EMERGENCE OF WAVE EQNS FROM QS

Shahn Majid, QMUL / Nov 2012 AEI

Classical geometry ⟵ Quantum diffl geometry ⟵ Quantum Gravity?

Lessons from 3D

Write the pair $A=(e^i,\omega^i)$ of 3-bein and spin connection as an $e_3=\mathbb{R}^3\rtimes su_2$-valued connection. Ad-invariant inner product on $e_3 \Rightarrow$

$$S_{\mathrm{Chern-Simons}}=\int_{\Sigma\times\mathbb{R}} A\dot{\wedge}(\mathrm{d}A+\frac{1}{3}[A\wedge A])=S_{\mathrm{Cartan-Weyl}}$$

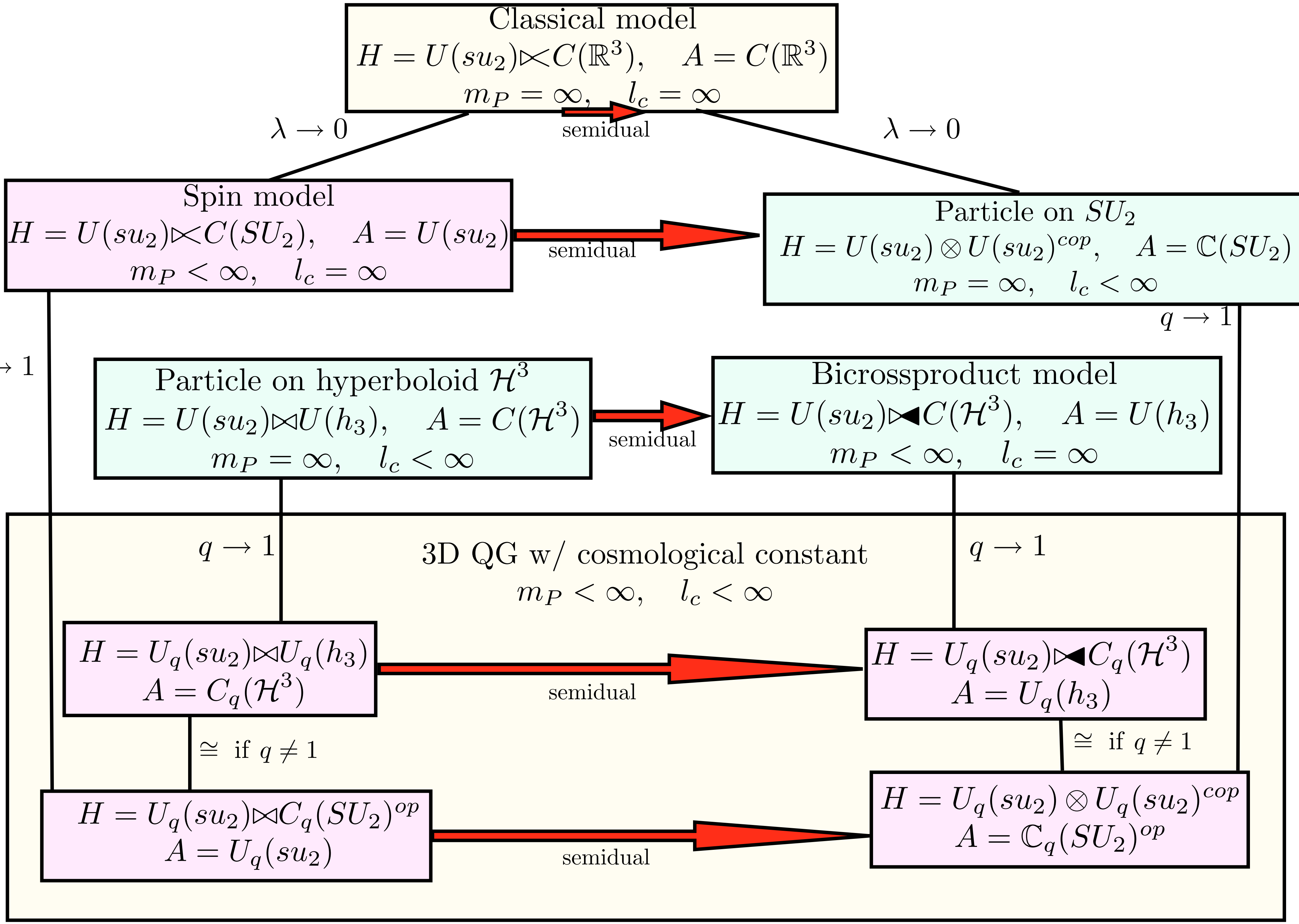
i.e. view gravity as a TFT.

$\Longrightarrow$ Theory described by topology of Σ and `local model' quantum group of motions $U(su_2)\ltimes C(SU_2)$ acting on $U(su_2)$ as quantum flat space, $[x_i,x_j]=2\imath\lambda\epsilon_{i,j,k}x_k$

$(H_1\bowtie H_2, H_2^*)$

With cosmological constant its instead $U_q(su_2)\ltimes C_q(SU_2)^{op}$ acting on $U_q(su_2)$ with $q\sim e^{-\frac{1}{m_p l_c}}$, where $l_c=\sqrt{-\Lambda}$

Different limits of 3D Quantum Gravity (w. B. Schroers)



Semidualization - quantum Born reciprocity (SM 1988)

Let $H = H_1 \bowtie H_2$ be a quantum group factorising into 'quantum rotations' H_1 and 'quantum momentum' H_2.

1. $H_1 \bowtie H_2$ acts canonically on H_2^* 'quantum spacetime'.
2. There is a new quantum group $H_2^* \blacktriangleright\!\!\triangleleft H_1$ (the 'semidual'). It acts canonically on H_2.
3. The Heisenberg-Weyl algebra $H_2^* \rtimes (H_1 \bowtie H_2)$ of the first model is the same as as the Heisenberg-Weyl algebra $(H_2^* \blacktriangleright\!\!\triangleleft H_1) \ltimes H_2$ of the second.
   i.e. the combined rotations-momentum-position algebra is invariant under position $\leftrightarrow$ momentum.
4. Applied to 3D quantum gravity we also swap $m_p \leftrightarrow l_c$.

Bicrossproduct model spacetime (SM+H. Ruegg '94)

$$A = U(\mathbb{R}\triangleright\!\!\!<\mathbb{R}^3)$$

$$H = U(so(1,3))\triangleright\!\!\blacktriangleleft\mathbb{C}[\mathbb{R}\triangleright\!\!\!<\mathbb{R}^3]$$

cf. Lukierski et al

$$[x_i, t] = \lambda x_i, \quad [x_i, x_j] = 0$$

$$x_i, t$$

space, time not simultaneously measurable

$$[p^i, N_j] = -\frac{\imath}{2}\delta^i_j\left(\frac{1-e^{-2\lambda p^0}}{\lambda} + \lambda\vec{p}^2\right) + \imath\lambda p^i p_j,$$

$$\Delta N_i = N_i \otimes 1 + e^{-\lambda p^0} \otimes N_i + \lambda\epsilon_{ij}{}^k p^j \otimes M_k,$$

$$\Delta p^i = p^i \otimes 1 + e^{-\lambda p^0} \otimes p^i$$

Wave operator on plane waves $e^{i\vec{x}\cdot\vec{p}}e^{itp_0}$ $\Rightarrow$ $||p||^2_\lambda = \vec{p}^2 e^{\lambda p^0} - \frac{2}{\lambda^2}(\cosh(\lambda p^0) - 1)$

$\Rightarrow$ Variable Speed Light $\quad |\frac{\partial p^0}{\partial p^i}| = e^{\lambda p^0}$

$$\Delta_T \sim \lambda\Delta_{p^0}\frac{L}{c} \sim 10^{-44}\,\mathrm{s} \times 100\,\mathrm{MeV} \times 10^{10}\,\mathrm{y} \sim 1\,\mathrm{ms},$$

Differential arrival time of gamma-ray bursts (SM+GAC'2000)

3. Quantum anomaly for differential calculus

Space of 1-forms, i.e. `differentials dx'

Ω^1 a((db)c)=(a(db))c `bimodule'

$\mathrm{d}: A \to \Omega^1$ d(ab)=(da)b+a(db) `Leibniz rule'

$\{a\mathrm{d}b\} = \Omega^1$ $\ker \mathrm{d} = \mathbb{C}.1$ connectedness(optional)

In quantum group case we ask it to be translation invariant:

E.g. $A = \mathbb{C}[x] \implies \Omega^1 = \mathbb{C}[x]\mathrm{d}x$

$$\mathrm{d}f(x) = \frac{f(x+\lambda) - f(x)}{\lambda}\mathrm{d}x$$

$$(\mathrm{d}x)f(x) = f(x+\lambda)\mathrm{d}x$$

Theorem (SM&E Beggs, 2004) For simple $\mathfrak{g}$ there do not exist associative differential calculi of classical dimensions (a) on $\mathbb{C}_q(G)$ that are bicovariant (b) on $U(\mathfrak{g})$ that are ad-covariant

=> extra cotangent dimensions. General feature of NCG!

Anomaly => extra dimension => Laplacian as conjugate

- If $\Omega(A)$ sufficiently noncommutative then expect d inner:

$$\exists\theta\in\Omega^1(A)\qquad [a,\theta]=\lambda\mathrm{d}a,\quad \forall a\in A$$ `non-classical equation'

Suppose $\Omega^1=\underbrace{\bar\Omega^1}_{\text{classical}}\oplus A.\theta$ ➤ $\mathrm{d}a=\bar{\mathrm{d}}a+\frac{\lambda}{2}(\Delta a)\theta$

$$\Delta: A\to A$$

Philosophy: Laplacian or wave operator arises out of the construction of the calculus

- Eg in bicrossproduct model $[x_i,x_j]=0,\quad [x_i,t]=\imath\lambda x_i$

poincare covariance has an anomaly, forces extra direction θ'

$$[\mathrm{d}x_i,x_j]=\imath\lambda\delta_{ij}\theta',\quad [\theta',x_i]=0,\quad [\theta',t]=\imath\lambda\theta'$$ cf Sitarz

$$[\mathrm{d}x_i,t]=0,\quad [x_i,\mathrm{d}t]=\imath\lambda\mathrm{d}x_i,\quad [\mathrm{d}t,t]=\imath\lambda\theta'-\imath\lambda\mathrm{d}t$$

$$\mathrm{d}\psi=\frac{\partial}{\partial x_i}\psi(x,t)\mathrm{d}x_i+\partial_0\psi(t)\mathrm{d}t+\frac{\imath\lambda}{2}(\Box\psi(x,t))\theta'\implies$$ same $\Box$ as before in VSL prediction

- What is the physical meaning of this new degree of freedom known as the the differential structure?

Fact: we can change to $[\mathrm{d}t, t] = \beta\imath\lambda\theta' - \imath\lambda\mathrm{d}t$ where β is any function on space, still gives calculus and Laplacian becomes:

$$\Box\psi = \bar{\Delta}\psi(t+\imath\lambda) + 2\Delta_0\psi, \quad \bar{\Delta} = \frac{\partial^2}{\partial x_i^2} - \frac{1}{2\beta}\frac{\partial\beta}{\partial x_i}\frac{\partial}{\partial x_i}$$

$$x_i\frac{\partial\mu}{\partial x_i} + 2\mu = \beta$$

$$x_i\frac{\partial\nu}{\partial x_i} + \nu = \mu$$

$$\Delta_0\psi(t) = \frac{\nu\psi(t+\imath\lambda) + \mu\psi(t-\imath\lambda(\frac{\beta}{\mu}-1)) - (\nu+\mu)\psi(t+\imath\lambda(1-\frac{\beta}{\nu+\mu}))}{(\imath\lambda)^2}$$

$\lim_{\imath\lambda\to 0} 2\Delta_0 = \beta\frac{\partial^2}{\partial t^2}$ so $\Box$ corresponds to $g = \frac{1}{\beta}\mathrm{d}t\otimes\mathrm{d}t + \mathrm{d}x_i\otimes\mathrm{d}x_i$

=> newtonian gravity arises out of a freedom for the quantum differential calculus on flat quantum spacetime

- $\beta = -\frac{1}{c^2}(1 - \frac{2\Phi}{c^2})$ where Φ is the newtonian potential. Can solve this in the quantum spacetime case for pt source

$$\beta = -\frac{1}{c^2}(1+\frac{\gamma}{r}), \quad \mu = -\frac{1}{c^2}(\frac{1}{2}+\frac{\gamma}{r}), \quad \nu = -\frac{1}{c^2}(\frac{1}{2}-\frac{\gamma}{r}\ln(\frac{\gamma}{r})) \qquad \gamma = \frac{2GM}{c^2}$$

$$\Rightarrow \quad \Box\psi(t) = \Box^{\beta=-1/c^2}\psi(t) - \frac{1}{2}\frac{\gamma}{r^3(1+\frac{\gamma}{r})}x_i\frac{\partial}{\partial x_i}\psi(t+\imath\lambda) - \frac{2\gamma}{c^2 r}\Delta_0^{hybrid}\psi(t+\imath\lambda)$$

where $\psi(x,t) = \sum\psi_n(x)t^n$ $\qquad \Delta_0^{hybrid} = \frac{1}{\imath\lambda}\left(\frac{\partial}{\partial t} - \partial_0\right)$

Now let $\psi = \Psi(x,t)e^{-\imath\frac{mc^2}{\hbar}t}$ $\qquad \tilde{m} = mc^2/\hbar.$

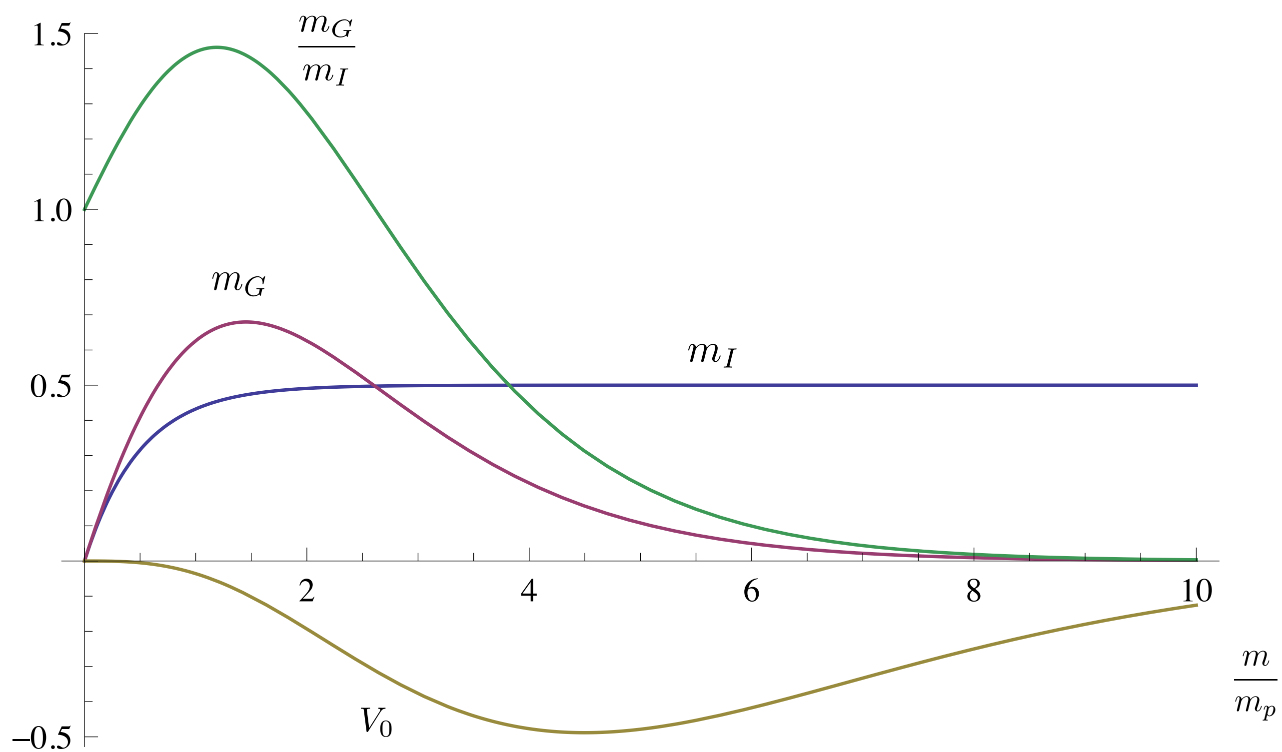
$$\imath\hbar\frac{\sinh(\tilde{m}\lambda)}{\tilde{m}\lambda}\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2 e^{\tilde{m}\lambda}}{2m}\bar{\Delta}^{flat}\Psi + \left(mc^2(1-\frac{\sinh(\frac{\tilde{m}\lambda}{2})}{\frac{\tilde{m}\lambda}{2}}) - \frac{GMm}{r}(\frac{\tilde{m}\lambda+e^{-\tilde{m}\lambda}-1}{\frac{\tilde{m}^2\lambda^2}{2}})\right)\Psi$$

which we write as $\imath\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m_I}\bar{\Delta}^{flat}\Psi + (V_0 - \frac{GMm_G}{r})\Psi$

$$m_I = m\frac{\sinh(\tilde{m}\lambda)}{\tilde{m}\lambda}e^{-\tilde{m}\lambda} \qquad m_G = m\left(\frac{\tilde{m}\lambda+e^{-\tilde{m}\lambda}-1}{\frac{\tilde{m}\lambda}{2}\sinh(\tilde{m}\lambda)}\right)$$

$$V_0 = mc^2 \frac{\tilde{m}\lambda}{\sinh(\tilde{m}\lambda)} \left(1 - \frac{\sinh(\frac{\tilde{m}\lambda}{2})}{\frac{\tilde{m}\lambda}{2}} \right) = -\frac{mc^2}{24} (\tilde{m}\lambda)^2 + o((\tilde{m}\lambda)^4))$$

suggests how vacuum energy might arise as a quantum geometry correction!



and suggests that macroscopic massive quantum states may behave differently approaching and above planck mass!

4. Minimally coupled quantum black hole

We take as before flat quantum spacetime $[x_i, t] = \imath\lambda_p x_i$ and

$\beta = -\frac{1}{c^2(1-\frac{\gamma}{r})}$ $\gamma = \frac{2GM}{c^2}$ Schwarzschild radius

We also `minimally couple' $\bar\Delta_{\mathbb{R}^3} \mapsto \bar\Delta_{LB}$ for BH spatial metric

Black hole quantum wave operator

$$\Box\psi(t) = 2\Delta_0\psi(t) + \bar\Delta_{LB}\psi(t+\imath\lambda_p) - \frac{1}{2\beta}(\bar{\mathrm{d}}\beta, \bar{\mathrm{d}}\psi)(t+\imath\lambda_p)$$

$$\Delta_0 e^{\imath\omega t} = \frac{1}{c^2}D(\omega, r)e^{\imath\omega t}$$

where

$$D(\omega, r) = \frac{1}{\lambda_p^2}\left(\sinh(\omega\lambda_p) + e^{-\omega\lambda_p}(1-\frac{\gamma}{r})\left(1 - e^{\omega\lambda_p} - \frac{\gamma}{r}\ln\left(\frac{e^{\omega\lambda_p}r-\gamma}{r-\gamma}\right)\right)\right)$$

$$\lim_{\lambda_p \to 0} D(\omega, r) = \frac{\omega^2}{2(1 - \frac{\gamma}{r})},$$

classical

$$\lim_{r \to \infty} D(\omega, r) = \frac{\cosh(\omega \lambda_p) - 1}{\lambda_p^2}$$

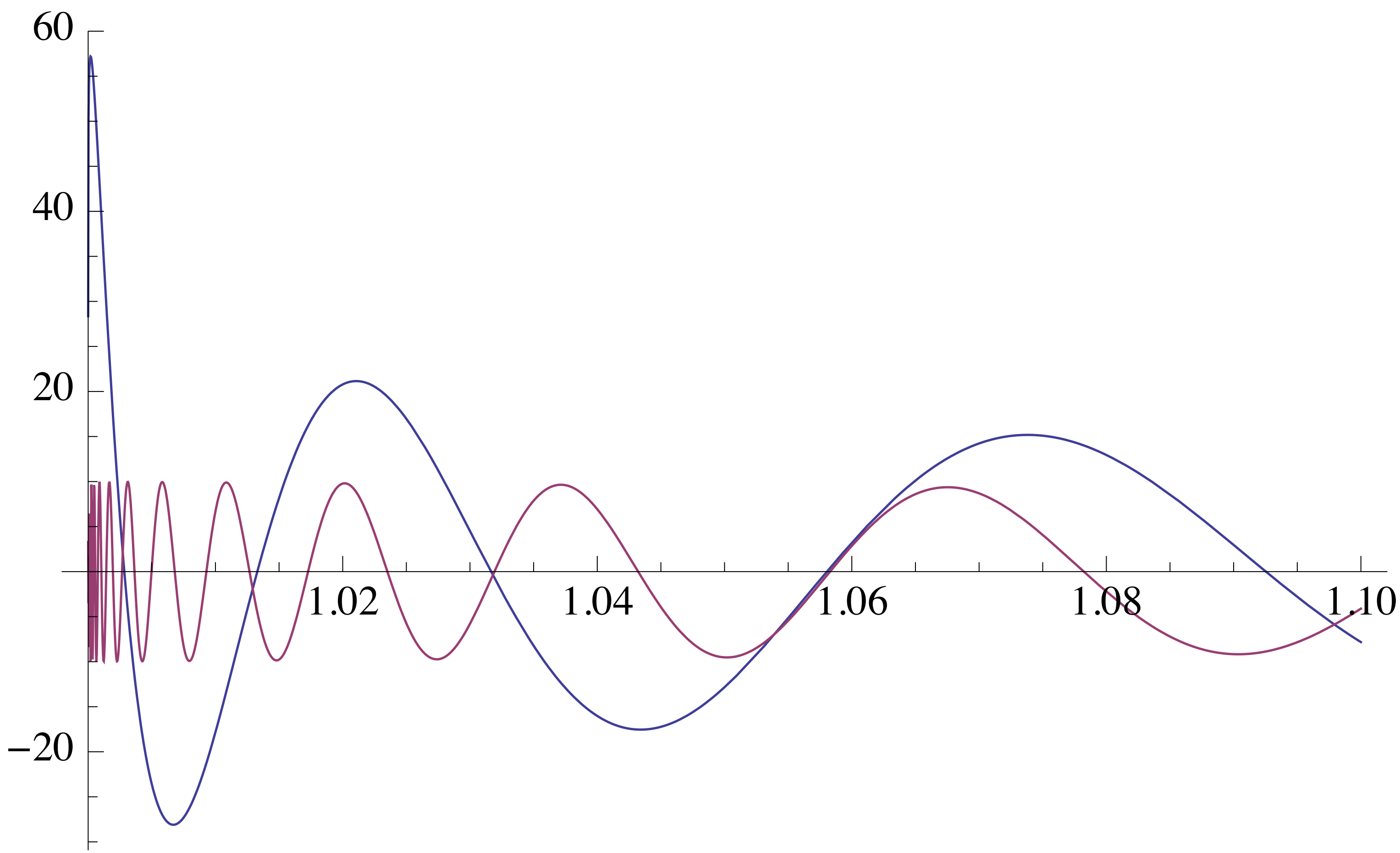
kappa-minkowski spacetime far from BH

- <u>Effect1</u>: The time dilation/redshift is finite at the event horizon

$$\lim_{r \to \gamma} D(\omega, r) = \frac{\sinh(\omega \lambda_p)}{\lambda_p^2},$$

e.g. $\omega = 10^{19}$ Hz

$z_{max} \approx 5 \times 10^{12}$



Effect 2: Gravl time dilation/redshift is frequency dependent

$$2D(\omega, r) = \frac{\omega^2}{(1-\frac{\gamma}{r})}\left(1 - \frac{2}{3}\frac{\omega\lambda_p\gamma}{r(1-\frac{\gamma}{r})} + O((\omega\lambda_p)^2)\right)$$

suggests that a higher frequency will be less redshifted.

An emission + n'th harmonic at radius r won't be a harmonic when received and this might be very sensitively detected. One cycle error accumulates after distance

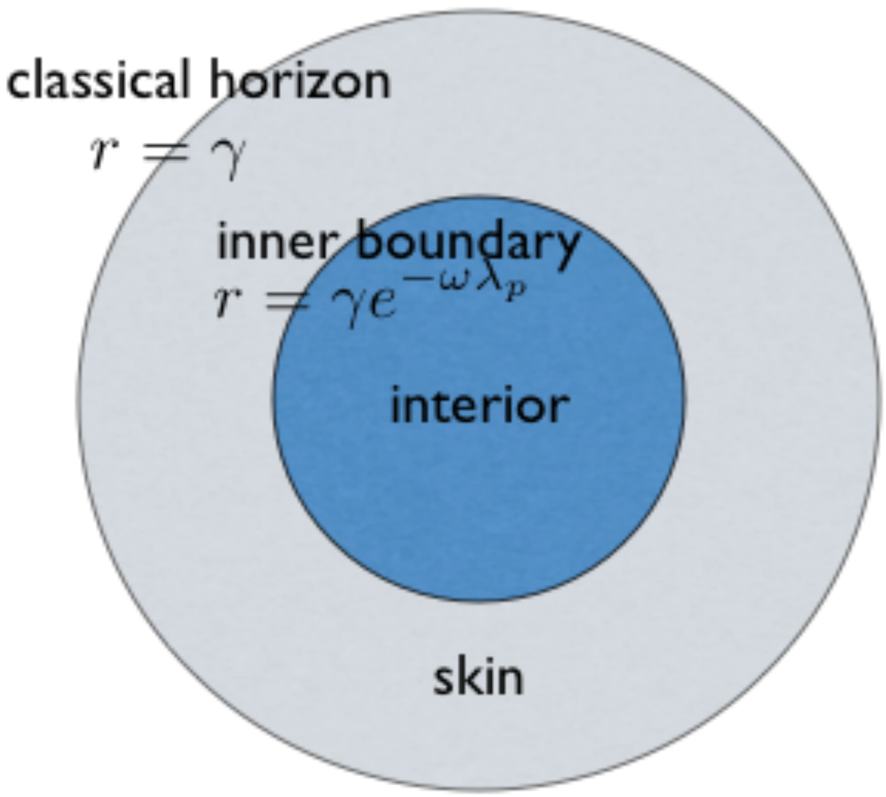
$$L \sim \frac{c^2}{\omega^2}\frac{3r}{n\gamma\lambda_p}$$

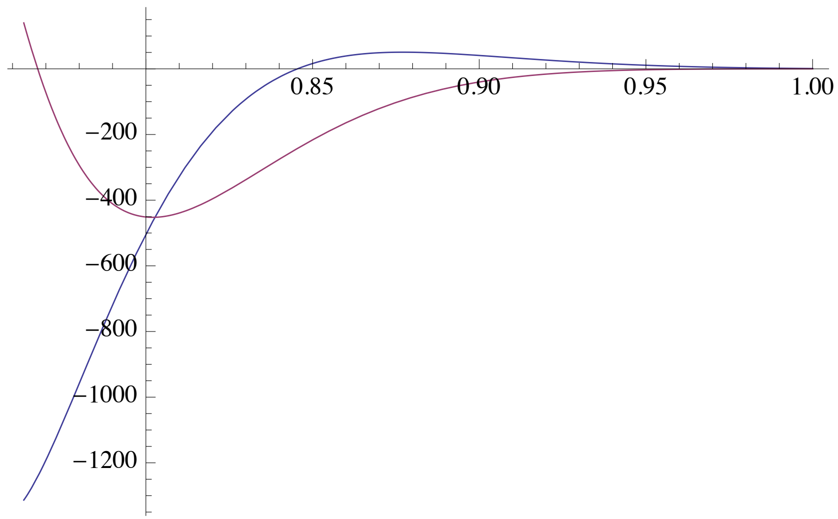
e.g. 0.1 nm (X ray), $\frac{\gamma}{r} = 0.1 \implies L \sim 0.1$ light years

Detect non-harmonicity by a resonant cavity?

Astrophysical harmonic emission?

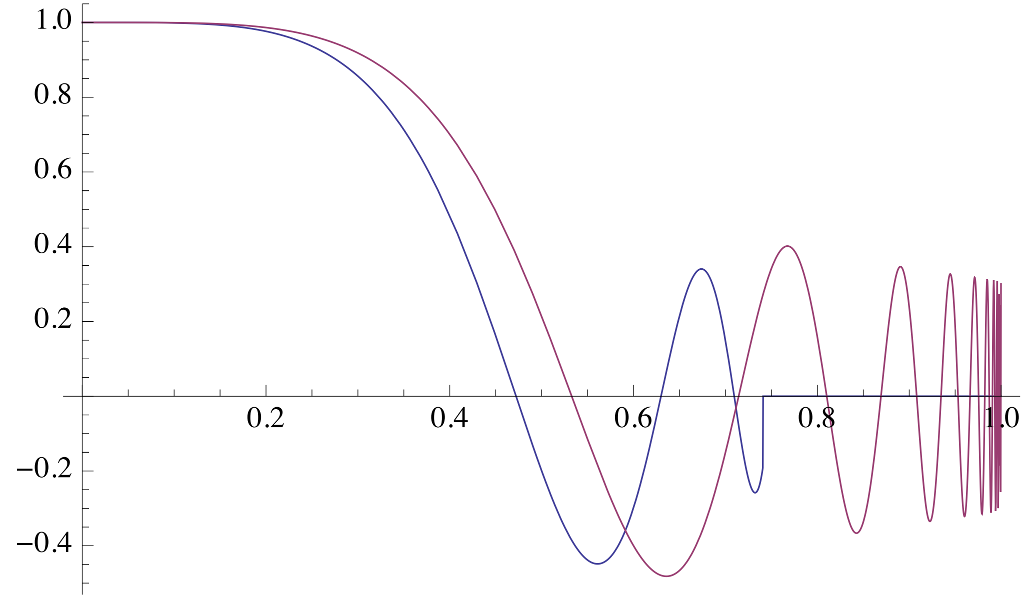
- <u>Effect 3</u>: For pos frequencies $\omega > 0$ a `skin' of width $\gamma(1-e^{-\omega\lambda_P})$ just below the event horizon where $\Im D(\omega, r) \neq 0$





(skin of the black hole just below the event horizon? Could be an artefact)

- Effect 4: Standing waves in the BH interior



(black holes may not evaporate but rather form bound states)

- Effect 5: singularity at origin?

$$D(\omega,r) = -\frac{(\cosh(\omega\lambda_p)-1)(1+2e^{\omega\lambda_p})}{3\lambda_p^2\gamma} r + O((\frac{r}{\gamma})^2)$$

(Has same exponential growth with frequency that leads to Planckian bound in spatial momentum in kappa-minkowski at large r)

- Effect 6: treats pos and neg frequencies differently

5. Quantization of $M\times\mathbb{R}$

Let $(M,\bar g)$ be a Riemannian manifold of dimension n and τ a vector field

$$A=C(M){\bowtie}\mathbb{R} \qquad\qquad [f,t]=\lambda\tau(f)$$

is a noncommutative version of $M\times\mathbb{R}$. Let $(\bar\Omega^1,\bar{\mathrm{d}})$ be classical, $\bar{\mathcal{L}}$ the Lie derivative, $\bar\Delta$ a 2nd order operator and $\alpha=\frac{2}{n}\mathrm{div}(\tau)-1$.

Thm (SM 2012) For any function β and conformal Killing vector field τ , extending $\bar\Omega^1(M)$ by $\mathrm{d}t,\theta'$ with relations

$$[f,\omega]=\lambda(\omega,\bar{\mathrm{d}}f)\theta',\quad \mathrm{d}f=\bar{\mathrm{d}}f+\frac{\lambda}{2}(\bar\Delta f)\theta' \qquad \forall f\in C(M),\ \omega\in\bar\Omega^1(M)$$

$$[\omega,t]=\lambda(\bar{\mathcal{L}}_\tau-\mathrm{id})\omega-\lambda^2(\tfrac{\mathrm{n}-2}{4})(\bar{\mathrm{d}}\alpha,\omega)\theta'-\frac{\lambda^2}{2}(\bar{\mathcal{L}}_\tau\zeta^*,\omega)\theta'$$

$$[\theta',t]=\alpha\lambda\theta',\quad [f,\mathrm{d}t]=\lambda\mathrm{d}f,\quad [\mathrm{d}t,t]=\beta\lambda\theta'-\lambda\mathrm{d}t$$

gives a differential calculus $\Omega^1(C(M)\rtimes\mathbb{R})$

$\zeta=-\frac{1}{2}\beta^{-1}\bar{\mathrm{d}}\beta \Rightarrow$ $\square$ quantises laplacian for metric $\bar g+\beta^{-1}\mathrm{d}t\bar\otimes\mathrm{d}t$

Insights into differential geometry

Let $(M, \bar{g})$ be a Riemannian manifold, inverse metric (,), levi-civita connection $\bar{\nabla}$ and Laplace-Beltrami operator $\bar{\Delta}$

Corollary Classical $\bar{\Omega}^1(M)$ has a noncommutative extension

$\Omega^1 = \bar{\Omega}^1 \oplus C(M)\theta'$ *with* θ' *central and*

$$f \bullet \omega = f\omega, \quad \omega \bullet f = \omega f + \lambda(\omega, \bar{\mathrm{d}} f)\theta', \quad \mathrm{d} f = \bar{\mathrm{d}} f + \frac{\lambda}{2}(\bar{\Delta} f)\theta'$$

$f \in C(M), \ \omega \in \bar{\Omega}^1$

Warning: this example is non-surjective \`generalised calculus'

Lemma There is a well-defined linear map

$$\phi : \bar{\Omega}^1 \bar{\otimes} \bar{\Omega}^1 \to \Omega^1 \hat{\otimes} \Omega^1, \quad \phi(\omega \bar{\otimes} \eta) = \omega \hat{\otimes} \eta - \lambda\theta' \hat{\otimes} \bar{\nabla}_\omega \eta, \quad \forall \omega, \eta \in \bar{\Omega}^1$$

from the classical $\bar{\otimes}$ over $C(M)$ to the new $\hat{\otimes}$ wrt $\bullet$

<u>Theorem</u> for any $K:\bar\Omega^1\to\bar\Omega^1$ and $\nabla\theta'$ central

$$\nabla\omega=\phi(\bar\nabla\omega)+\frac{\lambda}{2}\theta'\hat\otimes(\bar\Delta-K)\omega,\quad\forall\omega\in\bar\Omega^1\subset\Omega^1$$

$$\sigma(\omega\hat\otimes\eta)=\eta\hat\otimes\omega+\lambda\bar\nabla_\omega\eta\hat\otimes\theta'-\lambda\theta'\hat\otimes\bar\nabla_\eta\omega+\lambda(\omega,\eta)\nabla\theta'+\frac{\lambda^2}{2}(\mathrm{Ricci}_{\bar\Delta}+K^T)(\omega,\eta)\theta'\hat\otimes\theta'$$

$$\sigma(\omega\hat\otimes\theta')=\theta'\hat\otimes\omega,\quad\sigma(\theta'\hat\otimes\omega)=\omega\hat\otimes\theta',\quad\sigma(\theta'\hat\otimes\theta')=\theta'\hat\otimes\theta'$$

is a bimodule connection:

$$\nabla:\Omega^1\to\Omega^1\hat\otimes\Omega^1\qquad\nabla(f\omega)=\mathrm{d}f\hat\otimes\omega+f\nabla\omega$$

$$\sigma:\Omega^1\hat\otimes\Omega^1\to\Omega^1\hat\otimes\Omega^1\qquad\nabla(\omega f)=\sigma(\omega\hat\otimes\mathrm{d}f)+(\nabla\omega)f$$

<u>Propn</u> take $K=\mathrm{Ricci}$, $\nabla\theta'=0$ then

- $\sigma^2=\mathrm{id}$ *iff* $\mathrm{Ricci}=0$
- $\sigma_{12}\sigma_{23}\sigma_{12}=\sigma_{23}\sigma_{12}\sigma_{23}$ *iff* $(M,\bar g)$ is flat

(some kind of `braided 2-category' associated to any Riemannian manifold?)

6. Summary

1) Position-momentum duality visible in 2+1

	Position	Momentum
Gravity	Curved	Noncommutative
Cogravity	Noncommutative	Curved
Quantum Gravity	Both	Both

Einstein eqn ~ posn-mom symmetry (SM Class Quan Grav 1988)

2) Noncommutative space generates in own evolution out of an anomaly for differential calculus, wave operator is associated to an induced extra dimension

3) Differential calculus is a new degree of freedom, origin of gravity

4) BH model shows resolution of singularities, freq dept redshift

5) Get a deeper view of classical dgm as a shadow of ncg

THANK YOU!

Further Reading (not a bibliography)

kappa-Minkowski papers:

S. Majid and H. Ruegg, Bicrossproduct Structure of the k-Poincare Group and Non-Commutative Geometry, *Phys. Lett. B.* **334** (1994) 348-354

G. Amelino-Camelia and S. Majid, Waves on Noncommutative Spacetime and Gamma-Ray Bursts, *Int. J. Mod. Phys. A***15** (2000) 4301-4323

S. Majid, Quantum anomalies and Newtonian gravity on quantum spacetime, 9pp. *arXiv:1109.6190 (hep-th)*

Nogo theorem for quantum differential calculus:

E.J. Beggs and S. Majid, Semiclassical Differential Structures, *Pac. J. Math.* **224** (2006) 1-44

Black holes papers:

S. Majid, Almost commutative Riemannian geometry, I: wave operators, 50pp. *Commun. Math. Phys.* **310** (2012) 569-609

S. Majid, Scaling limit of the noncommutative black hole, *J. Phys. Conf. Ser.* **284** (2011) 012003 (12pp)

Born reciprocity and semidualisation:

S. Majid and B. Schroers, q-Deformation and semidualisation in 3D quantum gravity, *J. Phys A* **42** (2009) 425402 (40pp)