
Gabriel Catren

SPHERE (UMR 7219) - Université Paris Diderot/CNRS
Introduction

Cartan geometries $\leftrightarrow$ gravitation as a gauge theory?

- **Gauge theories of gravity**: Utiyama (1956), Sciama, Kibble, Trautman, Hehl, Ne’eman, Isham, Macdowell & Mansouri, Stelle & West, etc.

- The theory of Cartan connections seems to provide the adequate geometric framework for accomplishing this task (c.f. Wise, Randono).

- **Main bibliography**:
**General Relativity vs. Yang-Mills Theory**

- **General relativity:**

  \[
  [g] = \frac{g}{Diff(M)} \leadsto \text{Unique metric & torsionless connection (Levi-Civita connection)}
  \]

- **Yang-Mills theory:**

  \[
  [\omega_G] = \frac{\omega_G}{Aut_V(P_G)}
  \]

  where \(P_G \to M\) is a \(G\)-principal bundle with \(G\) a Lie group and \(\omega_G\) an Ehresmann conn.

  *Whereas a gauge field is represented by an Ehresmann connection on the internal spaces of a \(G\)-principal bundle over space-time,*...

  
  ...*the gravitational field is represented by a metric on the space-time itself.*

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Bridging the gap between GR and Y-M

- **“Metric” Einstein-Hilbert formulation:**
  
  \[ \text{metric } g. \]

- **“Tetrad-connection” Palatini formulation:**
  
  \[ \text{Ehresmann conn. } \omega \text{ for the local Lorentz group (called } \text{spin conn.}), \]
  
  \[ \text{tetrads, vierbeine, or moving frames } \theta \sim \sqrt{g}. \]

- **Cartan formulation:**
  
  \[ \omega + \theta \text{ is a connection (!)... for the local Poincaré group?} \]
  
  ... but not an Ehresmann conn., but rather a \textit{Cartan connection}.

- The difference between \textbf{Y-M theory} & \textbf{GR} is reduced to the difference between Ehresmann and Cartan connections.
Historical landmarks

- In the article

**Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)**,

(Annales scientifiques de l'E.N.S, 3e série, tome 40, pp. 325-412, 1923)

... E. Cartan introduced the **Cartan connections** and proposed a generalization of **GR** to geom. with non-zero torsion.

- In the article

**Les connexions infinitésimales dans un espace fibré différentiable**

(Seminaire N. Bourbaki, 1948-1951, exp. 24, pp. 153-168, 1950),

... C. Ehresmann formalized the notion of conn. by means of the theory of fiber bundles...

... and showed that Cartan conn. are a particular case of a more grl. notion, namely the notion of Ehresmann conn.

*Cartan's conception was overshadowed by the notion of Ehresmann connection.*
Motivations for “gauging” gravity

- Pragmatic motivations:

Since we know how to quantize Y-M theory, the reduction of the gap between Y-M and GR might be useful for quantizing gravity (c.f. LQG).

It might be helpful for unifying gravity with the other Y-M interactions.

- Conceptual motivations:

★ Y-M theory can be (partially) obtained from an astonishing heuristic argument, namely the gauge principle (GP):

\[ \text{Symmetry} \leftrightarrow \text{Locality} \leftrightarrow \text{Interactions} \]
Gauge Principle in Yang-Mills Theory

\[ \psi(x) \sim e^{i\alpha} \psi(x) \quad \text{Ehresmann connection} \quad \psi(x) \sim e^{i\alpha(x)} \psi(x) \]

In order to construct a locally invariant theory it is necessary to introduce physical interactions in the form of Ehresmann connections.

- **“Kretschmann” objection:**

A mere epistemic requirement regarding the permissible coordinate transformations seems to imply non-trivial new physics.

- **Solution:**

Local gauge invariance is the epistemic consequence of the ontological commitment of the theory regarding the fund. geom. structure that it presupposes: fiber bundles.
On Fiber Bundles

• Common conception:

*Fiber bundles = globally twisted generalizations of the Cartesian product of two spaces.*

• However, even locally a $G$-principal bundle $P_G \xrightarrow{\pi} M$ is not a product space $U_i \times G$, with $U_i \subset M$...

... since $\pi^{-1}(x) \neq G$, but is rather a $G$-*torsor* or a *principal homog. space*,...

... i.e. a set on which $G$ acts in a *free* and *transitive* manner.

• $\pi^{-1}(x)$ is isomorphic to $G$ *in a non-canonical way* since it does not have a privileged origin.

• Each fiber can be identified with $G$ only by fixing a local section $\sigma : U_i \rightarrow P_G$:

$$
\psi : U_i \times G \xrightarrow{\sim} \pi^{-1}(U_i) \quad \text{(local trivialization of $P_G$)}
$$

$$
\psi(x, g) \mapsto \sigma(x)g.
$$

*Internal states in different fibers cannot be *intrinsically* compared.*
On locality & interactions

- The requirement of local gauge invariance is just the “epistemic” counterpart of the fact...

  ... that internal states are not endowed with an intrinsic “qualitative suchness”.

- In Y-M theory, physical interactions in the form of Ehresmann conn. are necessary to overcome...

  ... (in a path-dependent or curved way)...

  ... the disconnection introduced by the spatio-temporal localization of matter fields.

A connection reconnects what space-time disconnects.
Different roles played by symmetry groups

- It is not the same to say...

  ... that the observables of the theory must be invariant under a symmetry group...

  ... than saying that the very degrees of freedom of the theory must be introduced in order to guarantee the invariance of the theory under a symmetry group.

- While the Y-M gauge fields are introduced in order to guarantee the invariance of the theory under $\text{Aut}_V(P_G)$...

  ... it is not clear to what extent the invariance under $\text{Diff}(M)$ plays such a “constructive” role in RG.
Towards a Gauge Principle for Gravity

- Is it possible to reformulate GR as a theory that describes a dynamical conn. on a fibration over $M$?

- And, what is the kind of locality guaranteed by such a gravitational connection?

- First evident answer:

Spacetime is endowed with a natural bundle, namely the tangent bundle $TM$...

... and the Levi-Civita conn. is a law for $\parallel$-transporting vectors in $TM$. 
Cartan’s criticism

- **Levi-Civita viewpoint:**

  \textit{Tangent local model of vacuum (LMV) = Minkowski vector space.}

- **Cartan viewpoint:** the Levi-Civita notion of connection has two flaws:

  ♠ Mink. S-T is a \textit{homog. space}, i.e. there is a group

  \[ \Pi(3, 1) = \mathbb{R}^4 \rtimes SO^+(3, 1) \quad \text{Poincaré group} \]

  that acts transitively on \( M \).

  However, the Levi-Civita conn. only takes into account the rotational part

  \( SO^+(3, 1) \) of \( \Pi(3, 1) \),...

  ... neglecting in this way the symmetry associated to the fact that flat S-T does

  not have a privileged origin.

  ♣ If the topology of \( M \) is not that of Mink. S-T, i.e. if Mink. S-T is not the ground

  state of the theory...

  ... why should we use it Mink. S-T as a LMV?
Generalized local models of vacuum

- Cartan bypasses these two flaws by using more general LMV.

  ♠ Cartan starts with an affine fiber bundle (rather than a vector bundle)...

  ... in which the local structural group is the whole affine group of the LMV...

  ... incorporating the fact that the ground state lacks a privileged origin.

  ♣ Rather than using Mink. S-T, Cartan uses as LMV a homog. space adapted to the topology of S-T,...

  ...models that are given by the so-called Klein geometries.

- All in all

  Tangent Minkowski vector space $\mapsto$ Tangent affine Klein geometry.
Klein Geometries

- An **homog. space** \((M, G)\) is a connected space \(M\) endowed with a **transitive action** of a Lie group \(G\).

- Given \(x_0 \in M\), the **surjective** map:
  \[
  \pi_{x_0} : G \rightarrow M \\
  g \mapsto g \cdot x_0.
  \]
  induces a bijection
  \[
  G/H_0 \rightarrow M,
  \]
  where \(H_0 = \pi_{x_0}^{-1}(x_0) \subset G\) is the isotropy group of \(x_0\).

- Whereas \(H_0\) leaves \(x_0\) invariant, \(G/H_0\) generates translations in \(M\).

- The pair \((G, H)\) with \(G/H\) a connected homog. space is called a **Klein geometry**.

- A **KG** \((G, H)\) induces a canonical \(H\)-fibration \(G \rightarrow G/H\), where \(G\) is called the **principal group** (“Hauptgruppe”) of the geometry.
The relevant examples of $\text{KG} \ (G, H)$ in the framework of gravitational theories are given by the vacuum solutions of the Einstein's equations with a cosmological constant $\Lambda$:

**Minkowski space-time ($\Lambda = 0$):**

$$ (\mathbb{R}^4 \rtimes SO^\dagger(3, 1), SO^\dagger(3, 1)),$$

**Anti-de Sitter space-time ($\Lambda < 0$):**

$$ (SO^\dagger(3, 2), SO^\dagger(3, 1)),$$

**de Sitter space-time ($\Lambda < 0$):**

$$ (SO^\dagger(4, 1), SO^\dagger(3, 1)).$$

**Note:** in order to obtain a Klein geometry $\text{KG} \ (G, H)$ from a homogeneous space, we have to choose an origin in the latter.
Limitations of Klein’s Erlangen Program

- Klein’s marriage between **geometry** and **group theory** only works for homog. (i.e. symmetric) spaces:

  “At first look, the notion of group seems alien to the geometry of Riemannian spaces, as they do not possess the homogeneity of any space with a principal group.”

  E. Cartan, *La théorie des groupes et les recherches récentes de géométrie différentielle*.

- This limitation of Klein’s Erlangen program can be bypassed by using the Klein symmetric spaces as local tangent models:

  “In spite of this, even though a Riemannian space has no absolute homogeneity, it does, however, possess a kind of infinitesimal homogeneity; in the immediate neighborhood it can be assimilated to a Kleinian space.”

Riemann + Klein = Cartan

- **Riemannian Geometry** is locally modeled on **Euclidean Geometry** but globally deformed by curvature.

- **Klein Geometries** provide more general symmetric spaces than **Euclidean Geometry**.

- **Cartan's twofold generalization**:

  **Cartan Geometries are locally modeled on Klein Geometries but globally deformed by curvature**

- In particular, a Riemannian geometry on $M$ is a torsion-free Cartan geometry on $M$ modeled on Euclidean space.
Summary (I)

- We shall start with a Y-M geometry (i.e., with a theory with purely internal affine symmetries):

\[
\begin{align*}
P_G & \downarrow \omega_G \\
M &
\end{align*}
\]

where \(\omega_G\) is an Ehresmann connection and \(G\) is the Poincaré, de Sitter or anti-de Sitter affine group that acts transitively on the vacuum solution of the theory.

- We shall then “externalize” some of the internal symmetries in order to induce geom. structures on \(M\) itself.

- We shall consider a KG \((G, H)\) with \(\text{dim}(G/H) = \text{dim}(M)\) where

\[
\begin{align*}
.G & = \text{Poincaré, de Sitter or anti-de Sitter affine group}. \\
.H & = \text{Lorentz group}. \\
.G/H & = \text{group of translations} \text{ of the vacuum solution}.
\end{align*}
\]
Summary (II)

- Since $G$ is now an affine group,...

... the fibers do not have a privileged point of attach. to $M$ as it is the case for tg. *vector* bundles.

- In order to *attach* the fibers to $M$, i.e. to *solder* the internal geometry to the geometry of $M$,...

... we have to “break” the Poincaré symmetry down to the Lorentz group $H$ by selecting a point of attach. in each fiber.

- This amounts to reduce the Ehresmann-connected $G$-bundle $P_G$ to a Cartan-connected $H$-bundle $P_H$:

$$P_H \xrightarrow{\iota} P_G$$

$$\iota^*(\omega_G) = \omega_H + \theta$$

$$\omega_G$$

$$M$$
Ehresmann connections

- An **Ehresmann conn.** on $P_G \to M$ is a **horizontal equivariant distribution** $H$ defined by means of a $G$-eq. and $g$-valued 1-form $\omega_G$ on $P_G$ such that

\[ H_p = \text{Ker} (\omega_G)_p \subset T_p P_G. \]

- The conn. form $\omega_G$ satisfies:

\[ R^*_h \omega_G = \text{Ad} (h^{-1}) \omega_G, \text{ where Ad is the adj. repr. of } G \text{ on } g. \]

\[ \omega_G (\xi^\#) = \xi \left( \text{"vertical parallelism"} \right) \text{ where } \]

\[ \sharp : g \to V_p P_G \]

\[ \xi \mapsto \xi^\# (f(p)) = \frac{d}{d\lambda} (f(p \cdot \exp (\lambda \xi))) |_{\lambda=0}. \]

**Important:** $\omega_G$ has values in the Lie algebra $g$ of the structural group $G$. 

\[ \frac{d}{d\lambda} (f(p \cdot \exp (\lambda \xi))) |_{\lambda=0}. \]
Associated bundle in homogeneous spaces

- It might be possible to reduce (non-canonically) $P_G$ to an $H$-bundle $P_H \to M$.

- To do so, we have to consider the associated $G$-bundle in homog. spaces

\[ P_G \times_G G/H \to M. \]

- This bundle is obtained by attaching to each $x$ a LMV $\simeq G/H$.

- It can be shown that

\[ P_G \times_G G/H \simeq P_G/H. \]
Reduced $H$-bundle

- The reduction of $P_G$ to $P_H$ can be defined either by a global section
  \[ \sigma : M \rightarrow P_G \times_G G/H \cong P_G/H \]
  or, equivalently, by an equivariant function
  \[ \varphi : P_G \rightarrow G/H, \quad \varphi(pg) = g^{-1}\varphi_\sigma(p). \]

- The reduced $H$-bundle $P_H \rightarrow M$ is given either by
  \[ P_H = \varphi_\sigma^{-1}([e]) \]
  or by the pullback of $P_G$ along the section $\sigma$:

\[
\begin{array}{c}
\end{array}
\]
Reduction in a nutshell

All in all, there is one-to-one correspondence between

Reduced $H$-subbundles $P_H$ of $P_G$

\[ \cong \]

Global sections $\sigma : M \rightarrow P_G/H$

or

Equivariant functions $\varphi : P_G \rightarrow G/H$

\[ P_H = \varphi_\sigma^{-1}([e]) = \sigma^* P_G \cong \varphi_\sigma \rightarrow G/H \]

\[ P_G/H \cong P_G \times_G G/H \]
The reduction amounts to select a \textit{point of attachment} \( \sigma(x) \) in each \( \text{LMV} \simeq G/H \) at each \( x \).

By doing so, we shall identify

- each \( \sigma(x) \) with \( x \),
- each \( T_x M \) to the vertical tangent space to \( \sigma(x) \).

In this way, the \text{LMV} attached to \( x \) will be tangent to \( M \) at \( \sigma(x) \).

By selecting a point of att. for each \( x \), we “break” the translational symm. of the \text{LMV}.

\( H = SO^\uparrow(3, 1) \) encodes the “unbroken” rotational symmetry.
Symmetry breaking or partial gauge fixing?

- Since the $\text{LMV} \sim G/H$ in which $\sigma$ is valued are analogous to the manifold that parameterizes the $\neq$ degenerated vacua in a theory with symmetry breaking....

... the field $\sigma$ is sometimes called a **Goldstone field**...

... and the reduction process is understood as a symmetry breaking (c.f. Stelle & West).

- However, the reduction $P_H \hookrightarrow P_G$ defined by $\sigma$ might also be understood as a...

  ... gauge fixing of the $G/H$-translational local invariance...

... that is, as a **partial** gauge fixing of the $G$-invariance.
In particular, a reduction of $P_G$ to a $\{id_G\}$-principal bundle is given either by a section

$$s : M \to P_G \times_G (G/\{id_G\}) \cong P_G/\{id_G\} = P_G$$

or by a $G$-equivariant function

$$\varphi : P_G \to G/\{id_G\} = G,$$

where the reduced $\{id_G\}$-bundle is

$$P_{\{id_G\}} = s^* P_G = \varphi^{-1}(id_G).$$

Hence, a complete reduction with $H = \{id_G\}$ is a trivialization $s : M \to P_G$ of $P_G$.

Instead of selecting a unique frame for each $x$ as the trivialization $s$ does...

... a $H$-reduction can be considered a sort of partial trivialization of $P_G$ that selects a non-trivial $H$-set of frames for each $x$. 
Canonical $G$-extension of an $H$-bundle

Whereas the reduction of the $G$-bundle $P_G$ depends on the existence of a global section

$$\sigma : M \to P_G / H \cong P_G \times_G G / H$$

... a $H$-bundle $P_H \to M$ can be canonically extended to the associated $G$-bundle

$$P_H \times_H G \to M,$$

where the $G$-action is given by

$$[(p, g)] \cdot g' = [(p, gg')]$$

... and where the inclusion is given by

$$\iota : P_H \leftrightarrow P_H \times_H G$$

$$p \leftrightarrow [(p, e)].$$

While the reduction of $P_G$ to $P_H$ is not canonical,

$P_H$ can always be extended to a $G$-bundle $P_H \times_H G$. 
Induced Cartan connection on $P_H$

- Let's suppose that $\omega_G$ satisfies
  \[ \text{Ker}(\omega_G) \cap \iota_* TP_H = 0, \]
  where $\iota : P_H \hookrightarrow P_G$...
  ... or, equivalently, that $\omega_G$ has no null vectors when restricted to $P_H$:
  \[ \text{Ker}(A \equiv \iota^*(\omega_G)) = 0. \]

- The 1-form $A : TP_H \to g$ defines a **Cartan connection** if
  . for each $p \in P_H$, $A$ induces a linear iso. $T_p P_H \cong g$ ("absolute parallelism")
  \[ (R^*_h A)_p = \text{Ad}_{(h^{-1})} A_p \text{ for all } p \in P_H \text{ and } h \in H. \]
  . $A(\xi^\sharp) = \xi$ for any $\xi \in g$ where $\# : g \to V_p P_G$.

- The 1-form $A$ cannot be an Ehresmann conn. on $P_H$ since it is not valued in $\mathfrak{h}$. 

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Reductive decomposition of \( g \)

- Let’s suppose that the KG \((G, H)\) is reductive, i.e. that there exists a \(Ad(H)\)-module decomposition

\[ g = h \oplus m, \quad Ad(H) \cdot m \subset m \]

(in what follows \( m \cong g/h \)).

- By composing with the projections, this decomp. of \( g \) induces a decomp. of \( A \):

\[ A = \omega_H + \theta, \]

where the so-called spin connection

\[ \omega_H : TP_H \xrightarrow{A} g \xrightarrow{\pi_h} h \]

is an Ehresmann conn. on \( P_H \) and the so-called soldering form

\[ \theta : TP_H \xrightarrow{A} g \xrightarrow{\pi_{g/h}} g/h \]

is a \( g/h \)-valued 1-form on \( P_H \) that is

- horizontal: \( \theta(\eta) = 0 \) for vertical vectors \( \eta \in VP_H \)

- \( H \)-eq.: \( R^*_h \theta = h^{-1} \theta \)
Coordinate & geometric soldering form

- While $\sigma$ attaches the LMV to $x \in M$ at $\sigma(x)$, the $g/h$-part $\theta$ of $A = \omega_H + \theta$ identifies each $T_x M$ to the vertical tangent space to the LMV at $\sigma(x)$.

- This is a consequence of $Ker(A) = 0$, since

$$A(v_h) = \omega_H(v_h) + \theta(v_h) = \theta(v_h) \neq 0 \in g/h,$$

where $v_h$ is a horizontal vector.

- Given the coordinate soldering form

$$\theta : TP_H \rightarrow g/h,$$

the isomorphism

$$\Omega^q_{hor}(P_H, g/h)^H \simeq \Omega^q(M, P_H \times_H g/h),$$

induces a geometric soldering form

$$\tilde{\theta} : TM \rightarrow P_H \times_H g/h$$

- The bundle $P_H \times_H g/h$ can be identified with the bundle of vertical tangent vectors to $P_G \times_G G/H$ along $\sigma : M \rightarrow P_G \times_G G/H$:

$$P_H \times_H g/h \simeq V_\sigma(P_G \times_G G/H).$$
Soldering the LMV to $M$

- The **geometric soldering form**

$$\tilde{\theta} : TM \rightarrow PH \times_H \mathfrak{g}/\mathfrak{h}$$

identifies each vector $v$ in $T_x M$ with a geometric vector $\tilde{\theta}(v)$ in $PH \times_H \mathfrak{g}/\mathfrak{h}$, ...

... that is with a vertical vector tangent to $P_G \times_G G/H$ at $\sigma(x)$.

- The tg. space $T_x M$ is thus **soldered** to the tg. space to the homog. fiber $\simeq G/H$ of $P_G \times_G G/H$ at $\sigma(x)$.

**The LMV are soldered to $TM$ along the section $\sigma$.**

- The **coordinate soldering form**

$$\theta_p : T_p PH \rightarrow \mathfrak{g}/\mathfrak{h}$$

defines the $\mathfrak{g}/\mathfrak{h}$-valued coordinates of $\tilde{\theta}(v)$ in the frame $p$. 

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Reducing the frame bundle

- The existence of a soldering form $\theta$ on $P_H$ implies that $P_H$ is isomorphic to a $GL(\mathfrak{g}/\mathfrak{h})$-structure,...

... that is to a subbundle of the $GL(\mathfrak{g}/\mathfrak{h})$-principal bundle $LM$ of linear frames on $M$.

- Indeed, $\theta$ defines an application

$$f^\theta : P_H \leftrightarrow LM$$

$$p \mapsto f^\theta(p) : \mathfrak{g}/\mathfrak{h} \rightarrow T_{\pi(p)}M$$

that identifies each element $p$ in $P_H$ with a frame $f^\theta(p) \in LM$ over $\pi(p) \in M$.

- Since $f^\theta : P_H \rightarrow LM$ is an $H$-morphism, $f^\theta(P_H)$ is a $H$-subbundle of $LM$.

- Such a reduction of $LM$ amounts to define a Lorentzian metric on $M$. 
Recovering the metric

- The metric $g^\theta$ on $M$ can be explicitly defined in terms of the $Ad(H)$-invariant scalar product $\langle \cdot, \cdot \rangle$ of $g/h$ by means of the expression

$$g^\theta(v,w) = \langle \tilde{\theta}_p(v), \tilde{\theta}_p(w) \rangle,$$

where $\tilde{\theta}_p(x) : T_x M \to g/h$.

- The $H$-invariance of $\langle \cdot, \cdot \rangle$ implies that $g^\theta(v,w)$ does not depend on the frame $p$ over $x$.

The translational part $\theta$ of the Cartan connection $A$...

... by inducing an isomorphism between $P_H$ and a $SO(3,1)$-subbundle of $LM$...

... induces a Lorentzian metric $g^\theta$ on $M$. 

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Soldering vs. canonical form

- The soldering form $\theta \in \Omega^1(P_H, \mathfrak{g}/\mathfrak{h})$ is the pullback by $f^\theta$ of the canonical form

$$\theta_c : T(LM) \rightarrow \mathfrak{g}/\mathfrak{h}$$

on $LM$ given by:

$$\theta_c : T_{e(x)}(LM) \rightarrow \mathfrak{g}/\mathfrak{h}$$

$$\bar{v} \mapsto e(x)^{-1}(\pi_*(\bar{v})).$$

where

$$e(x) : \mathfrak{g}/\mathfrak{h} \rightarrow T_x M$$

is a frame on $T_x M$.

- Contrary to the canonical form $\theta_c$ on $LM$, the soldering form $\theta$ on $P_H$ is not canonical...

... since it comes from the restriction of the arbitrary Ehresmann conn. $\omega_G$ on $P_G$ to $P_H$.

- This is consistent with the fact that $\theta$ defines a degree of freedom of the theory.
Curvature of the Cartan connection

- The **Cartan curvature** $F \in \Omega^2(P_H, g)$ of a Cartan geom. $(P_H, A)$ is given by

\[ F = dA + \frac{1}{2} [A, A] = F_\theta + F_{g/h} . \]

- The **curvature** $R \in \Omega^2(P_H, h)$ of a Cartan geom. $(P_H, A)$ is given by

\[ R = d\omega_H + \frac{1}{2} [\omega_H, \omega_H] = F_\theta - \frac{1}{2} [\theta, \theta]_h . \]

- The **torsion** $T \in \Omega^2(P_H, g/h)$ of a Cartan geom. $(P_H, A)$ is given by

\[ T = d\theta + \frac{1}{2} (\omega_H, \theta) + [\theta, \omega_H]) = F_{g/h} - \frac{1}{2} [\theta, \theta]_{g/h} . \]
On Cartan flatness

- In general, **Cartan flatness** does not imply $R = 0$ and $T = 0$:

\[
F = 0 \iff \begin{cases} 
R = -\frac{1}{2} [\theta, \theta]_h \\
T = -\frac{1}{2} [\theta, \theta]_g / h
\end{cases}
\]

- The standard for Cartan flatness $F = 0$ is given by the “curved” $\text{LMV} \simeq (G, H)$.

- The so-called **symmetric models** satisfy

\[
[g/h, g/h] \subseteq h,
\]

which implies

\[
T = F_{g/h} \leadsto F = (R + \frac{1}{2} [\theta, \theta]_h) + T.
\]

- $T$ naturally appears as the “translational” component of $F$. 

---

Let's consider the canonical $H$-fibration $G \to G/H$ of the KG $(G, H)$.

The **Maurer-Cartan form** $A_G$ of $G$ is given by

$$A_G(g) : T_g G \to \mathfrak{g}$$

$$\xi \mapsto (L_{g^{-1}})^{\ast} \xi,$$

where $L_{g^{-1}} : G \to G$ is the left translation defined by $L_{g^{-1}}(a) = g^{-1} a$ and it satisfies

$$R_g^{\ast} A_G = \text{Ad}(g) A_G$$

$$dA_G + \frac{1}{2} [A_G, A_G] = 0$$

**Maurer-Cartan form** = **Flat Cartan connection on** $G \to G/H$. 

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### $H$-parallel transports

- $\omega_H$ defines parallel transports in the associated bundle

$$P_H \times_H g/\eta \simeq V_\sigma(P_G \times_G G/H),$$

that is parallel transports of vectors tangent to the LMV along the section $\sigma : M \rightarrow P_G \times_G G/H$.

- Since the geometric soldering form $\tilde{\theta}$ defines an identification

$$TM \xrightarrow{\sim} P_H \times_H g/\eta,$$

the Ehresmann connection $\omega_H$ transports vectors tangent to $M$.

- The $\omega_H$-parallel transports coincide with the Levi-Civita parallel transports.

- Now, $A = \omega_H + \theta$ does not only parallel-transport "internal" states (tangent vectors in this case) as in Y-M theory (by means of $\omega_H$)...

... but also the spatiotemporal locations themselves (by means of $\theta$).
Development

- Let $\gamma : [0, 1] \to M$ be a curve on $M$ and $\tilde{\gamma} : [0, 1] \to P_H$ any lift of $\gamma$.

- Since $P_H \subset P_G$, the curve $\tilde{\gamma}$ is in $P_G$.

- If we use $\omega_G$ for $\|\|$-transporting $\tilde{\gamma}(t)$ to $\pi^{-1}(x_0)$ along $\gamma$ for all $t \in [0, 1]$, we obtain a curve $\hat{\gamma}$ in $\pi^{-1}(x_0)$.

- By using the projection

$$P_G \xrightarrow{\rho} P_G/H \simeq P_G \times_G G/H,$$

we can define a curve $\gamma^* = \rho(\hat{\gamma})$ in the fiber of $P_G \times_G G/H$ over $x_0$ called the development of $\gamma$ over $x_0$.

- In this way, any curve $\gamma : [0, 1] \to M$ can be “printed” on the LMV over $x_0$.

- It can be shown that:

  $\gamma^*$ only depends on $\gamma$ and is independent from the choice of $\tilde{\gamma}$.

  The development of a closed curve might fail to close by an amount given by $T$.

The torsion measures the non-commutativity of the translational parallel transports.
Infinitesimal developments

- Since the development is obtained by projecting a \(||\)-transport defined by \(\omega_G\) onto \(G/H\), ...

  ... the only relevant part of \(\omega_G\) is the \(g/\mathfrak{h}\)-valued part, namely \(\theta\).

- Given an infin. displacement \(v \in T_xM\), the geometric soldering form

\[
\tilde{\theta} : TM \to P_H \times_H g/\mathfrak{h} \simeq V_{\sigma}(P_G \times_G G/H)
\]

defines an infin. displacement in the \(LMV\) on \(x_0\) at the point of attachment \(\sigma(x)\).

- This means that the point of attachment at \(x + v\) will be developed in the fiber above \(x\) into the point \(\sigma(x) + \tilde{\theta}(v)\).

- In other terms, the translational part \(\theta\) of \(A\) defines the \(\gamma\)-dependent image of any \(x \in M\) in the \(LMV\) at \(x_0\).

**Translational locality**: this identification is dynamically defined by the translational component of the Cartan gauge field \(A\).
Conclusion (I)

- The theory of Cartan geometries allows us to put together $\omega_H$ and $\theta$ into a unique Cartan connection

$$A \begin{cases} \omega_H \text{ Gauges the local Lorentz symmetry} \\
\theta \begin{cases} \text{ Gauges the local translational symmetry} \\
\text{Induces a metric } g^\theta \text{ on } M \end{cases} \end{cases}$$

... that gauges the \textbf{local affine gauge invariance} defined by the affine group $G$ that acts transitively on the vacuum solution of the theory.

- This can be done by reducing a Y-M geometry

$$(P_G \to M, \omega_G)$$

by means of a partial gauge fixing

$$\sigma : M \to P_G \times_G G/H$$

that breaks the translational invariance of the $\text{LMV} \cong G/H$. 

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Conclusions (II)

- In GR (where $T = 0$), the consideration of a local *affine* symmetry instead of the smaller local Lorentz symmetry has no effects.

- If we relax the condition $T = 0$, then $\omega_H$ and $\theta$ are indep. geom. structures and the gravitational field must be described by the whole $A = \omega_H + \theta$.

- Since the LMV is not necessarily Minkowski S-T, the affine group $G$ is not necessarily the Poincaré group.
Further Research...

- Clarify the relationship between the *local translational invariance* gauged by $\theta$ and the *invariance under diffeomorphisms* of $M$...

\[ \tilde{\theta} : M \to V_{\sigma}(P_G \rtimes G G/H) \]

which identifies the external infinitesimal translations in $M$ with the internal translations in the internal $\text{LMV}$. 

- Clarify the nature of the reduction process: *dynamical symmetry breaking* or *partial gauge fixing*?

- Analyze the different actions $S$ that can be constructed from the Cartan connection $A$. 
Thanks for your attention !!!