Quasi-conformal realizations of exceptional groups, minimal representations, supersymmetry and AdS/CFT dualities

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HermannFest, Sept. 6-8, 2012
AEI, Golm

- Review of U-duality groups.
- Orbits of 5*d* extremal black holes and conformal extensions of 5*d* U-duality groups as spectrum generating symmetry group.
- Orbits of 4*d* extremal black hole solutions and quasiconformal extensions of 4*d* U-duality groups as spectrum generating symmetry group.
- Quantum spectra of extremal 4*d* black holes and unitary representations of their 3*d* U-duality groups as quasiconformal groups
- Quantization of quasiconformal group actions, minimal unitary representations and harmonic superspace
- Minimal unitary representations of noncompact groups and supergroups and AdS/CFT dualities
- Minimal unitary representations of 3*d* U-duality groups as "quarks" of the quantum spectra of 4*d* supergravities.
- Open problems

- Bosonic part of the $5D$ $N=2$ Maxwell-Einstein supergravity Lagrangian (MESGT) MG, Sierra and Townsend (1983)

$$e^{-1}\mathcal{L} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) + + \frac{e^{-1}}{6\sqrt{6}} C_{IJK}\,\varepsilon^{\mu\nu\rho\sigma\lambda}F^I_{\mu\nu}F^J_{\rho\sigma}A^K_\lambda$$

coupling of (n_V-1) vector multiplets $(A^a_\mu, \lambda^{ai}, \varphi^a)$ to $N=2$ supergravity $(g_{\mu\nu}, \psi^i_\mu, A_\mu)$ $\big($ $I,J,K=1,\ldots,n_V$, i=1,2 , $x,a=1,\ldots,(n_V-1)$ $\big)$

- $5D$, $N=2$ MESGT is uniquely determined by the constant symmetric tensor C_{IJK}.
- $5D$ MESGTs with symmetric scalar manifolds G/H such that G is a symmetry of the Lagrangian $\Longleftrightarrow$ C_{IJK} is given by the norm (determinant) $\mathcal{N}_3$ of a Euclidean Jordan algebra J of degree 3.

$$\mathcal{N}_3(J) = C_{IJK}h^Ih^Jh^K$$

Euclidean $J :\Longleftrightarrow X^2+Y^2=0 \Longrightarrow X=Y=0\ \forall\, X,\,Y\in J$

Symmetry Groups of Simple Jordan algebras

$n \times n$ Hermitian matrices over the division algebra $\mathbb{A}$ form a Jordan algebra $J_n^{\mathbb{A}}$ under the symmetric product $A \cdot B \equiv 1/2(AB + BA)$.

J	$Rot(J)$	$Lor(J)$	$Conf(J)$
$J_2^{\mathbb{C}}$	$SU(2)$	$SL(2,\mathbb{C})$	$SU(2,2)$
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$
$\Gamma_{(1,d)}$	$SO(d)$	$SO(d,1)$	$SO(d,2)$

Table: The complete list of simple Euclidean Jordan algebras and their rotation (automorphism), "Lorentz" (reduced structure) and "Conformal" (linear fractional) groups. The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. $J_n^{\mathbb{A}}$ denotes a Jordan algebra of $n \times n$ hermitian matrices over $\mathbb{A}$. $\Gamma_{(1,d)}$ denotes the Jordan algebra of Dirac gamma matrices.

- Unified $N = 2$ Maxwell-Einstein Supergravity theories in $5d \Leftrightarrow$ all the vectors fields including the graviphoton transform in an irreducible representation of a simple U-duality group of the action.
- There exist only four unified MESGTs in $d = 5$ with symmetric target spaces . They are defined by the four simple Euclidean Jordan algebras $J_3^{\mathbb{A}}$ of 3×3 Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ and describe the coupling of 5, 8, 14 and 26 vector multiplets to supergravity. Their symmetries in 5,4 and 3 dimensions give the groups of the Magic Square of Freudenthal, Rozenfeld and Tits $\Longrightarrow$ *Magical supergravity theories* (GST 1983)
- Scalar manifolds of five dimensional magical sugras are the irreducible symmetric spaces

$$\begin{array}{ccccccc} J & = & J_3^{\mathbb{R}} & J_3^{\mathbb{C}} & J_3^{\mathbb{H}} & J_3^{\mathbb{O}} \\ \mathcal{M}_5 & = & SL(3,\mathbb{R})/SO(3) & SL(3,\mathbb{C})/SU(3) & SU^*(6)/USp(6) & E_{6(-26)}/F_4 \end{array}$$

- The generic Jordan family defined by non-simple Jordan algebras $J = \Gamma_{(1,n-1)} \oplus \mathbb{R}$ has the scalar manifold $(SO(n-1,1) \times SO(1,1))\,/SO(n-1)$
- In addition there exist three infinite families of unified MESGT's in 5d whose scalar manifolds are neither symmetric nor homogeneous! They are defined by Lorentzian Jordan of degree $n \geq 3$ generated by $n \times n$ matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ Hermitian with respect to the Lorentzian metric η (MG, Zagermann 2003)

$$(A\eta)^\dagger = A\eta$$

- Under dimensional reduction of 5d MESGTs defined by Euclidean Jordan algebras to four dimensions one has

$$\mathcal{M}_5 = \frac{Lor(J)}{Rot(J)} \quad \Rightarrow \quad \mathcal{M}_4 = \frac{Conf(J)}{\widetilde{Lor}(J) \times U(1)}$$

where $\widetilde{Lor}(J)$ is the compact real form of the Lorentz group of J.

- The bosonic sector of dimensionally reduced Lagrangian is

$$\mathcal{L}^{(4)} = -\frac{1}{2}R - g_{I\bar{J}}(\partial_\mu z^I)(\partial^\mu \bar{z}^J) + \frac{1}{4}\mathrm{Im}(\mathcal{N}_{AB})F^A_{\mu\nu}F^{\mu\nu B} - \frac{1}{8}\mathrm{Re}(\mathcal{N}_{AB})\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma}$$

In 5D: Vector fields $A^\mu_I \Leftrightarrow$ Elements of Jordan algebra J
In 4D: $F^A_{\mu\nu} \oplus \tilde{F}^A_{\mu\nu} \Leftrightarrow$ Freudenthal triple system (FTS) $\mathcal{F}(J)$:

$$\mathcal{F}(J) \ni X = \begin{vmatrix} \mathbb{R} & \mathbf{J} \\ \tilde{\mathbf{J}} & \mathbb{R} \end{vmatrix} \Leftrightarrow \begin{vmatrix} F^0_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^I_{\mu\nu} & \tilde{F}^0_{\mu\nu} \end{vmatrix}$$

Automorphism group of $\mathcal{F}(J) \cong$ Conformal group of J

$N=2$ MESGTs reduce to $N=4$ supersymmetric sigma models coupled to gravity in $d=3$. The target manifolds of magical supergravity theories in $d=3$ are the exceptional quaternionic symmetric spaces:

- $\frac{F_{4(4)}}{Usp(6)\times USp(2)}$, $\frac{E_{6(2)}}{SU(6)\times SU(2)}$, $\frac{E_{7(5)}}{SO(12)\times SU(2)}$, $\frac{E_{8(-24)}}{E_7\times SU(2)}$
- The generic Jordan family of MESGT reduced to $d=3$ have the target spaces: $\frac{SO(n+2,4)}{SO(n+2)\times SO(4)}$
- Pure $N=2$ supergravity reduces to $N=4$ sigma model with target space $\frac{G_{2(2)}}{SO(4)}$.

Exceptional $N=2$ versus Maximal $N=8$ Supergravity:

- The exceptional $N=2$ supergravity is defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over real octonions $\mathbb{O}$. Its global invariance group in $5D$ is $E_{6(-26)}$ with maximal compact subgroup F_4.
- The C-tensor C_{IJK} of $N=8$ supergravity in five dimensions can be identified with the symmetric tensor given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ defined over split octonions $\mathbb{O}_s$. Its global invariance group in $5D$ is $E_{6(6)}$ with maximal compact subgroup $USp(8)$.
- In $D=4$ and $D=3$ the exceptional supergravity has $E_{7(-25)}$ and $E_{8(-24)}$ as its U-duality group while the maximal $N=8$ supergravity has $E_{7(7)}$ and $E_{8(8)}$, respectively.

U-duality Orbits of Extremal , Spherically Symmetric Stationary Black Hole Solutions of $5D$ Supergravity Theories with Symmetric Target Spaces: (MG and Ferrara, 1997)

- The black hole potential that determines the attractor flow takes on the following form for $N = 2$ MESGTs: (Ferrara, Gibbons, Kallosh, Strominger)

$$V(\phi, q) = q_I \overset{\circ}{a}{}^{IJ} q_J$$

where $\overset{\circ}{a}_{IJ}$ is the "metric" of the kinetic energy term of the vector fields. The $(n+1)$ dimensional charge vector in an extremal BH background is given by

$$q_I = \int_{S^3} H_I = \int_{S^3} \overset{\circ}{a}_{IJ} * F^J \qquad (I = 0, 1, ...n)$$

The entropy S of an extremal black hole solution of $N = 2$ MESGT with charges q_I is determined by the value of the black hole potential V at the attractor points

$$S_{BPS} = (V_{critical})^{3/4} = \left(C^{IJK} q_I q_J q_K\right)^{3/4}$$

The orbits of BPS black hole solutions of $5d$ Magical supergravities :

J	$\mathcal{O}_{BPS} = Str_0(J)/Aut(J)$	$\mathcal{O}_{non-BPS} = Str_0(J)/Aut(J_{(1,2)})$
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R})/SO(3)$	$SL(3,\mathbb{R})/SO(2,1)$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C})/SU(3)$	$SL(3,\mathbb{C})/SU(2,1)$
$J_3^{\mathbb{H}}$	$SU^*(6)/USp(6)$	$SU^*(6)/USp(4,2)$
$J_3^{\mathbb{O}}$	$E_{6(-26)}/F_4$	$E_{6(-26)}/F_{4(-20)}$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$SO(n-1,1) \times SO(1,1)/SO(n-1)$	$SO(n-1,1) \times SO(1,1)/SO(n-2,1)$

Table: Orbits of spherically symmetric stationary BPS and non-BPS black hole solutions with non-zero entropy in $5D$ MESGTs defined by Euclidean Jordan algebras J of degree three. U-duality and stability groups are given by the Lorentz (reduced structure) and rotation (automorphism) groups of J.

The orbits of BPS black hole solutions of $5d, N = 8$ supergravity :

$$\mathcal{O}_{1/8-BPS} = \frac{E_{6(6)}}{F_{4(4)}} \qquad \mathcal{O}_{1/4-BPS} = \frac{E_{6(6)}}{O(5,4) \circledS T_{16}} \qquad \mathcal{O}_{1/2-BPS} = \frac{E_{6(6)}}{O(5,5) \circledS T_{16}}$$

The entropy of $1/8$ BPS black holes is non-vanishing and entropies of $1/4$ and $1/2$ BPS black holes vanish. Vanishing entropy means vanishing cubic norm. Thus the black hole solutions corresponding to vanishing entropy has additional symmetries beyond the five dimensional U-duality group.

$\Longrightarrow$ They are invariant under the generalized special conformal transformations of the underlying Jordan algebras.

- The proposal: the conformal groups $Conf[J]$ of underlying Jordan algebras J of supergravity theories must act as spectrum generating symmetry groups of $5d$ supergravity theories defined by them.
  $Conf[J]$ leaves invariant light-like separations with respect to a cubic distance function $\mathcal{N}_3(J_1 - J_2)$ and admits a 3-grading with respect to their Lorentz subgroups
  $$Conf[J] = K_J \oplus Lor(J) \times \mathcal{D} \oplus T_J$$
  $Lor(J)$ is the $5D$ U-duality group that leaves the cubic norm invariant.
- $Conf[J] \Leftrightarrow$ U-duality group G_4 of corresponding $4D$ supergravity. $\Longrightarrow$ $4D$ U-duality groups G_4 must act as spectrum generating symmetry groups of corresponding five dimensional supergravity theories.
- U-duality group G_4 of a $4D$ Maxwell-Einstein supergravity defined by a Jordan algebra J of degree three $\Leftrightarrow$ $G_4 \equiv Aut(\mathcal{F}(J)) \equiv Conf[J]$
  Freudenthal triple system $\mathcal{F}(J)$ is endowed with an invariant symmetric quartic form $\mathcal{Q}_4(q, p)$ and a skew-symmetric bilinear

Extension of Orbit Analysis to $4d$ Black Holes :

- The entropy of an extremal dyonic black hole with charges (p^0, p^I, q_0, q_I) is given by the quartic invariant $\mathcal{Q}_4(q, p)$ of the Freudenthal triple system $\mathcal{F}(J)$.
- black hole attractor equations $\Rightarrow$ criticality conditions for black hole scalar potential in $4d$ are:
  $$V_{BH} \equiv |Z|^2 + G^{I\bar{J}}(D_I Z)(\overline{D}_{\bar{J}}\overline{Z})$$
- $$2\overline{Z} D_I Z + iC_{IJK} G^{J\bar{J}} G^{K\bar{K}} \overline{D}_{\bar{J}}\overline{Z}\overline{D}_{\bar{K}}\overline{Z} = 0$$

J	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-BPS}$	non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{non-BPS,Z\neq 0}$	non-BPS, $Z = 0$ orbits $\mathcal{O}_{non-BPS,Z=0}$
$-$	$\frac{SU(1,n+1)}{SU(n+1)}$	$-$	$\frac{SU(1,n+1)}{SU(1,n)}$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(1,1)\otimes SO(1,1+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2,n)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Table: Non-degenerate orbits of $N = 2$, $D = 4$ MESGTs with symmetric scalar manifolds. Except for the first row all such theories originate from five dimensions and are defined by Jordan algebras that are indicated in the first column. (MG & Ferrara, 1997) (Bellucci, Ferrara , MG, Marrani , 2006)

- The orbits of BH solutions of $4D$ $N=8$ supergravity under $E_{7(7)}$:

$$I_4 > 0 : \mathcal{O}_{\frac{1}{8}-BPS} = \frac{E_{7(7)}}{E_{6(2)}} \Longleftrightarrow \frac{1}{8}\text{-BPS};$$

$$I_4 < 0 : \mathcal{O}_{non-BPS} = \frac{E_{7(7)}}{E_{6(6)}} \Longleftrightarrow \text{non-BPS}.$$

Generic light-like orbit with 1 vanishing eigenvalue:

$$\frac{E_{7(7)}}{F_{4(4)} \circledS T_{26}}$$

Critical light-like orbit with 2 vanishing eigenvalues:

$$\frac{E_{7(7)}}{O(6,5) \circledS (T_{32} \oplus T_1)}$$

Doubly critical light-like orbit with 3 vanishing eigenvalues :

$$\frac{E_{7(7)}}{E_{6(6)} \circledS T_{27}}$$

- Question: Can the $3D$ U-duality groups act as spectrum generating conformal symmetries of corresponding $4D$ supergravity theories ? (MG, Koepsell, Nicolai 1997)
- Problem: $E_{8(8)}$ and $E_{8(-24)}$ appear as $3d$ U-duality groups. No conformal realization for any real forms of E_8, G_2 and $F_4 \Leftrightarrow$ No 3-grading with respect to a subgroup of maximal rank.
- However, all simple Lie algebras admit a 5-grading with respect to a subalgebra of maximal rank

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

such that the grade ± 2 subspaces are one-dimensional.

$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K$$

$A, B, C ... = 1, ...2N$ and $(K, \Delta, \tilde{K})$ form an $sl(2)$ subalgebra.
$U_A \in \mathfrak{g}^{+1}$ and $A \in \mathcal{F}$ where $\mathcal{F}$ is a Freudenthal triple system.

QUASICONFORMAL REALIZATION OF $E_{8(8)}$ MG, Koepsell, Nicolai, 2000

$$E_{8(8)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2}$$
$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K$$

over a space $\mathcal{T}$ coordinatized by the elements X of the exceptional FTS $\mathcal{F}(J_3^{\mathbb{O}^s})$ plus an extra singlet variable x: $56_{+1} \oplus 1_{+2} \Leftrightarrow (X, x) \in \mathcal{T}$:

$$K(X) = 0, \quad U_A(X) = A, \quad S_{AB}(X) = (A, B, X),$$
$$K(x) = 2, \quad U_A(x) = \langle A, X \rangle, \quad S_{AB}(x) = 2\langle A, B \rangle x,$$
$$\tilde{U}_A(X) = \frac{1}{2}(X, A, X) - Ax$$
$$\tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A \rangle + \langle X, A \rangle x$$
$$\tilde{K}(X) = -\frac{1}{6}(X, X, X) + Xx$$
$$\tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X \rangle + 2x^2$$

Freudenthal triple product $\Leftrightarrow (X, Y, Z)$
Skew-symmetric invariant form $\Leftrightarrow \langle X, Y \rangle = -\langle Y, X \rangle$
Quartic invariant of $E_{7(7)} \Leftrightarrow \langle (X, X, X), X \rangle$
$A, B, .. \in \mathcal{F}(J_3^{\mathbb{O}^s})$

- Geometric meaning of the quasiconformal action of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{T}$?
- Define a quartic norm of $\mathcal{X} = (X, x) \in \mathcal{T}$ as $\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2$
  $\mathcal{Q}_4(X)$ is the quartic norm of the underlying Freudenthal system and $X \in \mathcal{F}$.
- Define a quartic "distance" function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

$$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}))$$

  $\delta(\mathcal{X}, \mathcal{Y})$ is the "symplectic" difference of $\mathcal{X}$ and $\mathcal{Y}$:

$$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y \rangle) = -\delta(\mathcal{Y}, \mathcal{X})$$

- Light-like separations $d(\mathcal{X}, \mathcal{Y}) = 0$ are left invariant under quasiconformal group action.
  $\longrightarrow$ *Quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.*
- $E_{8(8)}$ is the invariance group of a quartic light-cone in 57 dimensions!

- Quasiconformal extensions of $4D$ U-duality groups of $N = 8$ sugra and $N = 2$ MESGTs defined by Jordan algebras $\equiv 3D$ U-duality groups.
- For an extremal black hole solution of $4D$, $N = 8$ sugra with 28 electric and 28 magnetic charges charges q_I, p^I choosing the extra singlet coordinate as entropy s one finds that light-like condition for vectors in this 57 dimensional charge-entropy space is:

$$\mathcal{N}_4(q_I, p^I, s) = 0 \Longrightarrow Q_4 = s^2$$

- Proposal: $3D$ U-duality groups must act as spectrum generating quasiconformal symmetry groups of the extremal black hole solutions of $4D$ supergravity theories. GKN 2000
- Work on $4d/5d$ lift of black hole solutions lend support to the proposal that conformal extensions of $5d$ U-duality groups act as spectrum generating symmetry groups.
- Composition of two proposals implies that $3d$ U-duality group must act as spectrum generating symmetry of the corresponding $5d$ supergravity theories. Recent work on using $3d$ U-duality groups to generate new solutions of $5d$ supergravity support these proposals.
  Galtsov, Scherbluk; Compere et.al; Palmkvist et.al...

J	$\mathcal{M}_5 =$ Lor (J) / Rot (J)	$\mathcal{M}_4 =$ Conf (J) / $\widetilde{\text{Lor}}\,(J) \times U(1)$	$\mathcal{M}_3 =$ QConf$(\mathcal{F}(J))/\widetilde{\text{Conf}}\,(J) \times \mathrm{SU}(2)$
$J_3^{\mathbb{R}}$	$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$	$\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$	$\mathrm{F}_{4(4)}/\mathrm{USp}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{C}}$	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$	$\mathrm{SU}(3,3)/\mathrm{S}\,(\mathrm{U}(3) \times \mathrm{U}(3))$	$\mathrm{E}_{6(2)}/\mathrm{SU}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{H}}$	$\mathrm{SU}^*(6)/\mathrm{USp}(6)$	$\mathrm{SO}^*(12)/\mathrm{U}(6)$	$\mathrm{E}_{7(-5)}/\mathrm{SO}(12) \times \mathrm{SU}(2)$
$J_3^{\mathbb{O}}$	$\mathrm{E}_{6(-26)}/\mathrm{F}_4$	$\mathrm{E}_{7(-25)}/\mathrm{E}_6 \times \mathrm{U}(1)$	$\mathrm{E}_{8(-24)}/\mathrm{E}_7 \times \mathrm{SU}(2)$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$\frac{\mathrm{SO}(n-1,1)\times\mathrm{SO}(1,1)}{\mathrm{SO}(n-1)}$	$\frac{\mathrm{SO}(n,2)\times\mathrm{SU}(1,1)}{\mathrm{SO}(n)\times\mathrm{SO}(2)\times\mathrm{U}(1)}$	$\frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2)\times\mathrm{SO}(4)}$

Table: Scalar manifolds $\mathcal{M}_d$ of $N = 2$ MESGT's defined by Jordan algebras J of degree 3 in $d = 3, 4, 5$ dimensions. $\widetilde{\text{Lor}}\,(J)$ and $\widetilde{\text{Conf}}\,(J)$ denote the compact real forms of the Lorentz group $\text{Lor}\,(J)$ and conformal group $\text{Conf}\,(J)$ of a Jordan algebra J. $\text{QConf}\,(\mathcal{F}\,(J))$ denotes the quasiconformal group associated with J.

- A concrete implementation of the proposal that $3D$ U-duality groups must act as spectrum generating quasiconformal groups of spherically symmetric stationary BPS black holes of $4D$ supergravity theories:

MG, Neitzke, Pavlyk, Pioline, Waldrom 2005,2006

- Equations of motion for a spherically symmetric stationary black hole of four dimensional supergravity theories are equivalent to equations for geodesic motion of a fiducial particle on the moduli space $\mathcal{M}_3^*$ of $3D$ supergravity obtained by reduction on a time-like circle.

Breitenlohner, Gibbons, Maison 1987

- $3D$ scalar manifold from compactification on a space-like (time-like) circle

$$\mathcal{M}_3 = \frac{G_3}{K_3} \qquad \left(\mathcal{M}_3^* = \frac{G_3}{H_3}\right)$$

where K_3 is the maximal compact subgroup of G_3 and H_3 is a noncompact real form of K_3. For $N = 2$ MESGTs defined by Jordan algebras $\mathcal{M}_3^*$ is a para-quaternionic symmetric space.

n_Q	n_V	M_4	$\mathcal{M}_3^*$	J
8	1	$\emptyset$	$\frac{U(2,1)}{U(1,1)\times U(1)}$	$\mathbb{R}$
8	2	$\frac{SL(2,\mathbb{R})}{U(1)}$	$\frac{G_{2,2}}{SO(2,2)}$	$\mathbb{R}$
8	7	$\frac{Sp(6,\mathbb{R})}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{Sp(6,\mathbb{R})\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{R}}$
8	10	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(3,3)\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{C}}$
8	16	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{H}}$
8	28	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{O}}$
8	$n+2$	$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(n,2)}{SO(n)\times SO(2)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$	$\mathbb{R}\oplus\Gamma_{(1,n-1)}$
16	$n+2$	$\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$	$\mathbb{R}\oplus\Gamma_{(5,n-5)}$
24	16	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SL(2,\mathbb{R}}$)	$J_3^{\mathbb{H}}$
32	28	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO^*(16)}$	$J_3^{\mathbb{O}_s}$

Table: Above we give the number of supercharges n_Q, $4D$ vector fields n_V , scalar manifolds of supergravity theories before and after reduction along a timelike Killing vector from $D=4$ to $D=3$, and associated Jordan algebras J. Isometry groups of $4D$ and $3D$ supergravity theories are given by $Conf(J)$ and $QConf(J)$, of J, respectively.

- The quantization of the motion of fiducial particle on $\mathcal{M}_3^*$ leads to quantum mechanical wave functions that provide the basis of a unitary representation of the isometry group G_3 of $\mathcal{M}_3^*$.
- BPS black holes correspond to a special class of geodesics which lift holomorphically to the twistor space $\mathcal{Z}_3$ of $\mathcal{M}_3^*$. Spherically symmetric stationary BPS black holes of $N = 2$ MESGT's are described by holomorphic curves in $\mathcal{Z}_3$
- The relevant unitary representations of the $3D$ isometry groups $QConf(J)$ for BPS black holes are those induced by their holomorphic actions on the corresponding twistor spaces $\mathcal{Z}_3$, which belong to quaternionic discrete series representations.
- For rank two quaternionic groups $SU(2,1)$ and $G_{2(2)}$ unitary representations induced by the geometric quasiconformal actions and their spherical vectors were studied in (MG, Neitzke, Pavlyk , Pioline , 2007).

- Construction of the quaternionic discrete series representations of $SU(2,1)$ and $G_{2(2)}$ using the quasiconformal realizations twisted with a unitary character ν : GNPP (2007)
  Starting point is the determination of the spherical vector $|\mathcal{O}\rangle$ of the QCG group G with the maximal compact subgroup K:

$$K|\mathcal{O}\rangle = |\mathcal{O}\rangle$$

  Consider the Verma module generated by the action of the noncompact generators P_a of G:

$$|\mathcal{O}\rangle \oplus P_a|\mathcal{O}\rangle \oplus P_a P_b|\mathcal{O}\rangle \oplus \cdots$$

  The quaternionic discrete series representations are obtained as submodules of the above Verma module for certain discrete values of ν.
  U-duality groups of magical sugras in d=3 are the quaternionic real forms of exceptional groups F_4, E_6, E_7 and E_8.

- Unified realization of $3D$ U-duality groups of all $N=2$ MESGTs defined by Jordan algebras as spectrum generating quasiconformal groups covariant with respect to their $5D$ U-duality groups. MG, Pavlyk (2009)
- Unified construction of the spherical vectors of quasiconformal realizations of $F_{4(4)}, E_{6(2)}, E_{7(-5)}, E_{8(-24)}$ and $SO(d+2,4)$ twisted by a unitary character ν with respect to their maximal compact subgroups and determination of their quadratic Casimir operators
- For $\nu = -(n_V+2) + i\rho$ the quasiconformal action induces unitary representations that belong to the principle series. For special discrete values of ν the quasiconformal action leads to unitary representations belonging to the quaternionic discrete series and their analytic continuations. (n_V = number of vector fields of the $5d$, $N=2$ MESGT.)
- Extension of the above results to $F_{4(4)}, E_{6(6)}, E_{7(7)}, E_{8(8)}$ and of $SO(n+3,m+3)$. as QCGs covariant with respect to $SL(3,R), SL(3,R)\times SL(3,R), SL(6,R), E_{6(6)}$ and $SO(n,m)\times SO(1,1)$, respectively.

  Quaternionic discrete series of $E_{8(-24)}$ obtained from its quasiconformal realization $\Longrightarrow$ "Octonionic discrete series" of $E_{8(8)}$ from its quasiconformal realization twisted by a unitary character.

Minimal Unitary Representations and Quasiconformal Groups:

- Quantization of the quasiconformal realization of a non-compact Lie group leads directly to its minimal unitary representation $\Rightarrow$ Unitary representation over an Hilbert space of square integrable functions of smallest number of variables possible.
- Minimal unitary representation of $E_{8(8)}$ over $L^2(\mathbb{R}^{29})$ from its geometric realization as a quasiconformal group MG, Koepsell & Nicolai 2000

$$E_{8(8)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(7)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2}$$

- Minimal unitary representation of $E_{8(-24)}$ over $L^2(\mathbb{R}^{29})$ from its geometric realization as a quasiconformal group MG, Pavlyk, 2004

$$E_{8(-24)} = 1_{-2} \oplus 56_{-1} \oplus E_{7(-25)} + SO(1,1) \oplus 56_{+1} \oplus 1_{+2}$$

56 of E_7 $\Rightarrow$ 28 coordinates and 28 momenta. These 28 coordinates plus the singlet coordinate yield the minimal number (29) of variables for E_8.

QCG Approach to Minimal Unitary Representations

GKN & MG, Pavlyk

- Lie algebra $\mathfrak{g}$ of a quasiconformal realization of a group G can be decomposed as :

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$$

$$\mathfrak{g} = E \oplus E^{\alpha} \oplus (J^a + \Delta) \oplus F^{\alpha} \oplus F$$

$\Delta = -\frac{i}{2}(yp + py)$ $([y, p] = i)$ determines the 5-grading and $\Omega^{\alpha\beta}$ is the symplectic invariant tensor of $\mathfrak{h}$ generated by J^a $(\alpha, \beta, .. = 1, 2, ..., 2n)$ and $\left[\xi^{\alpha}, \xi^{\beta}\right] = \Omega^{\alpha\beta}$

$$E = \frac{1}{2}y^2 \qquad E^{\alpha} = y\,\xi^{\alpha}, \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^{\alpha}\xi^{\beta}$$

$$F = \frac{1}{2}p^2 + \frac{\kappa I_4(\xi^{\alpha})}{y^2}, \qquad F^{\alpha} = [E^{\alpha}, F]$$

$I_4(\xi^{\alpha}) = S_{\alpha\beta\gamma\delta}\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \Leftrightarrow$ *quartic invariant of* $\mathfrak{h}$ generated by J^a
Choosing a polarization $\xi^{\alpha} = (x^i, p_j)$ one has $[x^i, p_j] = i\delta^i_j$ $(i, j = 1, 2, .., n) \Longrightarrow$
Minimal unitary representation in $n + 1$ coordinates (x^i, y) and their momenta.
Minimal Gelfand-Kirillov dimension $= n + 1$

- $(E, F, \Delta) \Longrightarrow SL(2, \mathbb{R})$ of conformal quantum mechanics with the quartic invariant I_4 playing the role of coupling constant.

- In HSS the metric on a quaternionic target space is given by a quartic potential $\mathcal{L}^{(+4)}$. The action is Galperin, Ogievetsky 1993

$$S=\int d\zeta^{(-4)}du\{Q^{+}_{\alpha}D^{++}Q^{+\alpha}-q^{+}_{i}D^{++}q^{+i}+\mathcal{L}^{(+4)}(Q^{+},q^{+},u^{-})\}$$

$\zeta, u^{\pm}_{i}$ are *analytic* superspace coordinates
$Q^{+}_{\alpha}(\zeta,u)$, $\alpha=1,...,2n$ are hypermultiplets and $q^{+}_{i}(\zeta,u)$, $(i=1,2)$ are supergravity compensators. Q^{+}_{α} and q^{+}_{i} are analytic $N=2$ superfields.
" Hamiltonian mechanics" with D^{++} playing the role of time derivative and Q^{+} and q^{+} corresponding to phase space coordinates under Poisson brackets

$$\{f,g\}=\frac{1}{2}\Omega^{\alpha\beta}\frac{\partial f}{\partial Q^{+\alpha}}\frac{\partial g}{\partial Q^{+\beta}}-\frac{1}{2}\epsilon^{ij}\frac{\partial f}{\partial q^{+i}}\frac{\partial g}{\partial q^{+j}},$$

Isometries are generated by Killing potentials $K_{A}(Q^{+},q^{+},u^{-})$ that obey the "conservation law" $\partial^{++}K_{A}+\{K_{A},\mathcal{L}^{(+4)}\}=0$

$$\mathcal{L}^{(+4)}=\frac{P^{(+4)}(Q^{+})}{(q^{+}u^{-})^{2}}$$

$P^{(+4)}(Q^{+})=\frac{1}{12}\,S_{\alpha\beta\gamma\delta}\,Q^{+\alpha}Q^{+\beta}Q^{+\gamma}Q^{+\delta}$
$S_{\alpha\beta\gamma\delta}$ is a completely symmetric invariant tensor of H.

The Killing potentials that generate the isometry group G are given by

$$\mathbf{Sp(2)}: \quad K^{++}_{ij} = 2(q^+_i q^+_j - u^-_i u^-_j \mathcal{L}^{(+4)}),$$

$$\mathbf{H}: \quad K^{++}_a = t_{a\alpha\beta} Q^{+\alpha} Q^{+\beta},$$

$$\mathbf{G/H} \times \mathbf{Sp(2)}: \quad K^{++}_{i\alpha} = 2q^+_i Q^+_\alpha - u^-_i (q^+ u^-) \partial^-_\alpha \mathcal{L}^{(+4)}$$

$N=2$ σ-model in HSS	Minrep of Isometry Grp G
compensator w	coordinate y
conjugate p^{++}	momentum p
$\{\ ,\ \}$	$i[\ ,\ ]$
harmonic superfields $Q^{+\alpha}$	symplectic coords ξ^α
$P^{(+4)}(Q^+)$	$I_4(\xi)$
$K^{a++} = t^a_{\alpha\beta} Q^{+\alpha} Q^{+\beta}$	$J^a = \lambda^a_{\alpha\beta} \xi^\alpha \xi^\beta$
$T^{+++}_\alpha = \{T^+_\alpha, M^{++++}\}$	$F_\alpha = [E_\alpha, F]$
$T^+_\alpha = -\sqrt{2} w Q^+_\alpha$	$E_\alpha = y\, \xi_\alpha$
$M^{++++} = \frac{1}{2}(p^{++})^2 - \frac{P^{(+4)}(Q^+)}{w^2}$	$F = \frac{1}{2}p^2 + \frac{\kappa(I_4(\xi^\alpha))}{y^2}$
$M^0 = \frac{1}{2}w^2$	$E = \frac{1}{2}y^2$
$M^{++} = \frac{1}{2}\left(wp^{++} + p^{++}w\right)$	$\Delta = -\frac{i}{2}\left(yp + py\right)$

Table: The correspondence between the harmonic superspace formulation of $N=2$ sigma models coupled to supergravity and the minimal unitary realizations of their isometry groups G obtained by quantization of quasiconformal action of G.

- The Poisson brackets (PB) $\{,\}$ in HSS formulation go over to i times the commutator $[,]$ in the minimal unitary realization and the harmonic superfields w, p^{++} corresponding to supergravity hypermultiplet compensators , that are canonically conjugate under PB map to the canonically conjugate coordinate and momentum operators y, p. Similarly, the harmonic superfields $Q^{+\alpha}$ that form n conjugate pairs under Poisson brackets map into the symplectic bosonic oscillators ξ^α on the MINREP side. One finds a normal ordering ambiguity in the quantum versions of the quartic invariants. The classical expression relating the quartic invariant polynomial $P^{(+4)}$ to the quadratic Casimir function in HSS formulation differs from the expression relating the quartic quantum invariant I_4 to the quadratic Casimir of H by an additive c-number depending on the ordering chosen.
- On the MINREP side we are working with a realization in terms of quantum mechanical coordinates and momenta, while in HSS side the corresponding quantities are classical harmonic analytic superfields. The above correspondence can be extended to the full quantum correspondence on both sides by reducing the $4D$ $N = 2$ σ model to one dimension and quantizing it to get a supersymmetric quantum mechanics (with 8 supercharges). The bosonic spectrum of the corresponding quantum mechanics must furnish a minimal unitary representation of the isometry group , which extends to a fully supersymmetric spectrum.

- The above mapping implies that the *fundamental spectra* of the "quantum" $N = 2$, quaternionic Kähler σ models coupled to sugra in $d = 4$ must fit into the minimal unitary representations of their isometry groups.
- The $N = 2$, $d = 4$ MESGT's , under dimensional reduction, lead to $d = 3$ supersymmetric σ models with quaternionic Kähler manifolds $\mathcal{M}_3$ (C-map). After T-dualizing the three dimensional theory one can lift it back to four dimensions, thereby obtaining an $N = 2$ sigma model coupled to supergravity that is in the mirror image of the original $N = 2$ MESGT.
- The above results suggest that there must be a correspondence between the fundamental and full quantum spectrum of a $4D$, $N = 2$ MESGT and the minimal unitary representation of its three dimensional U-duality group and those representations obtained by tensoring of its minrep.
- Important problem: decomposition of tensor products of minreps into irreducible representations of the spectrum generating symmetry groups. In particular, how to obtain quaternionic series representations by tensoring minimal unitary representations.

AdS/CFT: Aspen Summer 1984

- MG , Marcus (1984):
  The Kaluza-Klein spectrum of IIB supergravity on $AdS_5 \times S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$ AdS_5 superalgebra $PSU(2, 2\,|\,4)$.
- The CPT self-conjugate doubleton supermultiplet of $PSU(2, 2\,|\,4)$ does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes.
  The field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2, 2\,|\,4)$ lives on the boundary of AdS_5, which can be identified with $4D$ Minkowski space on which $SO(4, 2)$ acts as a conformal group, and the unique candidate for this theory is the *four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant.*
- MG, Warner (1984), MG, PvN, Warner (1984):
  The spectra of $11D$ supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$ were fitted into supermultiplets of the symmetry superalgebras $OSp(8\,|\,4, \mathbb{R})$ and $OSp(8^*|4)$. The entire Kaluza-Klein spectra over these two spaces were obtained by tensoring the singleton and doubleton supermultiplets of $OSp(8\,|\,4, \mathbb{R})$ and $OSp(8^*|4)$, respectively.
- The relevant singleton supermultiplet of $OSp(8\,|\,4, \mathbb{R})$ and doubleton supermultiplet of $OSp(8^*|4)$ do not have a Poincaré limit and decouple from the respective spectra as gauge modes. Again it was proposed that field theories of the singleton and scalar doubleton supermutiplets live on the boundaries of AdS_4 and AdS_7 as superconformally invariant theories.
- Singletons of $Sp(4, \mathbb{R})$ are the remarkable representations of Dirac (1963). Subsequent important work of Fronsdal and collaborators.

Minimal Unitary Representations versus Singletons & Doubletons and Supersymmetry

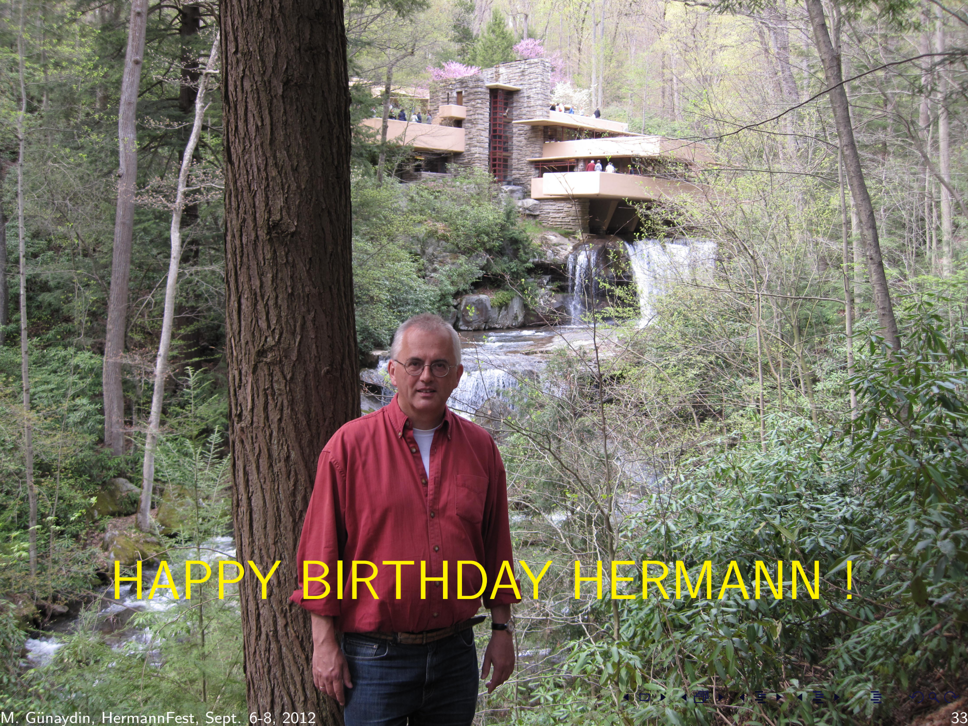
MG, Pavlyk (2006), MG, Fernando (2009/10)

- The minimal unitary representations of symplectic groups $Sp(2n,\mathbb{R})$ are the singletons and their generators can be written as bilinears of bosonic oscillators since their quartic invariants vanish. Tensoring procedure becomes simple for the symplectic groups. Minreps of $OSp(2n|2m,\mathbb{R})$ are supersingletons
- Minimal unitary representation of $SU(2,2) = SO(4,2)$ over L^2 functions in 3 variables $\Leftrightarrow$ conformal scalar = scalar doubleton
- Minrep of $SU(2,2)$ admits a one-parameter, ζ , family of deformations corresponding to massless conformal fields in $d = 4$ with helicity $\zeta/2$.
- Minrep of $PSU(2,2|4)$ is the $4D$ Yang-Mills supermultiplet (CPT-self-conjugate doubleton)
- Minrep of $SU(2,2|N)$ admits a one-parameter family of deformations $\Leftrightarrow$ Higher spin doubleton supermultipllets studied in MG, Minic , Zagermann (1998).
- Minrep of $SO^*(8) \simeq SO(6,2)$ realized over the Hilbert space of functions of five variables and its deformations labeled by the spin t of an $SU(2)$ subgroup correspond to massless conformal fields in six dimensions. Minimal unitary supermultiplet of $OSp(8^*|2N)$ admits deformations labeled by the spin t of an $SU(2)$ subgroup of the little group $SO(4)$ of lightlike vectors in $6D$.
- Minrep of $OSp(8^*|4)$ is the massless supermultiplet of $(2,0)$ conformal field theory that is dual to M-theory on $AdS_7 \times S^4$.

- To summarize: minreps and their deformations describe the CFT side of AdS/CFT dualities. For AdS supergroups the minreps act like "quarks" of positive energy unitary representations $\rightarrow AdS/CFT$
- Furthermore the above results imply that the minrep of any noncompact group G must admit deformations if it occurs as a factor in the even subgroup $G \times K$ of a supergroup. (For K compact such supergroups admit unitary representations in general.) Groups F_4, E_6, E_7 and E_8 do not occur as subgroups of any simple Lie supergroups !
- The mapping between the HSS formulation of $N = 2$ sigma models coupled to sugra in $d = 4$ and the minimal unitary realizations of their isometry group $\Longrightarrow$ Minimal unitary representations of $3d$ U-duality groups must act as the "quarks" of the spectra of these theories whose quantum completion may require extension to M/superstring theory.
  When reduced to one dimension they lead to sigma models with eight super symmetries. Indeed one finds that spectra of $N = 4$ superconformal quantum mechanical models with $D(2, 1; \lambda)$ symmetry studied by Fedoruk, Ivanov and Lechtenfeld and others fit into minimal unitary representations of $D(2, 1; \lambda)$ and its $SU(2)$ deformations by a pair of bosons. Govil & MG (2012)
  Minrep of $D(2, 1; \lambda)$ also admits deformations by arbitrary numbers of pairs of bosons and fermions with a dual symmetry algebra $OSp(2N^*|2M)$.

OPEN PROBLEMS

- Decomposition of tensor products of minreps of quasiconformal U-duality groups into irreps. Of particular interest are $E_{8(8)}$ and $E_{8(-24)}$.
- Physical meaning of the states of minrep of U-duality groups. For *AdS* supergroups the minreps act like "quarks" of positive energy unitary representations $\rightarrow$ *AdS*/*CFT*
- For $3D$ U-duality groups of MESGTs with 8 real supersymmetries the fundamental spectrum corresponds to "super BPS" states, i.e they preserve full susy!
- Explicit construction of the quaternionic discrete series for $F_{4(4)}$, $E_{6(2)}$, $E_{7(-5)}$ and $E_{8(-24)}$ and the corresponding discrete series for $E_{8(8)}$
- Construction of the full spectrum by tensoring of minreps and embedding into their quantum completion (M/Superstring theory).
- Supersymmetry of the black hole spectra & relation to the work of Townsend et.al. on the black holes and conformal quantum mechanics and Calogero models.
- At the non-perturbative level one expects only the discrete subgroups of U-duality groups to be symmetries of M/superstring theory compactified to various dimensions on tori or their orbifolds (Hull and Townsend) . How to extend the above results to the discrete subgroups? Automorphic representations ?


HAPPY BIRTHDAY HERMANN !