces' on Magnets, Phys. Rev. **171** (1968) 1370.

The acceleration dependence of the "force" may be removed by defining the canonical momenta $\mathbf{p}_a$ by

$$\begin{aligned}
\mathbf{p}_a &= \frac{\partial L}{\partial \mathbf{v}_a} \\
&= m_a \mathbf{v}_a + q_a \mathbf{A}_a \\
&= m_a \mathbf{v}_a + \frac{1}{2} \sum_{b \neq a} \frac{q_a q_b}{r_{ab}} \Big(\mathbf{v}_b + \hat{\mathbf{r}}_{ab} (\mathbf{v}_b \cdot \hat{\mathbf{r}}_{ab}) \Big)
\end{aligned}$$

where $\mathbf{A}_a$ is the approximate vector potential in Coulomb Gauge, due to all other particles, The equation of motion now becomes

$$\begin{aligned}
\dot{\mathbf{p}}_a &= \frac{\partial L}{\partial \mathbf{r}_a} \\
&= \sum_{b \neq a} \frac{q_a q_b}{r_{ab}^2} \hat{\mathbf{r}}_{ab} \\
&+ \sum_{b \neq a} \frac{q_a q_b}{r_{ab}^2} \frac{1}{2} \Big(\mathbf{v}_a (\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_b) + \mathbf{v}_b (\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_a) \Big) \\
&- \sum_{b \neq a} \frac{q_a q_b}{r_{ab}^2} \hat{\mathbf{r}}_{ab} \Big(\mathbf{v}_b \cdot (\mathbf{1} + 3 \hat{\mathbf{r}}_{ab} \hat{\mathbf{r}}_{ab}) \cdot \mathbf{v}_b \Big)
\end{aligned}$$

The kinetic part of Darwin's Lagrangian is quadratic in velocities and thus defines a curved metric on the configuration space of n particles in R^n. Passing to the Hamiltonian, which is quadratic in momenta and making the replacement $\mathbf{p}_a = \frac{h}{2\pi i}\frac{\partial}{\partial \mathbf{r}_a}$ gives the Breit-Darwin Hamiltonian [4], a non-relativistic, non-Galilean invariant Schrödinger equation on this configuration space. If $m_a = m\,, \forall a$

$$H \approx \sum_{1\leq a\leq n}\Big(m+\frac{\mathbf{p}_a^2}{2m}\Big)+\sum_{1\leq a\leq b\leq n}\frac{q_a q_b}{2r_{ab}}\Big(1-\frac{\mathbf{p}_a\cdot\mathbf{p}_b-(\widehat{\mathbf{r}}_{ab}\cdot\mathbf{p}_a)(\widehat{\mathbf{r}}_{ab}\cdot\mathbf{p}_b)}{2m}\Big)+\dots$$

[4] G. Breit, The effect of retardation on the interaction of two electron, *Phys. Rev.* **34** (1929) 553

The Force between current elements

Averaging over all charge carriers, dropping acceleration terms and assuming over-all neutrality we find Grassmann's Formula for the force between two current elements

$$
\begin{aligned}
< d^2\mathbf{F}_{ab}^{Grassmann} > &= i_a i_b \frac{1}{r_{ab}^2}(d\mathbf{r}_a \times (d\mathbf{r}_b \times \hat{\mathbf{r}}_{ab})) \\
&= -i_a i_b \frac{1}{r_{ab}^2}\Big((d\mathbf{r}_a \cdot \mathbf{r}_b)\hat{\mathbf{r}}_{ab} - (d\mathbf{r}_a \cdot \hat{\mathbf{r}}_{ab}) d\mathbf{r}_b\Big)
\end{aligned}
$$

The usual way of deriving Grassmann's Formula is to use the Biot-Savart law. Grassmann's Formula is not central and does not satisfy Newton's third law. However integrated around two current loops it satisfies both.

This gave to much discussion in nineteenth century. In fact Ampeŕe had originally proposed

$$< d^2\mathbf{F}_{ab}^{Ampere} >= -i_a i_b \frac{\mathbf{r}_{ab}}{r_{ab}^3}\Big(3(\hat{\mathbf{r}}_{ab}\cdot d\mathbf{r}_a)(\hat{\mathbf{r}}_{ab}\cdot d\mathbf{r}_b) - 2(d\mathbf{r}_a\cdot d\mathbf{r}_b)\Big)$$

which is both central and satisfies Newton's third law. Integrated around two current loops Ampére's Formula and Grassmann's Formula gives equivalent results.

$$< d^2\mathbf{F}_{ab}^{Ampere} > - < d^2\mathbf{F}_{ab}^{Grassmann} >= d\mathbf{r}_b\cdot\frac{\partial}{\partial\mathbf{r}_b}\Big(\frac{1}{2} i_a i_b \frac{\mathbf{r}_{ab}(\mathbf{r}_{ab}\cdot d\mathbf{r}_a)}{r_{ab}^3}\Big)$$

Weber's Lagrangian

Weber aimed to derive Ampeŕe's Formula from a Force Law coming from a Lagrangian which was conservative, central, satisfies Newton's third Law and Galilean invariant

$$
\begin{aligned}
L_{ab} &= -\frac{q_a q_a}{r_{ab}}\Big(1 + \frac{1}{2}(\mathbf{v}_{ab} \cdot \widehat{\mathbf{r}}_{ab})^2\Big) \\
&= -\frac{q_a q_a}{r_{ab}}\Big(1 + \frac{1}{2}\dot{r}_{ab}^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}_{ab}^{Weber} &= q_a q_b \frac{\mathbf{r}_{ab}}{r_{ab}^3}\Big(1 - \frac{1}{2}\dot{r}_{ab}^2 + r_{ab}\ddot{r}_{ab}\Big) \\
&= q_a q_b \frac{\mathbf{r}_{ab}}{r_{ab}^3}\Big(1 - \frac{3}{2}(\dot{\mathbf{r}}_{ab}\cdot\hat{\mathbf{r}}_{ab})^2 + (\dot{\mathbf{r}}_{ab})^2 + \hat{\mathbf{r}}_{ab}\cdot\ddot{\mathbf{r}}_{ab}\Big) \\
&= q_a q_b \frac{\mathbf{r}_{ab}}{r_{ab}^3}(1 + \mathbf{v}_a^2 + \mathbf{v}_b^2 - 2\mathbf{v}_a\cdot\mathbf{v}_b - \frac{3}{2}(\hat{\mathbf{r}}_{ab}\cdot\mathbf{v}_a)^2 - \frac{3}{2}(\hat{\mathbf{r}}_{ab}\cdot\mathbf{v}_b)^2 \\
+3 &\quad (\hat{\mathbf{r}}_{ab}\cdot v_a)(\hat{\mathbf{r}}_{ab}\cdot\mathbf{v}_b) + \hat{\mathbf{r}}_{ab}\cdot\ddot{\mathbf{r}}_{ab}\Big)
\end{aligned}
$$

Averaging over all charge carriers, dropping acceleration terms and assuming over-all charge neutrality we find Ampere's Formula for the force between two current elements

$$
< d^2\mathbf{F}_{ab}^{Ampere} >= -i_a i_b \frac{\mathbf{r}_{ab}}{r_{ab}^3}\Big(3(\hat{\mathbf{r}}_{ab}\cdot d\mathbf{r}_a)(\hat{\mathbf{r}}_{ab}\cdot d\mathbf{r}_b) - 2(d\mathbf{r}_a\cdot d\mathbf{r}_b)\Big)
$$

Others, including Gauss, proposed other Lagrangians which were not central but giving equivalent results when integrated around two current loops.

Clausius's Lagrangian

$$L_{ab}^{\text{Clausius}} = -\frac{q_a q_a}{r_{ab}}\left(1 + \frac{1}{2}\mathbf{v}_a \cdot \mathbf{v}_b\right)$$

$$\mathbf{F}_{ab}^{\text{Clausius}} = \frac{q_a q_a}{r_{ab}^2}\left((1 - \mathbf{v}_a \cdot \mathbf{v}_b)\hat{\mathbf{r}}_{ab} - \dot{r}_{ab}\mathbf{v}_b + r_{ab}\mathbf{a}_b\right)$$

Riemann's Lagrangian

$$L_{ab}^{\text{Riemann}} = -\frac{q_a q_a}{r_{ab}}\left(1 + \frac{1}{2}(\mathbf{v}_a - \mathbf{v}_b)^2\right)$$

$$\mathbf{F}_{ab}^{\text{Riemann}} = \frac{q_a q_a}{r_{ab}^2}\left((1 + \frac{1}{2}\mathbf{v}_{ab}^2)\hat{\mathbf{r}}_{ab} + \dot{r}_{ab}\mathbf{v}_{ab} - r_{ab}\mathbf{a}_{ab}\right)$$

Clausius's suggestion is not central and it does not satisfy Newton's Third law.

However Riemann's suggestion while not central it does satisfy Newton's third Law and is Galilean invariant.

Weber-Tisserand-Riemann-Levy Gravity

In the late nineteenth century Tisserand [5] and Levy [6] attempted to account for the anomalous precession of the perihelion of Mercury by replacing $q_a q_b$ by $-G m_a m_b$ and using a combination of Galilei invariant Weber and Riemann terms. These theories were not very successful and discarded after Einstein's 1915 paper on General Relativity.

[5] F. Tisserand, Sur le mouvement des planètes autour du Soleil d'apres la loi électrodynamique de Weber Comptes Rendus de l'Aacd de Sci de Paris **110** (1872) 760-763, F. Tisserand, Sur les mouvements des planètes en supposant l'attrction représentée par l'une des lois électrodynamique de Gauss ou de Weber Comptes Rendus de l'Aacd de Sci de Paris **110** (1872) 313-315

[6] M. Lévy, Sur l'application des lois électrodynamiques au mouvment des planètes Comptes Rendus de l'Aacd de Sci de Paris **110** (1872) 541-549

Similar models have been invoked to implement Mach's Principle and so-called Relational models of gravity [7] .However Droste, Einstein, Infeld and Hoffmann (EIH) and Fichtenholtz have shown that Einstein's theory leads to the analogue of Darwin's Lagrangian of the form

$$
\begin{aligned}
L &= \sum_{1\le a\le n} \frac{1}{2} m_a \mathbf{v}_a^2 \\
&+ \sum_{1\le a<b\le n} \frac{G m_a m_b}{r_{ab}} \Big\{ 1 \\
&+ \frac{3}{2}(\mathbf{v}_a^2 + \mathbf{v}_b^2) - \frac{7}{2} \mathbf{v}_a \cdot \mathbf{v}_b \\
&- \frac{1}{2} (\widehat{\mathbf{r}}_{ab} \cdot \mathbf{v}_a)(\widehat{\mathbf{r}}_{ab} \cdot \mathbf{v}_b) \Big\}
\end{aligned}
$$

[7] A. K. T. Assis, Weber's Law and Mach's Principle, in Mach's Principle: From Newton's Bucket to Quantum Gravity *Einstein Studies* **6**(1995) 159-171

The Droste-Fichtenholtz Lagrangian is neither central, relational nor Galilei invariant.

Despite this, it has recently been invoked [8] in an attempt to to use it to connect a precise form of Mach's principle with the mass-energy density of the universe: it should be twice the critical value.

[8] H. Essen, 'Mechanics, cosmology and Mach's principle arXiv:1208.3096 [physics.class-ph].

Non-Abelian Monopoles

In Manton 1985 [9] calculated the long range forces between two BPS dyons in $SU(2)$ Yang-Mills theory using the same methods as Darwin and included the effect of massless scalar exchange. His aim was to calculate the asymptotic metric on the 4-dimensional moduli space which he showed to be that of Taub-NUT with negative mass. The asymptotic metric on the n dimensional moduli space of n $SU(2)$ BPS monopoles was written down later. [10]

[9] N. S. Manton, Monopole Interactions at Long Range,'' Phys. Lett. B **154** (1985) 397 [Erratum-ibid. **157B** (1985) 475].

[10] G. W. Gibbons and N. S. Manton, The Moduli space metric for well separated BPS monopoles, Phys. Lett. B **356** (1995) 32

In 1986 [11] Gibbons and Ruback did the same for charged black holes in Einstein Maxwell theory and in Shiraishi extended this when scalars are present.

$$\boxed{\mathcal{L} = -me^{\frac{\sigma\phi}{m}}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + qA_\mu\frac{dx^\mu}{d\lambda}}$$

If

$$d\tau = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda\,,$$

then

$$\boxed{m\frac{d^2x^\mu}{d\tau^2} + m\Gamma_\rho{}^\mu{}_\nu\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau} = qg^{\mu\rho}F_{\rho\nu}\frac{dx^\nu}{d\tau} - \sigma\Big(g^{\mu\nu} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Big)\partial_\nu\phi}$$

[11] G. W. Gibbons and P. J. Ruback,The Motion Of Extreme Reissner-nordstrom Black Holes In The Low Velocity Limit," Phys. Rev. Lett. **57** (1986) 1492.

which is consistent with

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1\,.$$

$$\boxed{\mathcal{L} = -me^{\frac{\sigma\phi}{m}}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + qA_\mu\frac{dx^\mu}{d\lambda}}$$

Anti-Gravitating solutions have been given by many people, most completely [12]

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\Big(R - e^{-2\alpha\phi} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - 2 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Big)$$

$$\begin{aligned} ds^2 &= -H^{-\frac{2}{1+a^2}} dt^2 + H^{\frac{2}{1+a^2}} d\mathbf{x}^2 \,, \\ F &= \pm\sqrt{(1+a^2)}\, d\Big(\frac{dt}{H}\Big) \,, \\ e^{-2a\phi} &= H^{\frac{2a^2}{1+a^2}} \,. \end{aligned}$$

where H is a harmonic function on R^3 with n mass points

$$Gm_a : \sigma_a : q_a :: 1 : a : \pm\sqrt{1+a^2}$$

[12]K. Shiraishi, Multicentered solution for maximally charged dilaton black holes in arbitrary dimensions," J. Math. Phys. **34** (1993) 1480

SUSY and Anti-Gravity

$$
\begin{aligned}
L &= \sum_{1\le a\le n} \frac{1}{2} m_a \mathbf{v}_a^2 \\
&+ \sum_{1\le a<b\le n} \frac{1}{r_{ab}} \Big\{ (G m_a m_b + \sigma_a \sigma_b - q_a q_b) \\
&+ \frac{1}{2}(3 G m_a m_b - \sigma_a \sigma_b)(\mathbf{v}_a^2 + \mathbf{v}_b^2) \\
&+ \frac{1}{2}(q_q q_m + \sigma_a \sigma_b - 7 G m_a m_b)\mathbf{v}_a \cdot \mathbf{v}_b \\
&+ \frac{1}{2}(q_a q_b - \sigma_a \sigma_b - G m_a m_b)(\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_a)(\hat{\mathbf{r}}_{ab} \cdot \mathbf{v}_b) \Big\}
\end{aligned}
$$

If Scherk's anti-gravity condition [13] $Gm_am_b + \sigma_a\sigma_b - q_aq_b = 0$ holds then

$$\boxed{L = \frac{1}{2M}\Big(\sum_{1\le a\le n} m_a\mathbf{v}_a\Big)^2 + \frac{1}{2}\sum_{\le a<b\le n}\Big(\frac{Gm_am_b}{M} + \frac{3Gm_am_b - \sigma_a\sigma_b}{r_{ab}}\Big)(\mathbf{v}_a - \mathbf{v}_b)^2}$$

If $3Gm_am_b = \sigma_a\sigma_b$ then

$$\boxed{L = \frac{1}{2M}\Big(\sum_{1\le a\le n} m_a\mathbf{v}_a\Big)^2 + \frac{1}{2}\sum_{\le a<b\le n}\Big(\frac{Gm_am_b}{M}(\mathbf{v}_a - \mathbf{v}_b)^2}$$

[13] J. Scherk, Antigravity: A Crazy Idea?, Phys. Lett. B **88** (1979) 265.

We see that if the Force Balance Condition holds we recover Riemann's Galilei invariant Kinetic term.

If $G = 0$ we recover the results of Manton and Gibbons and Manton for $SU(2)$ Yang-Mills monopoles. In his case, the the kinetic energy can become negative, a sure sign that the asymptotic approximation breaks down at short distance. In fact the Taub-NUT metric is replaced by the Atiyah-Hitchin metric.

It is interesting to note that Hermann Ludwig Ferdinand von Helmholtz had raised this objection to the Lagrangians of Weber, Riemann and Clausius [14] . To satsify Helmholtz, we need

$$\boxed{3Gm_a m_b \geq \sigma_a \sigma_b}$$

The marginal case corresponds to Kaluza-Klein monopoles.

[14] M. Helmholtz ,On the theory of electrodynamics, Phil Mag **44** (1872) 530-537

Following the work of Gibbons and Ruback, Eardley and Ferrell[15] gave the exact metric on the the two maximally charged Einstein Maxwell black hole moduli space This was followed up by Shiraishi [16] [17]

[15] R.C.Ferrell and D.M. Eardley, Slow motion scattering and coalescence of maximally charged black holes, Phys.Rev. Lett. **59** (1987) 1617.

[16] K. Shiraishi, Moduli space metric for maximally charged dilaton black holes, Nucl. Phys. B **402** (1993) 399.

[17] K. Shiraishi, 'Classical and quantum scattering of maximally charged dilaton black holes,'' Int. J. Mod. Phys. D **2** (1993) 59.

Exact Moduli Space Metrics

$$
\begin{aligned}
ds^2 &= \sum_{1\le a\le n} m_a d\mathbf{r}_a^2 \\
&+ \frac{3-a^2}{4\pi(1+a^2)} \times \int d^3\mathbf{r} \Big\{ H^{\frac{2(1-a^2)}{1+a^2}}(\mathbf{r}) \\
&\quad \sum_{1\le a<b\le n} \frac{(\mathbf{r}-\mathbf{r}_a)\cdot(\mathbf{r}-\mathbf{r}_b)|d\mathbf{r}_a - d\mathbf{r}_b|^2 \mu_a\mu_a}{|\mathbf{r}-\mathbf{r}_a|^3|\mathbf{r}-\mathbf{r}_b|^3}, \Big\}
\end{aligned}
$$

where the mass m_a of a'th particle is given by

$$
m_a = \frac{1}{1+a^2}\mu_a \ ,
$$

This is Galilei invariant and the relative moduli space metric for two bodies is

$$
\begin{aligned}
ds^2_{\text{rel}} &= \gamma(r)\, d\mathbf{r} \cdot d\mathbf{r} \\
\gamma(r) &= 1 - \frac{(m_1 + m_2)^2}{m_2 m_2} - \frac{(3 - a^2)M}{r} \\
&+ \frac{m_1 + m_2}{m_1}\Big(1 + \frac{(1 + a^2)m_1}{r}\Big)^{\frac{3-a^2}{1+a^2}} + \frac{m_1 + m_2}{m_2}\Big(1 + \frac{(1 + a^2)m_2}{r}\Big)^{\frac{3-a^2}{1+a^2}}
\end{aligned}
$$

If $\frac{3-a^2}{1+a^2}$ is a non-negative integer, i.e. $a^2 = 3, 1, \frac{1}{3}, 0$. we have no-body, two-body or two and three-body forces or two, three and four-body forces.

In particular, the cases $a^2 = 3$ and $a^2 = 1$ corresponds exactly to the to the asymptotic metric and are flat or of Riemann form respectively.

As observed in [18] In 4 spacetime dimensions there is a family of regular black hole solutions depending upon four independent harmonic functions (H_1, H_2, H_3, H_4). These black-hole solutions can be lifted to solutions of eleven-dimensional supergravity which have the interpretation of intersecting branes preserving $1/8$ of the spacetime supersymmetry Each harmonic function is associated with a brane involved in the intersection.

[18] G. W. Gibbons, G. Papadopoulos and K. S. Stelle, 'HKT and OKT geometries on soliton black hole moduli spaces, Nucl. Phys. B **508** (1997) 623 [hep-th/9706207].

- (i) $a = 0 \equiv (H, H, H, H) \leftrightarrow 4$.

- (ii) $a = \frac{1}{\sqrt{3}} \equiv (1, H, H, H) \leftrightarrow 4$

- (iii) $a = 1 \equiv (1, 1, H, H) \leftrightarrow 8$.

- (iv) $a = \sqrt{3} \equiv (1, 1, 1, H) \leftrightarrow 16$.

The geometry of these moduli spaces is determined by SUSY quantum mechanics.

The Exact Yang-Mills Moduli spaces

In the case of Yang-Mills in the BPS limit, the moduli space is known to be a $4n$ dimensional Hyper-Kähler manifold which which splits as the product of the COM space $S^1 \times R^3$ and a $4(n-1)$ dimensional HyperKähler manifold. $SO(3)$ action. For $SU(2), n = 2$ this was constructed by Atiyah and Hitchin [19]It has been identified by Sen with the lift to M-theory of an orientifold plane.

[19]M. F. Atiyah and N. J. Hitchin,'Low-Energy Scattering of Nonabelian Monopoles,'' Phys. Lett. A **107** (1985) 21.

Fundamental Monopoles of Distinct Type

In the case of $SU(n+1)$ broken to $U(1)^n$ (i.e. for n so-called Distinct Fundamental Monopoles), the moduli space has a tri-holomorphic $U(1)^n$ action and an $SO(3)$) action. The exact metric was identified by Lee-Weinberg and Yi [20].and the relative space is a generalization of Taub-NUT (with positve masses) depending on $\frac{1}{2}n(n-1)$ harmonic functions $\frac{1}{|\mathbf{r}_a - \mathbf{r}_b|}$ on $(R^3)^n$. Setting the $U(1)^n$ charges to zero, gives the relative Shiraishi metric for $a^2 = 1$.

It coincides with the relative asymptotic metric of n $SU(2)$ monopoles execpt for the sign of the mass terms.

[20] K. -M. Lee, E. J. Weinberg and P. Yi, The Moduli space of many BPS monopoles for arbitrary gauge groups,' Phys. Rev. D **54**, 1633 (1996) [hep-th/9602167]

Homothetic Solutions and Central Configurations

Newton's equations for n gravitating point masses

$$m_a \ddot{\mathbf{r}}_a = -\sum_{b \neq a} \frac{G m_a m_b (\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}$$

admit homothetic solutions $\mathbf{r}_a(t) = S(t)\mathbf{x}_a, \; \dot{\mathbf{x}}_a = 0$ provided the scale factor $S(t)$ satisfies Raychaudhuri's equation for dust

$$\boxed{\frac{\ddot{S}}{S} = -\frac{\lambda}{S^3}}$$

provided the $\mathbf{x}_a$ form a Central Configuration[21]

$$\boxed{-\lambda m_a \mathbf{x}_a = \sum_{b \neq a} \frac{G m_a m_b (\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3}}$$

[21] R.A. Battye, G.W. Gibbons and P.M. Sutcliffe, Central Configurations in Three Dimensions *Proc Roy Soc A* **459** (2003)

Similar homothetic solutions exist in the anti-gravitating case . For example [22] in the Shiraishi/Lee-Weinberg-Yi case the scale factor $S(t)$ satisfies

$$\frac{\ddot{S}}{S} = -\lambda\Big(\frac{\dot{S}^2}{S^3} - \frac{\ddot{S}}{S^2}\Big) \tag{1}$$

$\lambda > 0$. The case $\lambda < 0$ corresonds to the Manton $SU(2)$ case Such homothetic solutions are not possible if $Gm_a m_b + \sigma_a \sigma_b - q_a q_b \neq 0$.

[22] R.A. Battye, G.W. Gibbons, P. Rychenkova, and P.M. Sutcliffe, Polyhedral scattering of fundamantal monopoles *J Math Phys* **44** (2003) 3532-3542

Testing the Moduli Space Approximation

Our story is not quite finished:

Since 2005 reliable numerical simulations of black hole collisions has become possible. Recently there has been numerical work by colliding charged black holes [23] . In this way it should be possible to test the accuracy of the moduli space geodesic approximation.

[23] M. Zilhao, V. Cardoso, C. Herdeiro, L. Lehner and U. Sperhake,'Collisions of charged black holes Phys. Rev. D **85** (2012) 124062 [arXiv:1205.1063 [gr-qc]].

Dear Hermann,

I have admired you deeply intellectual apparoach to physics and life in general, and for very many talents, mathematical, physical and musicai since our first encounter and and our (sadly only one) collaboration [24] on a topic which still remains topical today.

I hope I have convinced you even if one is condemed to repeat history, it can be be fun to learn about it after the fact.

HAPPY BIRTHDAY

[24] G. W. Gibbons and H. Nicolai, 'One Loop Effects On The Round Seven Sphere *Phys. Lett. B* **143** (1984) 108.