On Cartan Geometries and the Formulation of a Gravitational Gauge Principle.

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Introduction

Cartan geometries $\rightsquigarrow$ gravitation as a gauge theory?

● **Gauge theories of gravity**: Utiyama (1956), Sciama, Kibble, Trautman, Hehl, Ne'eman, Isham, Macdowell & Mansouri, Stelle & West, etc.

● The theory of Cartan connections seems to provide the adequate geometric framework for accomplishing this task (c.f. Wise, Randono).

● **Main bibliography**:

.**Mathematical**: E. Cartan, R.W. Sharpe, S. Kobayashi, P.W. Michor, and S. Sternberg.

.**Physical**: K.S. Stelle & P.C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, Phys. Rev. D, 21, 6, 1980.

General Relativity vs. Yang-Mills Theory

- **General relativity**:

$$[g] = \frac{g}{Diff(M)} \rightsquigarrow$$ Unique metric & torsionless connection (Levi-Civita connection)

- **Yang-Mills theory**:

$$[\omega_G] = \frac{\omega_G}{Aut_V(P_G)}$$

where $P_G \to M$ is a G-principal bundle with G a Lie group and ω_G an Ehresmann conn.

Whereas a gauge field is represented by an Ehresmann connection on the internal spaces of a G-principal bundle over space-time,...

...the gravitational field is represented by a metric on the space-time itself.

Bridging the gap between GR and Y-M

- **"Metric" Einstein-Hilbert formulation**:

  .metric g.

- **"Tetrad-connection" Palatini formulation**:

  .Ehresmann conn. ω for the local Lorentz group (called *spin conn.*),

  .tetrads, *vierbeine*, or moving frames $\theta \sim \sqrt{g}$.

- **Cartan formulation**:

  .$\omega + \theta$ is a connection (!)... for the local Poincaré group?

... but not an Ehresmann conn., but rather a *Cartan connection*.

- The difference between **Y-M** theory $\&$ **GR** is reduced to the difference between Ehresmann and Cartan connections.

Historical landmarks

- In the article

Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie),

(Annales scientifiques de l'E.N.S, 3^e série, tome 40, pp. 325-412, 1923)

... E. Cartan introduced the ***Cartan connections*** and proposed a generalization of **GR** to geom. with non-zero torsion.

- In the article

Les connexions infinitésimales dans un espace fibré différentiable

(Seminaire N. Bourbaki, 1948-1951, exp. $n^\circ 24$, pp. 153-168, 1950),

... C. Ehresmann formalized the notion of conn. by means of the theory of fiber bundles...

... and showed that Cartan conn. are a particular case of a more grl. notion, namely the notion of Ehresmann conn.

Cartan's conception was overshadowed by the notion of Ehresmann connection.

Motivations for “gauging” gravity

- **Pragmatic motivations**:

  .Since we know how to quantize **Y-M** theory, the reduction of the gap between **Y-M** and **GR** might be useful for quantizing gravity (c.f. ***LQG***).

  .It might be helpful for unifying gravity with the other **Y-M** interactions.

- **Conceptual motivations**:

  ♣ **Y-M** theory can be (partially) obtained from an astonishing heuristic argument, namely the ***gauge principle*** (**GP**):

Symmetry $\leftrightsquigarrow$ ***Locality*** $\leftrightsquigarrow$ ***Interactions***

Gauge Principle in Yang-Mills Theory

$$\psi(x) \sim e^{i\alpha}\psi(x) \xrightarrow{\text{Ehresmann connection}} \psi(x) \sim e^{i\alpha(x)}\psi(x)$$

In order to construct a locally invariant theory it is necessary to introduce physical interactions in the form of Ehresmann connections.

- **"Kretschmann" objection**:

A mere epistemic requirement regarding the permissible coordinate transformations seems to imply non-trivial new physics.

- **Solution**:

Local gauge invariance is the epistemic consequence of the ontological commitment of the theory regarding the fund. geom. structure that it presupposes: ***fiber bundles***.

On Fiber Bundles

- Common conception:

Fiber bundles $=$ ***globally twisted generalizations of the Cartesian product of two spaces***.

- However, even locally a G-principal bundle $P_G \xrightarrow{\pi} M$ is not a product space $U_i \times G$, with $U_i \subset M$...

... since $\pi^{-1}(x) \neq G$, but is rather a G***-torsor*** or a ***principal homog. space*** ,...

... i.e. a set on which G acts in a ***free*** and ***transitive*** manner.

- $\pi^{-1}(x)$ is isomorphic to G *in a non-canonical way* since it does not have a privileged origin.

- Each fiber can be identified with G only by fixing a local section $\sigma : U_i \longrightarrow P_G$:

$$\begin{aligned} \psi : U_i \times G &\xrightarrow{\simeq} \pi^{-1}(U_i) \qquad \text{(local trivialization of } P_G) \\ \psi(x, g) &\mapsto \sigma(x) g. \end{aligned}$$

Internal states in different fibers cannot be *intrinsically* compared.

On locality & interactions

- The requirement of local gauge invariance is just the "epistemic" counterpart of the fact...

  ... that internal states are not endowed with an intrinsic "*qualitative suchness*".

- In **Y-M** theory, ***physical interactions*** in the form of Ehresmann conn. are necessary to overcome...

  ... (in a path-dependent or curved way)...

  ... the disconnection introduced by the spatio-temporal ***localization*** of matter fields.

A connection reconnects what space-time disconnects.

Different roles played by symmetry groups

- It is not the same to say...

... that the observables of the theory must be invariant under a symmetry group...

... than saying that the very degrees of freedom of the theory must be introduced in order to guarantee the invariance of the theory under a symmetry group.

- While the **Y-M** gauge fields are introduced in order to guarantee the invariance of the theory under $Aut_V(P_G)$...

... it is not clear to what extent the invariance under $Diff(M)$ plays such a “constructive” role in **RG**.

Towards a Gauge Principle for Gravity

- Is it possible to reformulate GR as a theory that describes a dynamical conn. on a fibration over M?

- And, what is the kind of locality guaranteed by such a gravitational connection?

- ***First evident answer***:

Spacetime is endowed with a ***natural bundle***, namely the ***tangent bundle*** TM...

... and the Levi-Civita conn. is a law for $||$-transporting vectors in TM.

Cartan's criticism

- **Levi-Civita viewpoint**:

Tangent local model of vacuum (LMV) $=$ ***Minkowski vector space***.

- **Cartan viewpoint**: the Levi-Civita notion of connection has two flaws:

♠ Mink. **S-T** is a ***homog. space***, i.e. there is a group

$$\Pi(3,1) = \mathbb{R}^4 \rtimes SO^{\uparrow}(3,1) \qquad \textit{Poincaré group}$$

that acts transitively on M.

.However, the Levi-Civita conn. only takes into account the rotational part $SO^{\uparrow}(3,1)$ of $\Pi(3,1)$,...

... neglecting in this way the symmetry associated to the fact that flat **S-T** does not have a privileged origin.

♣ If the topology of M is not that of Mink. **S-T**, i.e. if Mink. **S-T** is not the ground state of the theory...

... why should we use it Mink. **S-T** as a **LMV**?

Generalized local models of vacuum

- Cartan bypasses these two flaws by using more general **LMV**.

♠ Cartan starts with an ***affine fiber bundle*** (rather than a vector bundle)...

... in which the local structural group is the whole affine group of the **LMV**...

... incorporating the fact that the ground state lacks a privileged origin.

♣ Rather than using Mink. **S-T**, Cartan uses as **LMV** a homog. space adapted to the topology of **S-T**,...

...models that are given by the so-called ***Klein geometries***.

- All in all

Tangent Minkowski vector space $\rightsquigarrow$ ***Tangent affine Klein geometry.***

Klein Geometries

- An ***homog. space*** (M, G) is a connected space M endowed with a ***transitive action*** of a Lie group G.

- Given $x_0 \in M$, the *surjective* map:

$$\begin{aligned} \pi_{x_0} : G &\longrightarrow M \\ g &\longmapsto g \cdot x_0 . \end{aligned}$$

induces a bijection

$$G/H_0 \longrightarrow M,$$

where $H_0 = \pi_{x_0}^{-1}(x_0) \subset G$ is the isotropy group of x_0.

- Whereas H_0 leaves x_0 invariant, G/H_0 generates translations in M.

- The pair (G, H) with G/H a connected homog. space is called a ***Klein geometry***.

- A **KG** (G, H) induces a canonical H-fibration $G \rightarrow G/H$, where G is called the ***principal group*** ("*Haugtgruppe*") of the geometry.

Locals models of gravitational vacuum

- The relevant examples of **KG** (G, H) in the framework of gravitational theories are given by the vacuum solutions of the Einstein's equations with a cosmological constant Λ:

.Minkowski space-time ($\Lambda = 0$):

$$(\mathbb{R}^4 \rtimes SO^{\uparrow}(3,1), SO^{\uparrow}(3,1)),$$

.Anti-de Sitter space-time ($\Lambda < 0$):

$$(SO^{\uparrow}(3,2), SO^{\uparrow}(3,1)),$$

.de Sitter space-time ($\Lambda < 0$):

$$(SO^{\uparrow}(4,1), SO^{\uparrow}(3,1)).$$

- **Note**: in order to obtain a Klein geometry **KG** (G, H) from a homogeneous space, we have to choose an origin in the latter.

Limitations of Klein’s Erlangen Program

● Klein’s marriage between ***geometry*** and ***group theory*** only works for homog. (i.e. symmetric) spaces:

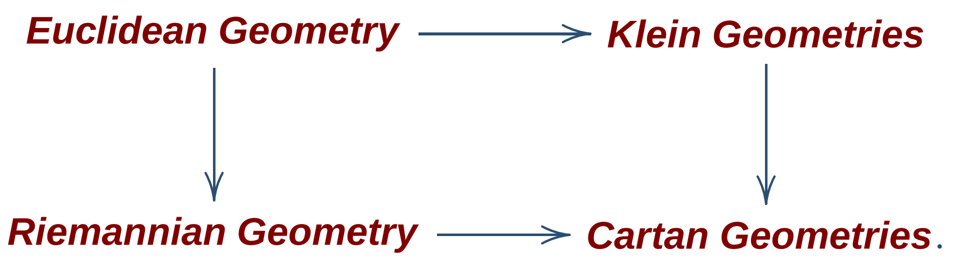
> “*At first look, the notion of group seems alien to the geometry of Riemannian spaces, as they do not possess the homogeneity of any space with a principal group.*”
>
> E. Cartan, *La théorie des groupes et les recherches récentes de géométrie différentielle*.

● This limitation of Klein’s Erlangen program can be bypassed by using the Klein symmetric spaces as local tangent models:

> “*In spite of this, even though a Riemannian space has no absolute homogeneity, it does, however, possess a kind of infinitesimal homogeneity; in the immediate neighborhood it can be assimilated to a Kleinian space.*”
>
> E. Cartan, *Ibid*.

● This generalization of Klein’s program requires to go beyond the stance according to which the infin. models of a curved geom. must be given by Euclidean space.

Riemann + Klein = Cartan



- ***Riemannian Geometry*** is locally modeled on ***Euclidean Geometry*** but globally deformed by curvature.

- ***Klein Geometries*** provide more general symmetric spaces that ***Euclidean Geometry***.

- ***Cartan's twofold generalization***:

Cartan Geometries are locally modeled on Klein Geometries but globally deformed by curvature

- In particular, a Riemannian geometry on M is a torsion-free Cartan geometry on M modeled on Euclidean space.

Summary (I)

• We shall start with a **Y-M** geometry (i.e., with a theory with purely *internal* affine symmetries):

$$\begin{array}{c} P_G \\ \Big\downarrow \omega_G \\ M \end{array}$$

where ω_G is an Ehresmann connection and G is the ***Poincaré, de Sitter or anti-de Sitter affine group*** that acts transitively on the vacuum solution of the theory.

• We shall then "*externalize*" some of the internal symmetries in order to induce geom. structures on M itself.

• We shall consider a **KG** (G, H) with $dim(G/H) = dim(M)$ where

.$G =$ ***Poincaré, de Sitter or anti-de Sitter affine group***.

.$H =$ ***Lorentz group***.

.$G/H =$ ***group of translations*** of the vacuum solution.

Summary (II)

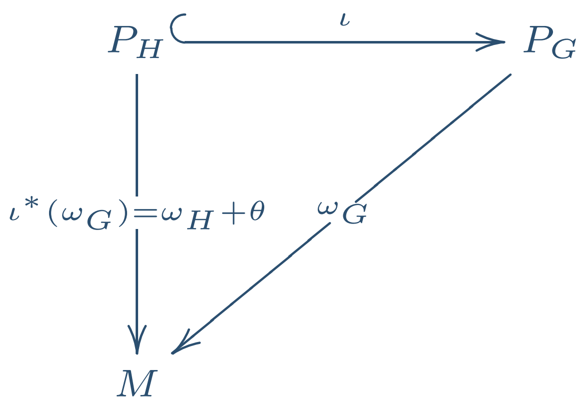
- Since G is now an affine group,...

... the fibers do not have a privileged point of attch. to M as it is the case for tg. *vector* bundles.

- In order to *attach* the fibers to M, i.e. to **solder** the internal geometry to the geometry of M,...

... we have to "break" the Poincaré symmetry down to the Lorentz group H by selecting a point of attch. in each fiber.

- This amounts to reduce the Ehresmann-connected G-bundle P_G to a Cartan-connected H-bundle P_H:



Ehresmann connections

- An ***Ehresmann conn.*** on $P_G \rightarrow M$ is a ***horizontal equivariant distribution*** H defined by means of a G-eq. and $\mathfrak{g}$-valued 1-form ω_G on P_G such that

$$H_p = \text{Ker } (\omega_G)_p \subset T_p P_G .$$

- The conn. form ω_G satisfies:

$.R_h^* \omega_G = Ad_{(h^{-1})} \omega_G$, where Ad is the adj. repr. of G on $\mathfrak{g}$.

$.\omega_G(\xi^\sharp) = \xi$ ("***vertical parallelism***") where

$$\begin{aligned} \sharp : \mathfrak{g} &\rightarrow V_p P_G \\ \xi &\mapsto \xi^\sharp(f(p)) = \frac{d}{d\lambda}(f(p \cdot exp(\lambda\xi)))_{|\lambda=0} . \end{aligned}$$

Important: ω_G has values in the Lie algebra $\mathfrak{g}$ of the structural group G.

Associated bundle in homogeneous spaces

- It might be possible to reduce (non-canonically) P_G to an H-bundle $P_H \longrightarrow M$.

- To do so, we have to consider the associated G-bundle in homog. spaces

$$P_G \times_G G/H \rightarrow M.$$

- This bundle is obtained by attaching to each x a **LMV** $\simeq G/H$.

- It can be shown that

$$P_G \times_G G/H \cong P_G/H.$$

Reduced H-bundle

- The reduction of P_G to P_H can be defined either by a global section

$$\sigma : M \to P_G \times_G G/H \cong P_G/H$$

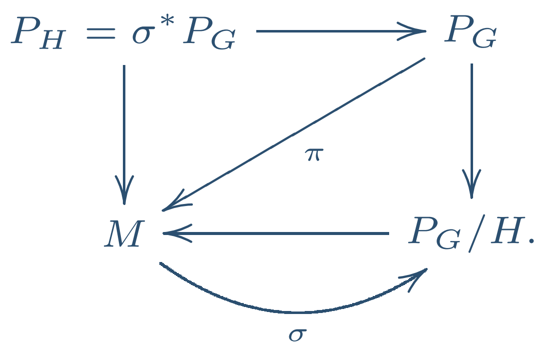
or, equivalently, by an equivariant function

$$\varphi : P_G \to G/H, \qquad \varphi(pg) = g^{-1}\varphi_\sigma(p).$$

- The reduced H-bundle $P_H \to M$ is given either by

$$P_H = \varphi_\sigma^{-1}([e])$$

or by the pullback of P_G along the section σ:



Reduction in a nutshell

- All in all, there is one-to-one correspondence between

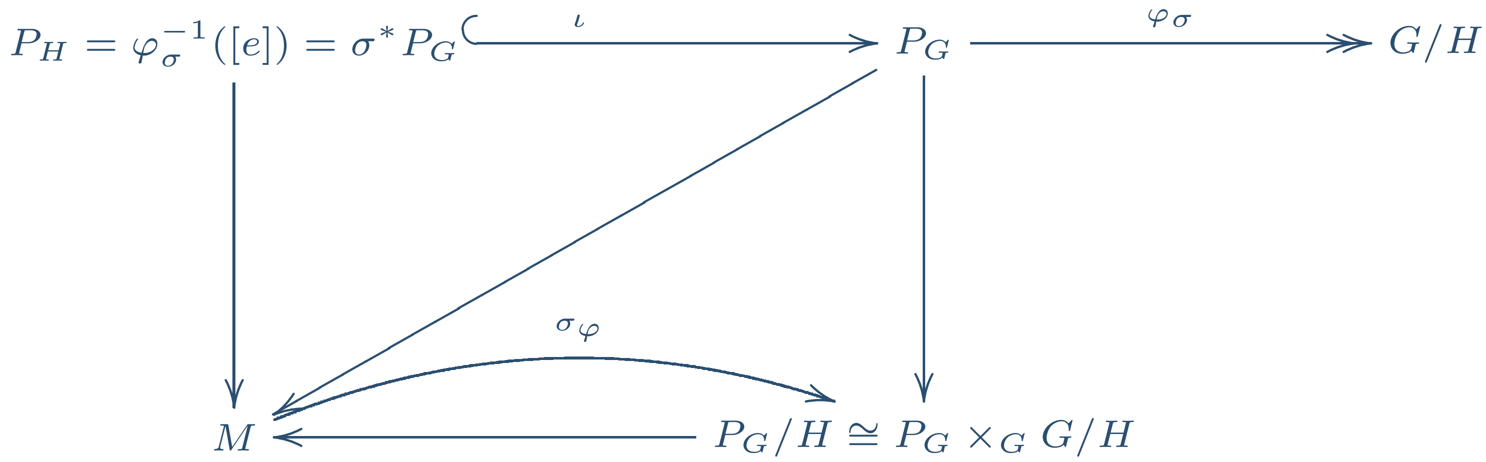
Reduced H-subbundles P_H of P_G

$$\Updownarrow$$

Global sections $\sigma : M \to P_G/H$

or

Equivariant functions $\varphi : P_G \to G/H$



Attaching the LMV to M

- The reduction amounts to select a ***point of attachment*** $\sigma(x)$ in each **LMV** $\simeq G/H$ at each x.

- By doing so, we shall identify

  .each $\sigma(x)$ with x,

  .each T_xM to the vertical tangent space to $\sigma(x)$.

- In this way, the **LMV** attached to x will be tangent to M at $\sigma(x)$.

- By selecting a point of att. for each x, we "break" the translational symm. of the **LMV**.

- $H = SO^{\uparrow}(3,1)$ encodes the "unbroken" rotational symmetry.

Symmetry breaking or partial gauge fixing?

- Since the **LMV** $\simeq G/H$ in which σ is valued are analogous to the manifold that parameterizes the $\neq$ degenerated vacua in a theory with symmetry breaking....

... the field σ is sometimes called a ***Goldstone field***...

... and the reduction process is understood as a symmetry breaking (c.f. Stelle & West).

- However, the reduction $P_H \hookrightarrow P_G$ defined by σ might also be understood as a...

... gauge fixing of the G/H-translational local invariance...

... that is, as a *partial* gauge fixing of the G-invariance.

H-reductions as *partial* trivializations

- In particular, a reduction of P_G to a $\{id_G\}$-principal bundle is given either by a section

$$s : M \to P_G \times_G (G/\{id_G\}) \cong P_G/\{id_G\} = P_G$$

or by a G-equivariant function

$$\varphi : P_G \to G/\{id_G\} = G,$$

where the reduced $\{id_G\}$-bundle is

$$P_{\{id_G\}} = s^* P_G = \varphi^{-1}(id_G).$$

- Hence, a *complete reduction* with $H = \{id_G\}$ is a trivialization $s : M \to P_G$ of P_G.

- Instead of selecting a unique frame for each x as the trivialization s does...

... a H-reduction can be considered a sort of *partial trivialization* of P_G that selects a non-trivial H-set of frames for each x.

Canonical G-extension of an H-bundle

- Whereas the reduction of the G-bundle P_G depends on the existence of a global section

$$\sigma : M \longrightarrow P_G/H \cong P_G \times_G G/H$$

... a H-bundle $P_H \longrightarrow M$ can be *canonically* extended to the associated G-bundle

$$P_H \times_H G \longrightarrow M,$$

where the G-action if given by

$$[(p,g)] \cdot g' = [(p, gg')]$$

... and where the inclusion is given by

$$\begin{array}{rcl} \iota : P_H & \hookrightarrow & P_H \times_H G \\ p & \mapsto & [(p,e)]. \end{array}$$

While the reduction of P_G to P_H is not canonical,

P_H can always be extended to a G-bundle $P_H \times_H G$.

Induced Cartan connection on P_H

- Let's suppose that ω_G satisfies

$$Ker(\omega_G) \cap \iota_* TP_H = 0,$$

where $\iota : P_H \hookrightarrow P_G$...

... or, equivalently, that ω_G has no null vectors when restricted to P_H:

$$Ker(A \doteq \iota^*(\omega_G)) = 0.$$

- The 1-form $A : TP_H \longrightarrow \mathfrak{g}$ defines a ***Cartan connection*** if

  .for each $p \in P_H$, A induces a linear iso. $T_pP_H \cong \mathfrak{g}$ ("***absolute parallelism***")

  .$(R_h^* A)_p = Ad_{(h^{-1})} A_p$ for all $p \in P_H$ and $h \in H$.

  .$A(\xi^\sharp) = \xi$ for any $\xi \in \mathfrak{g}$ where $\sharp : \mathfrak{g} \longrightarrow V_pP_G$.

- **The 1-form A cannot be an Ehresmann conn. on P_H since it is not valued in $\mathfrak{h}$.**

Reductive decomposition of $\mathfrak{g}$

● Let's suppose that the **KG** (G, H) is ***reductive***, i.e. that there exists a $Ad(H)$-module decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad Ad(H) \cdot \mathfrak{m} \subset \mathfrak{m}$$

(in what follows $\mathfrak{m} \doteq \mathfrak{g}/\mathfrak{h}$).

● By composing with the projections, this decomp. of $\mathfrak{g}$ induces a decomp. of A:

$$A = \omega_H + \theta,$$

where the so-called ***spin connection***

$$\omega_H : TP_H \xrightarrow{A} \mathfrak{g} \xrightarrow{\pi_{\mathfrak{h}}} \mathfrak{h}$$

is an Ehresmann conn. on P_H and the so-called ***soldering form***

$$\theta : TP_H \xrightarrow{A} \mathfrak{g} \xrightarrow{\pi_{\mathfrak{g}/\mathfrak{h}}} \mathfrak{g}/\mathfrak{h}$$

is a $\mathfrak{g}/\mathfrak{h}$-valued 1-form on P_H that is

.horizontal: $\theta(\eta) = 0$ for vertical vectors $\eta \in VP_H$)

.H-eq.: $R_h^* \theta = h^{-1}\theta$

Coordinate & geometric soldering form

• While σ attaches the **LMV** to $x \in M$ at $\sigma(x)$, the $\mathfrak{g}/\mathfrak{h}$-part θ of $A = \omega_H + \theta$ identifies each $T_x M$ to the vertical tg. space to the **LMV** at $\sigma(x)$.

• This is a consequence of $Ker(A) = 0$, since

$$A(v_h) = \omega_H(v_h) + \theta(v_h) = \theta(v_h) \neq 0 \in \mathfrak{g}/\mathfrak{h},$$

where v_h is a horizontal vector.

• Given the ***coordinate soldering form***

$$\theta : TP_H \rightarrow \mathfrak{g}/\mathfrak{h},$$

the isomorphism

$$\Omega^q_{hor}(P_H, \mathfrak{g}/\mathfrak{h})^H \simeq \Omega^q(M, P_H \times_H \mathfrak{g}/\mathfrak{h}),$$

induces a ***geometric soldering form***

$$\tilde{\theta} : TM \rightarrow P_H \times_H \mathfrak{g}/\mathfrak{h}$$

• The bundle $P_H \times_H \mathfrak{g}/\mathfrak{h}$ can be identified with the bundle of vertical tg. vectors to $P_G \times_G G/H$ along $\sigma : M \rightarrow P_G \times_G G/H$:

$$P_H \times_H \mathfrak{g}/\mathfrak{h} \simeq V_\sigma(P_G \times_G G/H).$$

Soldering the LMV to M

- The ***geometric soldering form***

$$\tilde{\theta} : TM \rightarrow P_H \times_H \mathfrak{g}/\mathfrak{h}$$

identifies each vector v in $T_x M$ with a geometric vector $\tilde{\theta}(v)$ in $P_H \times_H \mathfrak{g}/\mathfrak{h}$,...

... that is with a vertical vector tangent to $P_G \times_G G/H$ at $\sigma(x)$.

- The tg. space $T_x M$ is thus ***soldered*** to the tg. space to the homog. fiber $\simeq G/H$ of $P_G \times_G G/H$ at $\sigma(x)$.

The LMV are soldered to TM along the section σ.

- The ***coordinate soldering form***

$$\theta_p : T_p P_H \rightarrow \mathfrak{g}/\mathfrak{h}$$

defines the $\mathfrak{g}/\mathfrak{h}$-valued coordinates of $\tilde{\theta}(v)$ in the frame p.

Reducing the frame bundle

- The existence of a soldering form θ on P_H implies that P_H is isomorphic to a $GL(\mathfrak{g}/\mathfrak{h})$-structure,...

... that is to a subbundle of the $GL(\mathfrak{g}/\mathfrak{h})$-principal bundle LM of linear frames on M.

- Indeed, θ defines an application

$$\begin{array}{rcl} f^{\theta}: P_H & \hookrightarrow & LM \\ p & \mapsto & f^{\theta}(p): \mathfrak{g}/\mathfrak{h} \rightarrow T_{\pi(p)}M \end{array}$$

that identifies each element p in P_H with a frame $f^{\theta}(p) \in LM$ over $\pi(p) \in M$.

- Since $f^{\theta}: P_H \rightarrow LM$ is an H-morphism, $f^{\theta}(P_H)$ is a H-subbundle of LM.

- Such a reduction of LM amounts to define a Lorentzian metric on M.

Recovering the metric

- The metric g^{θ} on M can be explicitly defined in terms of the $Ad(H)$-invariant scalar product $\langle \cdot , \cdot \rangle$ of $\mathfrak{g}/\mathfrak{h}$ by means of the expression

$$g^{\theta}(v,w) = \left\langle \tilde{\theta}_p(v), \tilde{\theta}_p(w) \right\rangle ,$$

where $\tilde{\theta}_p(x) : T_x M \rightarrow \mathfrak{g}/\mathfrak{h}$.

- The H-invariance of $\langle \cdot , \cdot \rangle$ implies that $g^{\theta}(v,w)$ does not depend on the frame p over x.

The translational part θ of the Cartan connection A...

... by inducing an isomorphism between P_H and a $SO(3,1)$-subbundle of LM...

... induces a Lorentzian metric g^{θ} on M.

Soldering vs. canonical form

- The soldering form $\theta \in \Omega^1(P_H, \mathfrak{g}/\mathfrak{h})$ is the pullback by f^θ of the ***canonical form***

$$\theta_c : T(LM) \to \mathfrak{g}/\mathfrak{h}$$

on LM given by:

$$\begin{array}{rcl} \theta_c : T_{e(x)}(LM) & \longrightarrow & \mathfrak{g}/\mathfrak{h} \\ \tilde{v} & \longmapsto & e(x)^{-1}(\pi_*(\tilde{v})). \end{array}$$

where

$$e(x) : \mathfrak{g}/\mathfrak{h} \to T_x M$$

is a frame on $T_x M$.

- Contrary to the canonical form θ_c on LM, the soldering form θ on P_H is not canonical...

... since it comes from the restriction of the arbitrary Ehresmann conn. ω_G on P_G to P_H.

- This is consistent with the fact that θ defines a degree of freedom of the theory.

Curvature of the Cartan connection

- The ***Cartan curvature*** $F \in \Omega^2(P_H, \mathfrak{g})$ of a Cartan geom. (P_H, A) is given by

$$F = dA + \frac{1}{2}[A, A] = F_{\mathfrak{h}} + F_{\mathfrak{g}/\mathfrak{h}}.$$

- The ***curvature*** $R \in \Omega^2(P_H, \mathfrak{h})$ of a Cartan geom. (P_H, A) is given by

$$R \doteq d\omega_H + \frac{1}{2}[\omega_H, \omega_H] = F_{\mathfrak{h}} - \frac{1}{2}[\theta, \theta]_{\mathfrak{h}}.$$

- The ***torsion*** $T \in \Omega^2(P_H, \mathfrak{g}/\mathfrak{h})$ of a Cartan geom. (P_H, A) is given by

$$T \doteq d\theta + \frac{1}{2}([\omega_H, \theta] + [\theta, \omega_H]) = F_{\mathfrak{g}/\mathfrak{h}} - \frac{1}{2}[\theta, \theta]_{\mathfrak{g}/\mathfrak{h}}.$$

On Cartan flatness

- In general., ***Cartan flatness*** does not imply $R = 0$ and $T = 0$:

$$F = 0 \Leftrightarrow \begin{cases} R = -\frac{1}{2}[\theta, \theta]_{\mathfrak{h}} \\ T = -\frac{1}{2}[\theta, \theta]_{\mathfrak{g}/\mathfrak{h}} \end{cases}$$

- The standard for Cartan flatness $F = 0$ is given by the "curved" **LMV** $\simeq (G, H)$.

- The so-called ***symmetric models*** satisfy

$$[\mathfrak{g}/\mathfrak{h}, \mathfrak{g}/\mathfrak{h}] \subseteq \mathfrak{h},$$

which implies

$$T = F_{\mathfrak{g}/\mathfrak{h}} \rightsquigarrow F = (R + \frac{1}{2}[\theta, \theta]_{\mathfrak{h}}) + T.$$

- T naturally appears as the "translational" component of F.

(Maurer-)Cartan flat connection

- Let's consider the canonical H-fibration $G \rightarrow G/H$ of the **KG** (G, H).

- The ***Maurer-Cartan form*** A_G of G is given by

$$\begin{aligned} A_G(g) : T_g G &\longrightarrow \mathfrak{g} \\ \xi &\longmapsto (L_{g^{-1}})_* \xi, \end{aligned}$$

where $L_{g^{-1}} : G \rightarrow G$ is the left translation defined by $L_{g^{-1}}(a) = g^{-1}a$ and it satisfies

$$\begin{aligned} R_g^* A_G &= Ad(g) A_G \\ dA_G + \frac{1}{2}[A_G, A_G] &= 0 \end{aligned}$$

Maurer-Cartan form $=$ ***Flat Cartan connection on*** $G \rightarrow G/H$.

- ω_H defines parallel transports in the associated bundle

$$P_H \times_H \mathfrak{g}/\mathfrak{h} \simeq V_\sigma(P_G \times_G G/H),$$

that is parallel transports of vectors tg. to the **LMV** along the section

$$\sigma : M \to P_G \times_G G/H.$$

- Since the geom. soldering form $\tilde{\theta}$ defines an identification

$$TM \xrightarrow{\simeq} P_H \times_H \mathfrak{g}/\mathfrak{h},$$

the Ehresmann conn. ω_H transports vectors tg. to M.

- The ω_H-parallel transports coincide with the levi-Civita parallel transports.

- Now, $A = \omega_H + \theta$ does not only $||$-transport "internal" states (tg. vectors in this case) as in **Y-M** theory (by means of ω_H)...

... but also the spatiotemporal locations themselves (by means of θ).

Development

- Let $\gamma : [0,1] \longrightarrow M$ be a curve on M and $\tilde{\gamma} : [0,1] \longrightarrow P_H$ any lift of γ.
- Since $P_H \subset P_G$, the curve $\tilde{\gamma}$ is in P_G.
- If we use ω_G for $\|$-transporting $\tilde{\gamma}(t)$ to $\pi^{-1}(x_0)$ along γ for all $t \in [0,1]$, we obtain a curve $\hat{\gamma}$ in $\pi^{-1}(x_0)$.
- By using the projection

$$P_G \xrightarrow{\varrho} P_G/H \simeq P_G \times_G G/H,$$

we can define a curve $\gamma^* = \varrho(\hat{\gamma})$ in the fiber of $P_G \times_G G/H$ over x_0 called the ***development of γ over x_0***.

- In this way, any curve $\gamma : [0,1] \longrightarrow M$ can be "*printed*" on the **LMV** over x_0.
- It can be shown that:

  .γ^* only depends on γ and is independent from the choice of $\tilde{\gamma}$.

  .The devel. of a closed curve might fail to close by an amount given by T.

The *torsion* measures the non-commutativity of the *translational* parallel transports.

Infinitesimal developments

• Since the development is obtained by projecting a $\|$-transport defined by ω_G onto G/H,...

... the only relevant part of ω_G is the $\mathfrak{g}/\mathfrak{h}$-valued part, namely θ.

• Given an infin. displacement $v \in T_x M$, the ***geometric soldering form***

$$\tilde{\theta} : TM \to P_H \times_H \mathfrak{g}/\mathfrak{h} \simeq V_\sigma(P_G \times_G G/H)$$

defines an infin. displacement in the LMV on x_0 at the point of attachment $\sigma(x)$.

• This means that the point of attachment at $x + v$ will be developed in the fiber above x into the point $\sigma(x) + \tilde{\theta}(v)$.

• In other terms, the translational part θ of A defines the γ-dependent image of any $x \in M$ in the **LMV** at x_0.

Translational locality: this identification is dynamically defined by the translational component of the Cartan gauge field A.

Conclusion (I)

- The theory of Cartan geometries allows us to put together ω_H and θ into a unique Cartan connection

$$A \begin{cases} \omega_H & \text{Gauges the local Lorentz symmetry} \\ \theta & \begin{cases} \text{Gauges the local translational symmetry} \\ \text{Induces a metric } g^\theta \text{ on } M \end{cases} \end{cases}$$

... that gauges the ***local affine gauge invariance*** defined by the affine group G that acts transitively on the vacuum solution of the theory.

- This can be done by reducing a **Y-M** geometry

$$(P_G \rightarrow M, \omega_G)$$

by means of a partial gauge fixing

$$\sigma : M \rightarrow P_G \times_G G/H$$

that breaks the translational invariance of the **LMV**$\simeq G/H$.

Conclusions (II)

- In GR (where $T = 0$), the consideration of a local *affine* symmetry instead of the smaller local Lorentz symmetry has no effects.

- If we relax the condition $T = 0$, then ω_H and θ are indep. geom. structures and the gravitational field must be described by the whole $A = \omega_H + \theta$.

- Since the **LMV** is not necessarily Minkowski **S-T**, the affine group G is not necessarily the Poincaré group.

Further Research...

- Clarify the relationship between the ***local translational invariance*** gauged by θ and the ***invariance under diffeomorphisms*** of M...

... being these symmetries related by the soldering

$$\tilde{\theta} : M \to V_\sigma(P_G \times_G G/H)$$

which identifies the external infinitesimal translations in M with the internal translations in the internal **LMV**.

- Clarify the nature of the reduction process: ***dynamical symmetry breaking*** or ***partial gauge fixing***?

- Analyze the different actions S that can be constructed from the Cartan connection A.

The End

Thanks for your attention !!!