Asymptotic freedom, Confinement and triviality in classical $\lambda\varphi^4$

with G. Dvali and C. Gomez

Set up

$$S = \int \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \lambda_0 \phi^4 + 4\pi Q \phi \right) d^4 x$$

Static spherically symmetric solution

$$\Delta\varphi - \lambda\varphi^3 = -4\pi Q \delta(x)$$

Anti-screening

$$\phi_0 = \frac{Q}{r}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi Q\delta(\mathbf{x}) + \lambda_0\frac{Q^3}{r^3}$$

If $\lambda_0 > 0 \Rightarrow$ screening, if $\lambda_0 < 0 \Rightarrow$ *anti* – screening

The perturbative expansion

$$\phi(r) = \frac{Qf(r)}{r}$$

$$f(x) = 1 + \alpha_0 e^x \int_x^\infty \left(\int_0^{x'} f^3(x'')dx'' \right) e^{-x'} dx' - N(\alpha_0),$$

$$\alpha_0 \equiv -\lambda_0 Q^2 > 0 \qquad x = \ln(r/r_0)$$

$$\begin{aligned} f = {} & 1 + \alpha_0 x + \tfrac{3}{2}(\alpha_0 x)^2 + \tfrac{5}{2}(\alpha_0 x)^3 + \tfrac{35}{8}(\alpha_0 x)^4 + \tfrac{63}{8}(\alpha_0 x)^5 + \tfrac{231}{16}(\alpha_0 x)^6 \\ & + \frac{429}{16}(\alpha_0 x)^7 + \frac{6435}{128}(\alpha_0 x)^8 + \frac{12\,155}{128}(\alpha_0 x)^9 + \frac{46\,189}{256}(\alpha_0 x)^{10} + \frac{88\,179}{256}(\alpha_0 x)^{11} + \frac{676\,039}{1024}(\alpha_0 x)^{12} \\ & + \frac{1\,300\,075}{1024}(\alpha_0 x)^{13} + \frac{5\,014\,575}{2048}(\alpha_0 x)^{14} + \frac{9\,694\,845}{2048}(\alpha_0 x)^{15} + \frac{300\,540\,195}{32\,768}(\alpha_0 x)^{16} \\ & + \frac{583\,401\,555}{32\,768}(\alpha_0 x)^{17} + \frac{2\,268\,783\,825}{65\,536}(\alpha_0 x)^{18} + \frac{4\,418\,157\,975}{65\,536}(\alpha_0 x)^{19} + \frac{34\,461\,632\,205}{262\,144}(\alpha_0 x)^{20} \ldots, \end{aligned}$$

$$\frac{1}{\sqrt{1 - 2\alpha_0 x}}$$

$$\alpha_{eff}(x) = -\lambda_0 Q^2 f^2(x) = \alpha_0 f^2(x)$$

$$x = \ln(r/r_0)$$

$$\alpha_{eff}(x) = \alpha_0[1 + 2\tilde{x} + 4 \quad \boxed{\frac{1}{1-2\tilde{x}}} \quad + 32\tilde{x}^6 + \ldots]$$

$$+ \alpha_0^2\Big[6\tilde{x} + 30 \quad \boxed{\frac{3\ln(1-2\tilde{x})}{(1-2\tilde{x})^2}} \quad \tilde{x}^4 + \frac{4176}{5}\tilde{x}^5 + \frac{10\,704}{5}\tilde{x}^6 \ldots\Big]$$

$$+ \alpha_0^3\Big[4 \quad \boxed{\frac{9(\ln(1-2\tilde{x}))^2 - 9\ln(1-2\tilde{x}) + 30\tilde{x}}{(1-2\tilde{x})^3}} \quad \frac{7\,248}{5}\tilde{x}^5 + 59\,248\tilde{x}^6 \ldots\Big] + \ldots$$

$$\tilde{x} = \alpha_0 x$$

Renormalization group and asymptotic freedom

$$\alpha_{eff}(x) = \sum_{n=0}^{\infty} \alpha_0^{n+1} g_n(x)$$

$$x = \ln(r/r_0)$$

$\alpha_0(r_0)$?????

$$g_0(x) = 1,\ g_1(x) = 2x,\ g_2(x) = 4x^2 + 6x,\ g_3(x) = 8x^3 + 30x^2 + 48x,$$

$$g_4(x) = 16x^4 + 104x^3 + 342x^2 + 570x,$$

$$g_5(x) = 32x^5 + 308x^4 + 1572x^3 + 4998x^2 + 8568x,$$

$$g_6(x) = 64x^6 + \frac{4176}{5}x^5 + 5880x^4 + 27\,612x^3 + 86\,832x^2 + 151\,956x,$$

RG properties

$$\alpha_{eff}(r) = \alpha(r_0) + \alpha^2(r_0) g_1\left(\frac{r}{r_0}\right) + \ldots = \sum_{n=0}^{\infty} \alpha^{n+1}(r_0) g_n\left(\frac{r}{r_0}\right)$$

$$\frac{d}{dr_0}\left(\sum_{n=0}^{\infty} \alpha^{n+1}(r_0) g_n\left(\frac{r}{r_0}\right)\right) = 0$$

$$\frac{d\alpha(r_0)}{d\ln r_0} = \alpha^2(r_0) \frac{\sum_{k=0}^{\infty} g'_{k+1}(x)\alpha^k(r_0)}{\sum_{k=0}^{\infty} (k+1) g_k(x)\alpha^k(r_0)}$$

$$\frac{dg_{n+1}(x)}{dx} = \sum_{k=0}^{n} (k+1) g'_{n+1-k}(0) g_k(x)$$

In our case satisfied!!!

Gell-Mann-Low equation

$$\frac{d\alpha_{eff}(x)}{dx} = \alpha_{eff}^2(x) \sum_{k=0}^{\infty} g'_{k+1}(0)\alpha_{eff}^k(x)$$

$$\beta \equiv \frac{d\alpha_{eff}(x)}{dx}$$

$$\beta(\alpha) = \sum_{k=1}^{\infty} \beta_k \alpha^{k+1} = 2\alpha^2 + 6\alpha^3 + 48\alpha^4 + 570\alpha^5$$

$$+ 8568\alpha^6 + 151956\alpha^7 + \ldots,$$

Partial resummations

$$\frac{d\alpha(x)}{dx} = 2\alpha^2(x) + 6\alpha^3 + 48\alpha^4$$

$$\alpha(x) = \frac{\alpha_0}{1-2\alpha_0 x} - 3\left(\frac{\alpha_0}{1-2\alpha_0 x}\right)^2 \ln(1-2\alpha_0 x)$$
$$+ 9\left(\frac{\alpha_0}{1-2\alpha_0 x}\right)^3 \left(\ln^2(1-2\alpha_0 x) - \ln(1-2\alpha_0 x) + \frac{30}{9}\alpha_0 x\right) + O(\alpha_0^4),$$

Dimensional transmutation and asymptotic freedom

$$\alpha(r) = \frac{-\lambda_0 Q^2}{1+2\lambda_0 Q^2 \ln(r/r_0)}$$

We can define the renormalization group invariant physical scale R_c via

$$\ln\frac{R_c}{r_0} = -\frac{1}{2\lambda_0 Q^2}$$

$$\alpha(r) \equiv -\lambda(r)Q^2 = \frac{1}{2\ln(R_c/r)}$$

$$\Downarrow$$

Asymptotic freedom

Beyond perturbation theory and asymptotic behavior

Exact classical β function

$$\phi = \frac{Qf(r)}{r}$$

$$f'' - f' + \alpha_0 f^3 = 0 \qquad x = \ln(r/R_c)$$

$$\alpha(x) = \alpha_0 f^2(x) \qquad \alpha_0 = -\lambda_0 Q^2$$

$$\beta \equiv \alpha'$$

$$\beta = 2\alpha^2 + \frac{1}{2}\left(\frac{d\beta^2}{d\alpha} - \frac{\beta^2}{\alpha}\right)$$

Weak coupling expansion and renormalons

$$\beta(\alpha) = \sum_{k=1}^{\infty} \beta_k \alpha^{k+1}$$

$$\beta_1 = 2, \quad \beta_k = \sum_{m=1}^{k-1} \left(m + \tfrac{1}{2}\right) \beta_{k-m} \beta_m \text{ for } k \geq 2,$$

$$\beta_1 = 2,\ \beta_2 = 6,\ \beta_3 = 48,\ \beta_4 = 570,\ \beta_5 = 8568,\ \beta_6 = 151956, \ldots$$

$$\beta_k \simeq (k+1)!\,\beta_1^k$$

renormalon

"Nature" of renormalon

$$\beta = 2\alpha^2(1+\varepsilon)$$

$$2\alpha^2 \frac{d\varepsilon}{d\alpha} = \frac{\varepsilon}{1+\varepsilon} - 3\alpha(1+\varepsilon) = \varepsilon - 3\alpha + O(\varepsilon^2, \varepsilon\alpha)$$

$$\varepsilon(\alpha) = \left(\frac{3}{\beta_1}\mathrm{Ei}\left(\frac{1}{\beta_1\alpha}\right) + C\right)e^{-\frac{1}{\beta_1\alpha}} + O\left(\left(e^{-\frac{1}{\beta_1\alpha}}\right)^2\right)$$

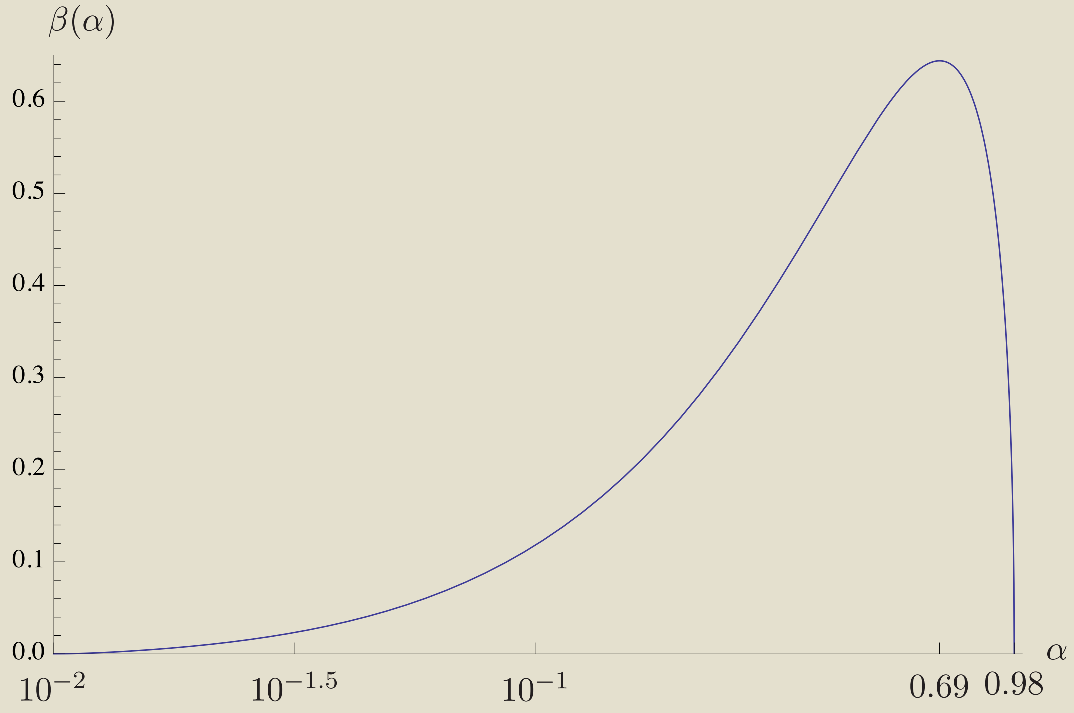
$$\mathrm{Ei}(z) = \frac{e^z}{z}\left(\sum_{k=0}^{n}\frac{k!}{z^k} + O\left(\frac{1}{z^{n+1}}\right)\right)$$

$$e^{-\frac{1}{\beta_1\alpha}} \sim \frac{r}{R_c}$$

The nonperturbative solution and confinement

The infrared coupling constant

$$\frac{d\beta}{d\alpha} = \frac{2\alpha\beta - 4\alpha^3 + \beta^2}{2\alpha\beta}$$



For $\alpha>1$ it is more convenient to work with

$$f'' - f' + \alpha_0 f^3 = 0 \qquad x = \ln(r/R_c)$$

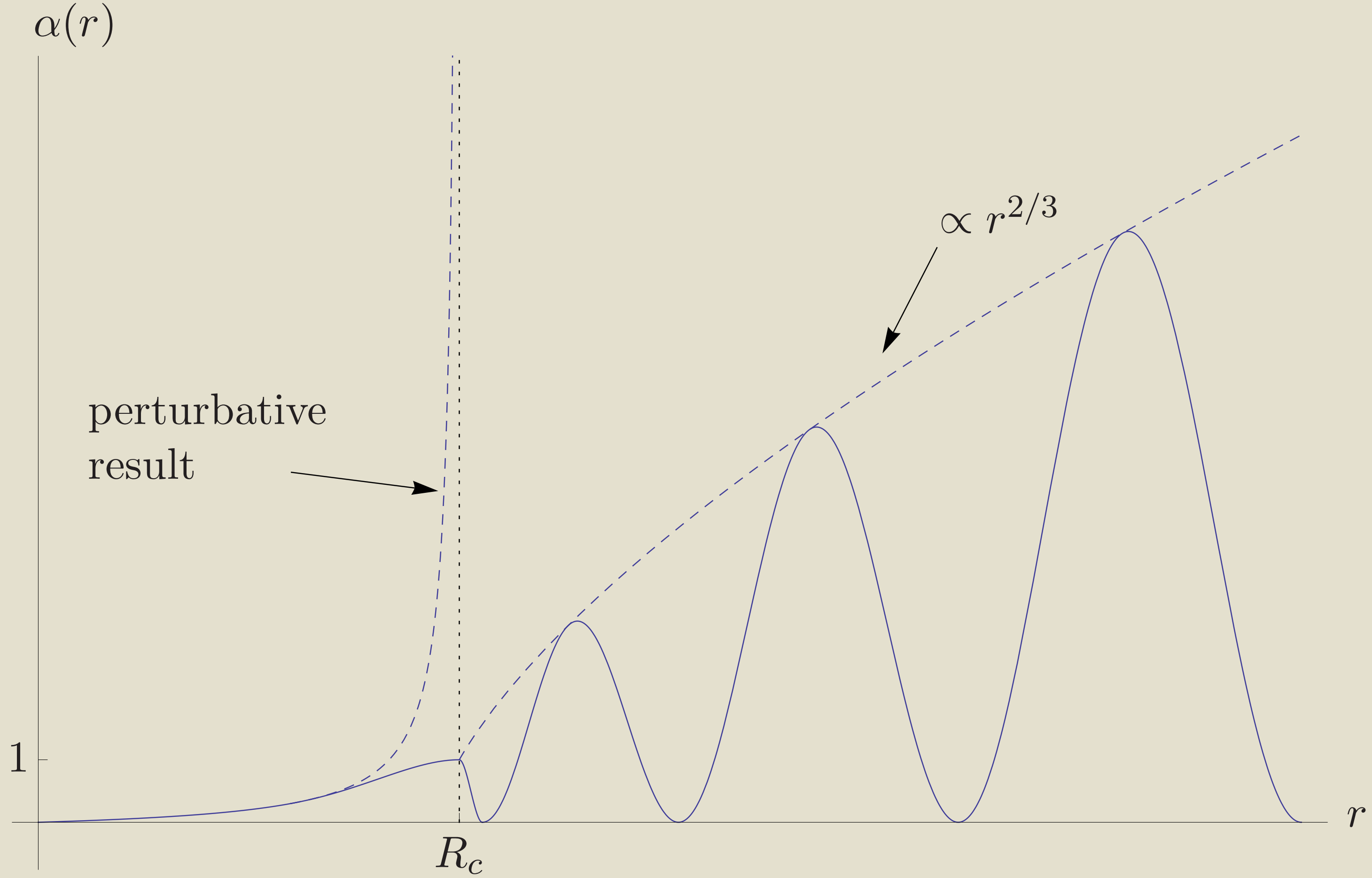
$$(ff')' - f'^2 - \frac{1}{2}(f^2)' + \alpha_0 f^4 = 0$$

$$\langle f'^2 \rangle = \alpha_0 \langle f^4 \rangle \quad \textit{for} \quad \Delta x \sim \sqrt{\frac{1}{\alpha_0 f^2}} \sim \sqrt{\frac{1}{\alpha(x)}} \ll 1$$

$$\langle f^4 \rangle = C \exp\left(\frac{4x}{3}\right) = C\left(\frac{r}{R_c}\right)^{4/3}$$

$$\alpha(r) = \alpha_0 f^2 \simeq \alpha_0 \sqrt{\langle f^4 \rangle} \cos^2\left(\int \sqrt[4]{\alpha_0^2 \langle f^4 \rangle}\, dx\right)$$

$$\simeq O(1)\left(\frac{r}{R_c}\right)^{2/3} \cos^2\left(\frac{r}{R_c}\right)^{2/3}, \textit{ for } \alpha>1$$



$\alpha(r)$
perturbative
result
$\propto r^{2/3}$
1
R_c
r

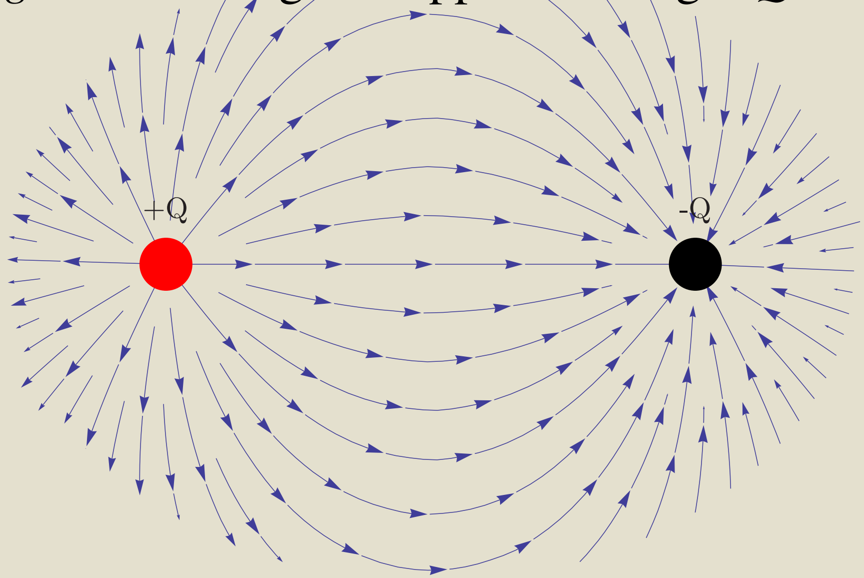
Confinement

$$V(r) \sim \frac{\alpha^2(r)}{r} \propto r^{1/3}$$

$$E = \frac{1}{2}\int\left((\nabla\phi)^2 + \frac{\lambda_0}{2}\phi^4\right)d^3x = \pi Q^2\int_{R_c}^{r}\alpha_0\langle f^4\rangle\frac{dr}{r^2} \sim O(1)Q^2\frac{1}{R_c}\left(\frac{r}{R_c}\right)^{1/3}$$

The energy of the isolated charge diverges as $r^{1/3}$ and therefore it cannot exist as a free *asymptotic state*. This can be interpreted as a hint of confinement of isolated sources.

colorless configuration using two opposite charges Q and $-Q$

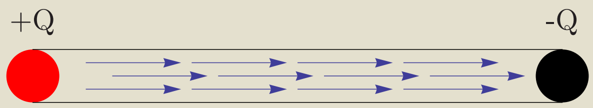


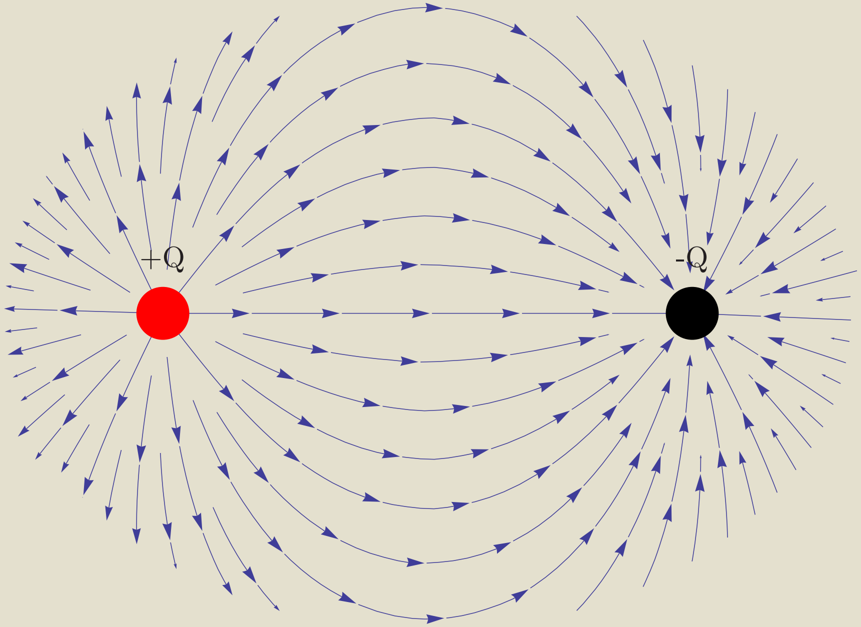
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi Q\delta(\mathbf{x}) + \lambda_0\frac{Q^3 l^3}{r^6}$$

$$E \sim O(1)R_c^{-1}\left(\frac{l}{R_c}\right)^{1/3}$$

confinement in strong sense

$$m \sim O(1)R_c^{-1}$$





String vs. *Dipole*

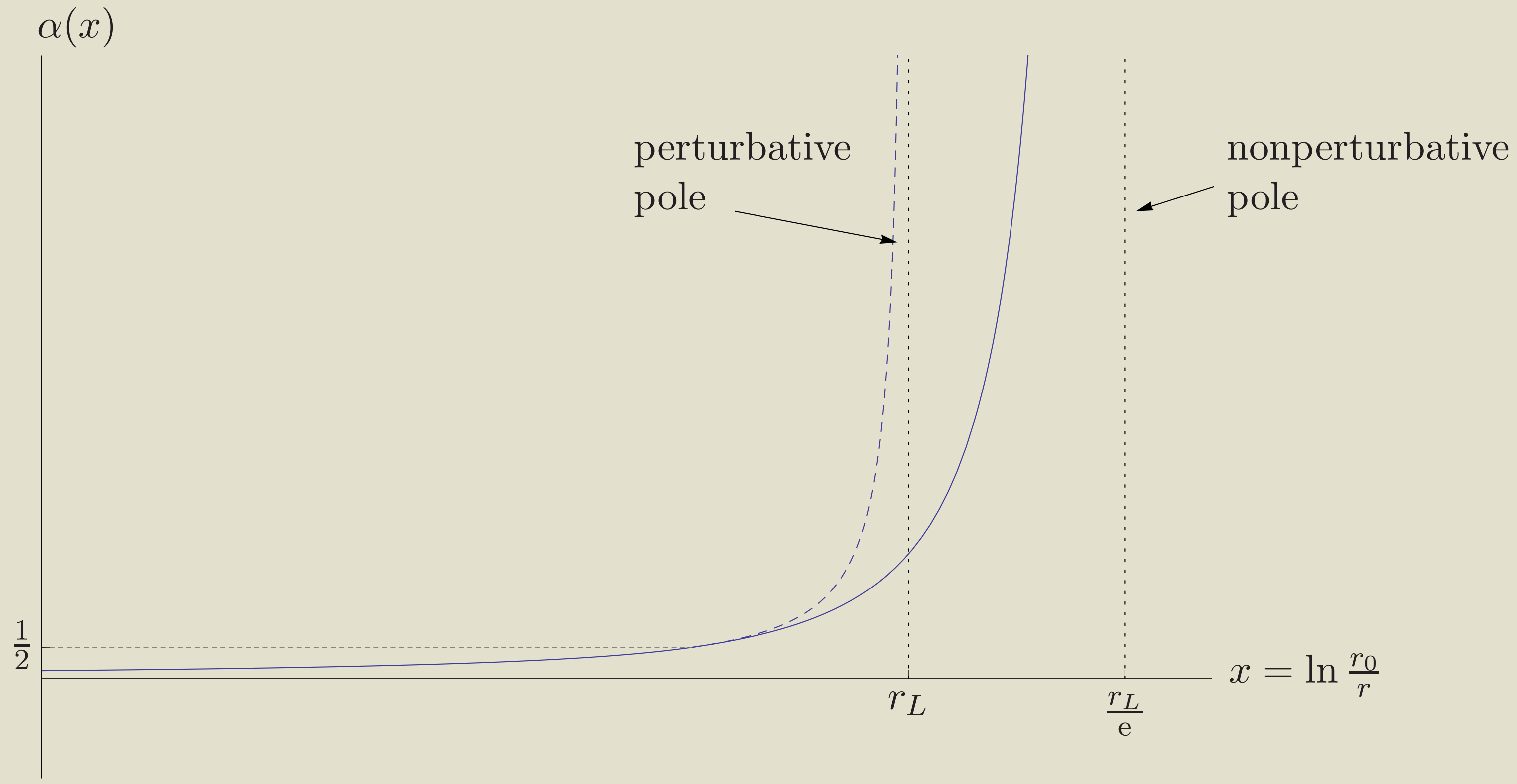
On the triviality of $\lambda\varphi^4$ theory with positive λ

$$\beta = 2\alpha^2 - 6\alpha^3 + 48\alpha^4 - 570\alpha^5 + \dots$$

$$\alpha(r) = \frac{\alpha_0}{1-2\alpha_0 x} = \frac{\lambda_0 Q^2}{1-2\lambda_0 Q^2 \ln(r_0/r)}$$

$$\beta = \sqrt{2}\,\alpha^{3/2}\left(1 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{\alpha}} + O\left(\left(\frac{1}{\sqrt{\alpha}}\right)^2\right)\right)$$

$$\alpha(r) = 8\left(\frac{\alpha_0}{1-2\alpha_0(x-1)}\right)^2 = 8\left(\frac{\lambda_0 Q^2}{1-2\lambda_0 Q^2 \ln(r_0/re)}\right)^2$$



$\alpha(x)$
perturbative pole
nonperturbative pole
$\frac{1}{2}$
$x = \ln \frac{r_0}{r}$
r_L
$\frac{r_L}{\mathrm{e}}$

Conclusions

- HAPPY BIRTHDAY HERMANN!!!