A semiclassical intrinsic canonical approach to quantum gravity

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Overview

1. INTRODUCTION

Introduction

The twentieth century founders of quantum mechanics drew on a rich classical mechanical tradition in their effort to incorporate the quantum of action into their deliberations. The phase space formulation of mechanics proved to be the most amenable to adaptation, and in fact the resulting quantum theory proved to not be so distant from the classical theory within the framework of the associated Hamilton-Jacobi theory.

Introduction

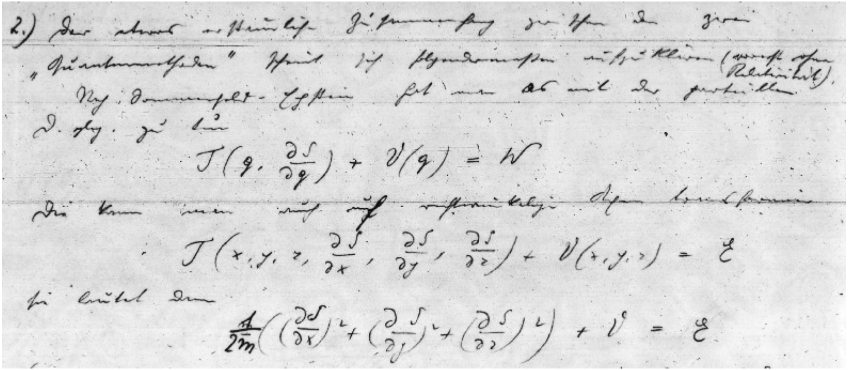
We will briefly review these efforts and their relevance to attempts over the past few decades to construct a quantum theory of gravity. We emphasize in our overview the heuristic role of the action, and in referring to observations of several of the historically important players we witness frequent expressions of hope, but also misgivings and even misunderstandings that have arisen in attempts to apply Hamilton-Jacobi techniques in a semi-classical approach to quantum gravity. In particular we will address the origins and misinterpretations of the Wheeler-DeWitt equation.

Introduction

We claim that that the deficiencies and misuse of the Wheeler-DeWitt equation originated in an incomplete understanding of the role of constraints in the classical phase space formalism. We then propose a general framework which both explains the limited successes of the Wheeler DeWitt approach to date and offers a procedure which can in principle yield semi-classical solutions in generic general relativity.

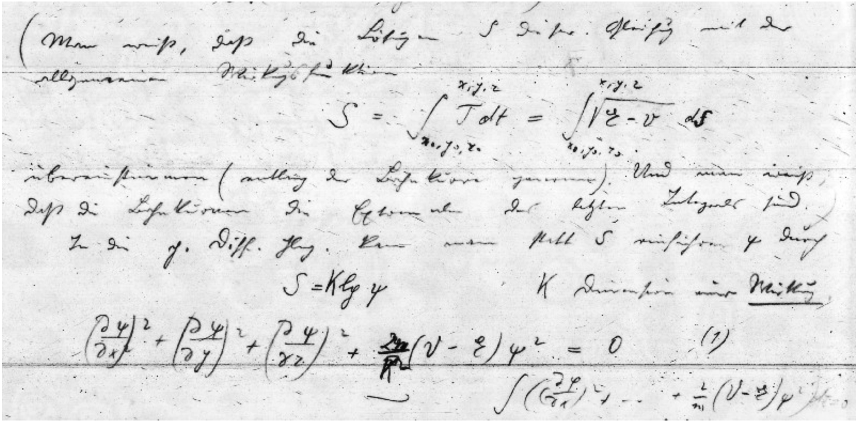
2. AN HISTORICALLY MOTIVATED HEURISTICS

The old quantum theory as semiclassical quantum theory



Pages from Schrödinger's notebook.

The old quantum theory as semiclassical quantum theory



Pages from Schrödinger's notebook.

The role of Hamilton-Jacobi theory in anticipating structures of quantum theory

- The optical mechanical analogy has its roots in Hamilton's original introduction of his characteristic function in 1837. This is the same function that appears in the Sommerfeld-Epstein rule.
- The Hamilton principal function S as a phase is consistent with the Schrödinger wave equation in the limit $\hbar \to 0$ The power series expansion of S in $\hbar$ was first introduced independently by Wentzel and Brillouin in 1926. They in turn with Kramers in 1926 established the general conditions under which Sommerfeld-Epstein quantization agreed with wave mechanics. See Pauli's 1933 Handbuch der Physik article for an overview.

The role of Hamilton-Jacobi theory in anticipating structures of quantum theory

- As also noted 1926, semi-classical wave packets that satisfy the correct classical equations of motion may be constructed through the superposition over complete principal function solutions of the Hamilton-Jacobi equation. These superpositions are of the form

$$\int d\alpha e^{iS(x,t;\alpha)/\hbar}$$

The result follows as a consequence of the Hamiltonian dynamical equations

- These observations served as a point of departure of Peter Weiss's groundbreaking extension of the Hamilton-Jacobi formalism to field theory in 1936. [Wei36, Wei38]

Dirac in 1951 on the significance of Hamilton-Jacobi

From Dirac's 1951 foundational paper on constrained Hamiltonian dynamics, "The Hamiltonian form of field dynamics" [Dir51]

1. Introduction. In classical dynamics one has usually supposed that when one has solved the equations of motion one has done everything worth doing. However, with the further insight into general dynamical theory which has been provided by the discovery of quantum mechanics, one is lead to believe that this is not the case. It seems that there is some further work to be done, namely to group the solutions into families (each family corresponding to one principal function satisfying the Hamilton-Jacobi equation). The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood.

Dirac in 1951 on the significance of Hamilton-Jacobi

The importance of the family is brought out by the Schrödinger form of quantum mechanics and not by the Heisenberg form. The latter is in direct analogy with the classical Hamiltonian equations of motion and does not require any grouping of the solutions. The Schrödinger form goes beyond this in ascribing importance to the concept of a quantum state, subject to the principle of superposition and described by a solution of Schrödinger's wave equation, and this concept requires the introduction of families of solutions for its analogue in classical mechanics, the Schrödinger equation itself being the analogue of the Hamilton-Jacobi equation.

3. THE STANDARD APPROACH TO SEMICLASSICAL CANONICAL QUANTIZATION

The standard approach to semiclassical canonical quantization of gravity

Let us now look more closely at the promise and at the limitations of the standard approach to a semi-classical canonical quantization of general relativity via the Wheeler-DeWitt equation. It was inspired by a so-called Einstein-Hamilton-Jacobi equation first written down by Peres in 1962 [Per62]

IL NUOVO CIMENTO VOL. XXVI, N. 1 1° Ottobre 1962

On Cauchy's Problem in General Relativity – II.

A. PERES (*) (**)

Palmer Physical Laboratory, Princeton University - Princeton, N. J.

Einstein-Hamilton-Jacobi equation

Peres replaced the canonical momenta p^{ab} in the Hamiltonian constraint

$$\mathcal{H} = -\sqrt{g}\,{}^3R + \frac{1}{\sqrt{g}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) = 0,$$

with a functional derivative of a Hamilton principal function S with respect to the spatial metric field g_{ab}. It is important to note that this replacement was not derived by Peres from a variation of the gravitational action.

This equation inspired Bryce DeWitt. In his own words, from his 1999 paper "The Quantum and Gravity: The Wheeler-DeWitt equation": [DeW99]

Origins of the Wheeler-DeWitt Equation according to DeWitt

"John Wheeler, the perpetuum mobile of physics, called me one day in the early sixties. I was then at the University of North Carolina in Chapel Hill, and he told me that he would be at the Raleigh-Durham airport for two hours between planes. He asked if I could meet with him there and spend a while talking quantum gravity. John was pestering everybody at the time with the question: What are the properties of the quantum mechanical state functional Psi and what is its domain? He had fixed in his mind that the domain must be the space of 3-geometries, and he was seeking a dynamical law for Psi."

Origins of the Wheeler-DeWitt Equation according to DeWitt

"I had recently read a paper by Asher Peres which cast Einstein's theory into Hamilton-Jacobi form, the Hamilton-Jacobi function being a functional of 3-geometries. It was not difficult to follow the path already blazed by Schrödinger, and write down a corresponding wave equation. This I showed to Wheeler, as well as an inner product based on the Wronskian for the functional differential wave operator. Wheeler got tremendoulsy excited at this and began to lecture about it on every occasion."

Origins of the Wheeler-DeWitt Equation according to DeWitt

"I wrote a paper on it in 1965, which didn't get published until 1967 because my Air Force grant was terminated and the Physical Review in those days was holding up publication of papers whose authors couldn't pay the page charges. My heart wasn't really in it [...] But I thought I should at least point out a number of intriguing features of the functional differential equation, to which no one had yet begun to devote much attention: [...] The fact that the wave functional is a wave function of the universe and therefore cannot be understood except within the framework of a many-worlds view of quantum mechanics [...] In the long run one has no option but let the formalism provide its own interpretation. And in the process of discovering this interpretation one learns that time and probability are both phenomenological concepts."

Wheeler-DeWitt equation

It follows as a consequence of the Einstein-Hamilton-Jacobi equation in the limit as $\hbar \to 0$ that the quantum wave function

$$\Psi\left[g_{ab}\right] \propto e^{iS/\hbar},$$

satisfies the Wheeler-DeWitt equation:

$$-g\,{}^{3}R\Psi + g_{ab}g_{cd}\left(\frac{\delta^2}{\delta g_{ac}\delta g_{bd}} - \frac{1}{2}\frac{\delta^2}{\delta g_{ab}\delta g_{cd}}\right)\Psi = 0.$$

4. LIMITATIONS OF THE STANDARD APPROACH TO SEMICLASSICAL CANONICAL QUANTIZATION

DeWitt's later view

DeWitt himself famously disavowed the significance of his equation. We quote from his posthumously published comments from 2009: [DeW09]

DeWitt's attitude toward the Wheeler-DeWitt equation

"This equation should be confined to the dustbin of history for the following reasons: 1) By focussing on time sclices it violates the very spirit of relativity. 2) Scores of man-years have been wasted by researchers trying to extract from it a natural time parameter. 3) Since good path integral techniques exist for basing Quantum Theory on gauge invariant observables only, it seems a pity to drag in the paraphernalia of constrained hamiltonian systems."

Wheeler DeWitt and phase space

DeWitt's observations are symptomatic of a widespread dismissive attitude within the relativity community of canonical phase space approaches to diffeomorphism symmetry. And it turns out that one of the first perceived consequences of this symmetry appeared not directly in the phase space context, but in the related path integral approach to quantum gravity. We witness it in Misner's discovery of "frozen time". In an interview with Misner and Brill conducted at the University of Maryland on March 16, 2011 we have the following response to my query on diffeomorphism symmetry.

Wheeler DeWitt and phase space

SALISBURY: So this problematic with gauge invariance, then, is maybe first encountered in this context [of path integral quantization]?
MISNER: Well it was, because the way it comes up there is that I was able to show that the propagator as defined four dimensionally was zero, the Hamiltonian zero. And I got a letter from Pauli objecting to that, which I eventually lost –

Misner and frozen time

RICKLES: I wonder whether you noticed any connections to the issues that Bergmann was dealing with in the 57 paper, in fact, when you were talking about the propagator being the identity, and you get the zero Hamiltonian. Did you notice any similarities to the canonical work?
MISNER: No, I don't think I had absorbed the canonical work at that time.

5. HISTORY OF CONSTRAINED HAMILTONIAN DYNAMICS

The problem of preferred time foliations

Perhaps the most persuasive argument amongst relativists against a Hamiltonian approach to general relativity was that it seemed from the start to grant a preferred status to the initial choice of time slicing.

The disputed role of classical canonical constraint analysis

Let us then look at the history of constrained Hamiltonian analysis, its relation to diffeomorphism covariance, and the relevance of this work to early quantum gravity. Three individuals were the significant players in the early development:

- Rosenfeld pioneered in this field in 1930 [Ros30, Sal09], but his results were largely ignored both by his mentor Pauli and by himself in his later work.
- Bergmann and collaborators began publishing work on the canonical phase space realization of diffeomorphism symmetry in 1949. [Ber49]
- Dirac, whose first published work on constrained Hamiltonian dynamical formalism appeared in 1950, was never concerned with constraints as generators of diffeomorphism transformations. [Dir50]

The disputed role of classical canonical constraint analysis

This work apparently had little or no impact on the subsequent development of the ADM formalism of Arnowitt, Deser, and Misner - who independently produced a canonical formalism for dealing with the initial value problem in general relativity. (Much remains to be done to sort out the historical interrelationships of Rosenfeld, Bergmann, Dirac, and ADM.)

We get some flavor of Deser's attitude regarding the desirability of understanding the relationship between constraints and diffeomorphisms from an interview conducted with Deser on March 12, 2011. We begin with Deser's presentation of linearized gravity at Neuchâtel in 1958 in a meeting that was co-organized by Bergmann and by Deser's father-in-law, Oskar Klein.

The disputed role of classical canonical constraint analysis

DESER: ... The thing is actually we were of course ahead of our time because relativists didn't understand what we were doing. First of all, it was linearized, and secondly, it was all this quantum field theory language. Rosenfeld would have understood it, but not, you know – And Bergmann was doing very strange stuff. Bergmann was always orthogonal to us...

Deser continued

DESER: ... Yes, right. Peter and I always had, what shall I say, a frosty relationship because Peter thought he owned the field, that is, what should I say, the formal development of GR. We got to know him and his students. I did because in those days you actually went to APS meetings and people actually talked at APS meetings, so we got to know people like Jim Anderson and then Ralph Schiller and the whole crew. They would all give these to us incomprehensible talks full of letters with subscripts and superscripts and all sorts of dots. It was all, we felt – we were very superior; after all, we came from ... Whereas Peter felt that that was his domain and this was not something to be encroached on lightly. So basically we decided to just disagree.

Deser continued

SALISBURY: A related question has to do with operators or generators that you identify as being related, at least, to general coordinate transformations, to the gauge symmetries. One thing I've wondered about is since you were from the very start working in an operator formulism, you were not really in a position to be able to relate the variations that were generated by those objects to coordinate transformations.

Deser continued

DESER: Its about, in fact, the incredible jungle that's opened up when you quantize, because then you can make coordinate transformations with horrible operator properties, ... That was formal. You see, you've got to be careful. There's a difference between formal quantum discussion and a real one, and it's a fundamental difference. In fact, that was a problem with people, like I think Bergmann in particular and Dirac also. You can say quantum and that's a great word, but either you're talking at the tree level, in which case you're really talking about classical theory, or you don't know anything because the full theory is such a general ... So that there are all sorts of paradoxes that have come up already in this word quantum, apparent paradoxes.

Deser continued

Like operator coordinate transformation – you read that paper [with Arnowitt and Misner in 1961 on the Heisenberg representation] and you'll see everything in that paper is correct, and it just shows you that if you try to be too general, you just get totally lost because there's a fundamental conflict between the classical formalism and blindly pushing it to the quantum level. SALISBURY: So the position you would represent then, I guess the one you have represented historically, is that there's no benefit to be gained from having pursued this classical theory.

Deser continued

DESER: I would say that. Yes, that's right, I would say that. I mean look, the general idea is the theory – you know, it's like in poker, the theory speaks for itself. It is what it is, and your job is – I mean loading it down with loaded words does not really do it a service. You have a theory and you write down an action, and that action at the classical level, it's hard enough. Even at the classical level, of course, you have generators. But there are two thing you have to understand. One is the general covariance; and secondly, that unless you then work in a particular gauge, youre not doing anything at all. In Maxwell theory, of course, not only can you work on a particular gauge, you can work gauge invariantly even though it looks as though you've chosen a gauge ... So there are all these subtleties and gradations, which a lot of people sort of do not take into consideration.

Anderson on the Bergmann program

We continue with this review of reactions to the constrained Hamiltonian dynamics program by quoting an observation of one of Bergmann's closest early collaborators, Jim Anderson.

Anderson on the Bergmann program

From interview with James Anderson conducted on March 19, 2011

ANDERSON: ... He [Bergmann] was mainly interested in setting up a quantum formalism so that he could apply the rules of quantum mechanics to put classical relativity into quantum form. And constraints reared their ugly head, in all their ugliness. And thats mainly what I worked on.

...

ANDERSON: ... Peter's approach was always get a consistent Hamiltonian theory of quantum mechanics and then quantize it. And once I convinced myself that there was no factor sequence that worked with the constraints, I realized that wasn't going to work. And then I sort of lost interest, because I didn't see any way out of the problem.

Anderson continued

ANDERSON: ... That's why I sort of distanced, and after I left Peter and the more I worked, I tried to do physics and not the formalism. In particular he and Art Komar had very elaborate programs. But I felt the physics was missing.

Status of Hamiltonian diffeomorphism covariance in the 1950's through the 1970's

We need to stress that in this era the relationship between Hamiltonian constraints and diffeomorphism symmetry was either misunderstood or in some cases, outright rejected. Kuchar, for example, still maintained in the mid-1980's that the full four-dimensional diffeomorphism group was not realizeable as a canonical transformation group. One consequence of this lack of understanding was the earlier naive form of the "problem of time".

The problem of frozen time in the canonical approach

The naive classical argument for "frozen time" is based on the supposition that since a global rigid translation in time is a diffeomorphism, and physically meaningful objects must be invariant under this symmetry, then nothing can depend on the time. Time must be "frozen"!
As far as we can tell this expression first appeared in a Stevens meeting in 1959

Frozen time in the canonical approach

Dear Professor Dirac:

I have just studied your paper that appeared in the May 1 issue of the Physical Review. I am writing you, first to ask you for a reprint when they are available, but I should also like to make a few comments.

(1) The objections that Professor Lichnerowicz and I raised at the end of your lecture at Royaumont, whether or not they were valid then, certainly do not apply to the work that you have published here. Regardless of the motive of introducing the metric g_{rs} on the initial hypersurface, the canonical transformation that you first published a year ago to simplify and kill the primary constraints, is both legimate and sucessful. At this stage the total number of canonical field variables is reduced from twenty to twelve.

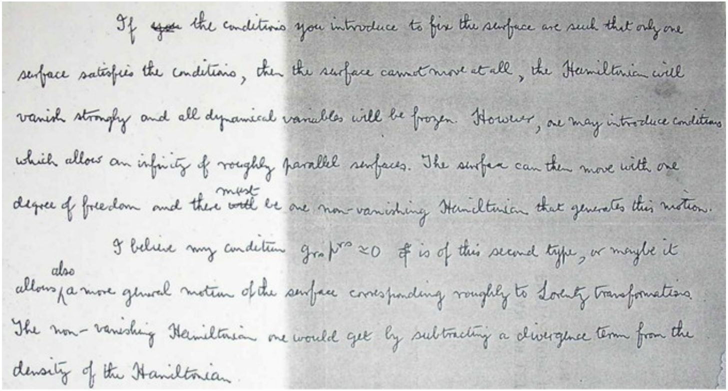
Letter from Bergmann to Dirac dated October 9, 1959.

Frozen time in the canonical approach

(3) When I discussed your paper at a Stevens conference yesterday, two more questions arose, which I should like to submit to you: To me it appeared that because you use the Hamiltonian constraint H_L to eliminate one of the non-substantive field variables, κ, in the final formulation of the theory your Hamiltonian vanishes strongly, and hence all the final field variables, i.e. $\tilde{e}^{rs}$ $\tilde{p}^{rs}$, are "frozen" (constants of the motion). I should not consider that as a source of embarrassment, but Jim Anderson says that in talking to you he found that you now look at the situation a bit differently. Could you enlighten me? If you have no objection, I should communicate your reply to Anderson and a few other participants in the discussion.

Continuation of letter from Bergmann to Dirac dated October 9, 1959.

Frozen time in the canonical approach



If ~~you~~ the conditions you introduce to fix the surface are such that only one surface satisfies the conditions, then the surface cannot move at all, the Hamiltonian will vanish strongly and all dynamical variables will be frozen. However, one may introduce conditions which allow an infinity of roughly parallel surfaces. The surface can then move with one degree of freedom and there ~~will~~ must be one non-vanishing Hamiltonian that generates this motion.

I believe my condition $g_{rs} p^{rs} \approx 0$ is of this second type, or maybe it also allows a more general motion of the surface corresponding roughly to Lorentz transformations. The non-vanishing Hamiltonian one would get by subtracting a divergence term from the density of the Hamiltonian.

Dirac's response dated November 11, 1959.

Frozen time in the Hamilton-Jacobi approach

Komar and Bergmann also concluded that time is frozen and true diffeomorphism invariants must be time-independent in papers published in the 1960's and early 1970's on the Hamilton-Jacobi approach [Ber66, Ber71, Kom67, Kom68, Kom71] We shall see that this issue is resolved through the recognition that time evolution and gauge symmetry are distinct in canonical gravity. See the discussion in the most recent edition of Kiefer's textbook on quantum gravity.

Recent progress in canonical gravity

This is part of the new light that has been shed on most of the issues we have just addressed in a series of papers by Pons, Salisbury, Shepley, and Sundermeyer. [PSS97, PS05, PSS09a] In particular, this work has shown that a form of full four-dimensional diffeomorphism covariance is retained in the constrained Hamiltonian dynamical treatment of general relativity. And most importantly for our present discussion, this group can be employed to construct time-dependent diffeomorphism invariants.

6. A BRIDGE BETWEEN SPACETIME AND PHASE SPACE FORMULATIONS OF GENERAL RELATIVITY

The realization of the full diffeomorphism group in phase space

A misunderstanding of the role of constraints has led to many of the errors in both the formulation and in the interpretation of the Wheeler-DeWitt equation. We will show that the appropriate phase space formalism that will admit the incorporation of the quantum of action and retain the full four-dimensional diffeomorphism symmetry is a formalism that retains the lapse and shift as configuration variables. This will in turn yield a fruitful fully covariant semi-classical approach to quantum gravity.

The diffeomorphism-induced canonical transformation group

- Global translations in time (time evolution) are not realizeable in general relativity as a canonical phase space transformations. This is commonly known - though not fully appreciated - as the decomposition of infinitesimal diffeomorphims into hypersurface tangential and perpendicular transformations

$$\delta x^\mu = \delta^\mu_a \xi^a + n^\mu \xi^0.$$

- The notion of "multi-fingered" time was introduced by Kuchar in 1972 [Kuc72] before it was understood that the full 4-diffeomorphism-induced group could be realized as a canonical transformation group.

The diffeomorphism-induced canonical transformation group

Global rigid translation in time is generated in a fixed gauge by the Rosenfeld-Bergmann-Dirac Hamiltonian (also known as the ADM Hamiltonian)

$$H_{RBD} = \int d^3x \left(N^\mu \mathcal{H}_\mu + \lambda^\mu \pi_\mu \right),$$

where the λ^μ are spacetime functions, related via the Hamiltonian equations of motion to time rates of change of the lapse and shift,

$$\frac{\partial N^\mu}{\partial t} = \lambda^\mu .$$

The diffeomorphism-induced canonical transformation group

General infinitesimal diffeomorphism-induced transformations of the full 4-metric and conjugate momenta are generated by

$$G_\xi(t) = \int d^3x \left(P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + N^\rho C^\nu_{\mu\rho} P_\nu)\xi^\mu \right).$$

Taking into account the time-dependence of this generator, a standard calculation demonstrates that even though the spacetime functions λ^μ in the H_{RBD} Hamiltonian are not dependent on the phase space variables, the formalism yields the correct variation of these functions under an arbitrary infinitesimal four-dimensional diffeomorphism. In other words, the phase space formalism (retaining lapse and shift as canonical phase space variables) is fully covariant under arbitrary time coordinate foliations.

The diffeomorphism-induced canonical transformation group

Thus the Hamiltonian and true Hamilton-Jacobi formalism is covariant under the full four-dimensional diffeomorphism group.

This leads us to reconsider the Hamilton-Jacobi approach within the framework of the enlarged phase space.

As a first step we derive what we call the true Hamilton-Jacobi equation which is fully covariant under four-dimensional diffeomorphism-induced canonical transformations.

The true Hamilton-Jacobi approach

This calculation does not appear in the cited literature: One finds that the gravitational action varies in the following manner under independent variations $\delta_0 g_{\mu\nu}$ of the metric and ϵ^μ of the spacetime coordinates

$$\delta S = \int d^3x \left[p^{ab}\delta_0 g_{ab} + \pi_\mu \delta_0 N^\mu \right.$$

$$\left. + \left(p^{ab}\dot{g}_{ab} - \mathcal{L} + \pi_\mu \dot{N}^\mu \right) \epsilon^0 + \left(-p^{ab}_{|c} g_{ab} - \pi_{0,c} N - \pi_{a,c} N^a \right) \epsilon^c . \right]$$

The true Hamilton-Jacobi approach

Thus the "true" Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + H_{RBD}\left[g_{ab}, N^{\mu}, \frac{\delta S}{\delta g_{ab}}, \frac{\delta S}{\delta N^{\mu}}, \lambda(x)\right] = 0,$$

where

$$H_{RBD} = \int d^3x \left(N^{\mu}\mathcal{H}_{\mu}(g_{ab}, p^{ab}) + \lambda^{\mu}(x)\pi_{\mu}\right)$$

is the vanishing Rosenfeld-Bergmann-Dirac Hamiltonian generator of time evolution.
Therefore $\frac{\partial S}{\partial t}$ vanishes.

Comparison with the standard Hamilton-Jacobi approach

We have called this the "true" or "proper" Hamilton-Jacobi equation to distinguish it from the so-called Einstein-Hamilton-Jacobi equation of Peres (the forerunner of the Wheeler-DeWitt equation)

$$\mathcal{H}_0\left(g_{ab}, \frac{\delta S}{\delta g_{ab}}\right) = 0.$$

How do these two approaches compare?

Pros and cons of the standard H-J approach

One advantage of the standard approach is that it avoids an off-shell problem that occurs if one does not automatically satisfy the constraints.

But a further implication is that it does not access all possible coordinate gauge choices, limiting as a consequence the choice of spacetime foliations. This also suggests a preferred role of the spatial metric.

In its role as a classical principal function the standard principal function cannot generate all possible solutions of Einstein's equations due to the fact the constraints are satisfied identically.

Pros and cons of the true H-J approach

The key advantage of the true approach is that at least at the classical level it generates solutions of Einstein's equations in all possible gauges, and hence does not favor any special time foliation.

But this presents a problem in its use in a semi-classical approach to quantum gravity since the gauge choices are contained in factors multiplying the constraints.

Ways to salvage the true H-J approach in semiclassical gravity

The off-shell problem is well known in approaches to quantum gravity, most notably in string-membrane theory in which symmetry generators do not annihilate quantum states.

Hence one might try to satisfy the on-shell conditions only as expectation values. This would offer an approach in which one encounters fluctuations around classical temporal and spatial coordinates.

Alternatively, one might impose quantum conditions on classical solutions as in the Sommerfeld-Epstein approach.

Ways to salvage the true H-J approach in semiclassical gravity

But here we propose a different approach inspired by the limited successes of the standard approach which may be interpreted as having made implicitly a choice of intrinsic coordinates that follows from the satisfaction of constraints.

We thus develop a program of introducing general intrinsic coordinates in order to maintain the full diffeomorphism covariance of the true Hamilton-Jacobi equation.

7. INTRINSIC COORDINATES AND A HAMILTON-JACOBI APPROACH TO SEMICLASSICAL QUANTUM GRAVITY

Origins and history of intrinsic coordinates

Einstein himself encountered the problem of the physical significance of coordinates when he tried to justify a non-covariant precursor theory of General Relativity by means of his famous "hole argument." In resolving this hole argument he recognized that only spacetime event coincidences (produced by dynamical fields) possess physical meaning. This seemed to suggest that coordinates cannot have any physical meaning. This perspective changed, however, when it became clear that the gravitational field itself acts as a dynamical field giving rise to such spacetime coincidences.

Origins and history of intrinsic coordinates

- Several authors in the 1960's and 1970's proposed that spacetime correlations could be established with the aid of spacetime scalars constructed with dynamical fields.
- Komar and Bergmann showed in 1960 that Weyl scalars could be written exclusively in terms of the 3-metric and conjugate momenta. [BK60] Thus the possibility arose that a purely gravitational phase space approach to intrinsic coordinates could be realized.

Definition of intrinsic coordinates

Intrinsic coordinates are function[al]s of dynamical phase space variables of the theory that are identified as measures of time and space. The dynamics can then be understood as correlations between the values of these function[al]s and the remaining dynamical variables of the theory.

Intrinsic variables then qualify as “clocks” and “rods” in the context of general relativity since it will in principle be possible to show at least locally that the “clocks” define a Cauchy surface. And in addition the corresponding metric will define a spacelike measure on the constant intrinsic time foliations.

Construction of diffeomorphism invariants through the finite action of the diffeomorphism group

- The diffeomorphism group can now be deployed to construct diffeomorphism invariants using intrinsic coordinates as follows
  - Choose appropriately behaved spacetime scalar functions of the phase space variables as intrinsic coordinates $X^{\mu}[g, p]$, with gauge condition $x^{\mu} = X^{\mu}[g, p]$.
  - Find the finite symmetry transformation that transforms solutions in any coordinate system to solutions that satisfy the gauge conditions.
- Note that all resulting phase space functionals - including the lapse and shift - are diffeomorphism invariants

Summary of key observations

Let us summarize the key elements that will enter into our general proposal

- Full canonical phase space diffeomorphism-induced symmetry
- A true Hamilton-Jacobi equation
- Intrinsic coordinates

The general proposal

Finally we give our general proposal for constructing a semi-classical limit of quantum gravity. It consists of the following steps

- Identify appropriately behaved functionals of the dynamical fields that can serve as intrinsic coordinates.
- Determine the corresponding complete Hamilton-Jacobi principal function S
- Employ this principle function in an appropriate semiclassical approach to quantum gravity

Examples

We now illustration the implementation of this method with examples.

Free relativistic particle example

Choosing $\theta = t/c$ as the intrinsic coordinate parameter we obtain the following invariants (under arbitrary reparameterizations of θ, [PSS09b]

$$x^a = x^a(\theta) - \frac{p^a}{p^0} x^0(\theta) + \frac{p^a}{p^0} t,$$

where the momenta p^μ, being independent of θ in any gauge are already invariants.

We witness here a general feature of the invariants that are determined in this manner. They will generally consist of coefficients built of invariant functionals of the dynamical fields, appearing in power series of the intrinsic coordinates.

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Evolving constants of the motion

This expansion is closely related to Rovelli's notion of "evolving constants of the motion." [Rov90] The coefficients of the powers of intrinsic time are constant functions of phase space variables.

Free relativistic particle example

It also happens to be true in this case that we can make this power series substitution directly into the principal function solution of the true Hamilton-Jacobi equation. This is true because of our good fortune in selecting an intrinsic coordinate that happens to be a canonical dynamical variable!

A mini-superspace example

This good fortune continues to hold in mini-superspace isotropic expansion models where either the expansion factor a or matter fields can serve as intrinsic time. This is the reason why such a time can be isolated in the Wheeler-DeWitt equation.

This leads us to a somewhat less trivial example - Einstein-Rosen cylindrical waves.

A midisuperspace model

We consider an Einstein-Rosen spacetime with metric of the form

$$g_{\mu\nu} = \begin{pmatrix} -n^2 + e^{\gamma-\psi} m^2 & e^{\gamma-\psi} m & 0 & 0 \\ e^{\gamma-\psi} m & e^{\gamma-\psi} & 0 & 0 \\ 0 & 0 & e^{-\psi} R^2 & 0 \\ 0 & 0 & 0 & e^{\psi} \end{pmatrix}$$

where the lapse function n, the shift m, and the scalar fields R and ψ are functions of the coordinate time t and the radial coordinate r. We assume no dependence on the azimuthal angle ϕ or on the cylindrical coordinate z.

Kuchar canonical transformations

First make change in canonical variables (first introduced by Kuchar [Kuc71])

$$T = -\int_{\infty}^{r} d\bar{r}\bar{\pi}_{\gamma},$$

$$p_R = \pi_R + \left(\ln\left(\frac{R' - \pi_\gamma}{R' + \pi_\gamma}\right)\right)',$$

$$p_T = -\gamma' + \left(\ln\left(R'^2 - \pi_\gamma^2\right)\right)'.$$

Einstein-Rosen constraints and intrinsic coordinates

Then we choose intrinsic coordinates $t = T$ and $r = R$.

Note that unless this transformation is undertaken before the corresponding Wheeler-DeWitt equation is written down and solved the Wheeler-DeWitt equation cannot access these choices of intrinsic time and position.

Invariant ψ field

Just to frighten you we write down the expression for the invariant scalar field ψ, constructed using the finite diffeomorphism transformation:

$$\mathcal{I}_\psi \approx exp\left(\{-, \bar{\xi}^I \overline{\mathcal{H}}_I\}\right)\psi\Big|_{\bar{\xi}=\chi} = \psi + \chi^I\{\psi, \overline{\mathcal{H}}_I\}$$

$$+\frac{1}{2!}\chi^I\chi^J\{\{\psi, \overline{\mathcal{H}}_I\}, \overline{\mathcal{H}}_J\} + \frac{1}{3!}\chi^I\chi^J\chi^K\{\{\{\psi, \overline{\mathcal{H}}_I\}, \overline{\mathcal{H}}_J\}, \overline{\mathcal{H}}_K\} + \dots$$

$$=: \sum_{n=0}^{\infty}\frac{1}{n!}\chi^n\{\psi, \overline{\mathcal{H}}\}_{(n)}\,.$$

where $\chi^0 := t - T$ and $\chi^1 := r - R$. The construction employs a trick that was first introduced by Dittrich. [Dit06]

Hamilton-Jacobi equation

ψ can profitably be expanded in terms of Bessel functions in this gauge as

$$\psi(t,r) = \int_0^\infty d\omega A_\omega(t) J_0\left(\frac{\omega}{c} r\right).$$

Then the complete Hamilton principal function takes the form

$$S[A_\omega, t] = \int_0^\infty d\omega W(A_\omega) - Et,$$

with

$$W(A_\omega) = \int^{A_\omega} \left(2E(\omega) - A_\omega'^2\right)^{1/2} dA_\omega'.$$

Hamilton-Jacobi and functional Schrödinger equation

The Hamilton-Jacobi equation is

$$\int_0^\infty d\omega\omega \left(A_\omega^2 + \frac{\delta S}{\delta A_\omega}\frac{\delta S}{\delta A_\omega} \right) + \frac{\partial S}{\partial t} = 0,$$

with a corresponding functional Schrödinger equation

$$\left(\int_0^\infty d\omega\omega \left[-\hbar^2 \frac{\delta^2}{\delta A_\omega^2} + A_\omega^2 \right] \right) \Psi = i\hbar \frac{\partial}{\partial t}\Psi.$$

Spacetime quantization

This equation can be solved exactly since it describes independent oscillator modes whose quantization yields discrete excitations in ψ.

In addition, the timelike component of the original Einstein-Rosen metric also depends on this ψ and we obtain therefore a fluctuating light cone.

These results are consistent with the work of Ashtekar and Pierri in 1996 which were obtained without reference to intrinsic time. [AP96]

Spacetime quantization

Finally, we note that in the generic case it will not be possible to isolate intrinsic coordinates that can be obtained through a canonical phase space transformation. This means that these generic choices will generally be inaccessible via the Wheeler-DeWitt equation.

We also observe that the notion of superspace itself is questionable. It is by no means certain from the point of view of intrinsic coordinates that the isolation of an independent complete set (an enterprise on which we have not engaged ourselves) would yield a categorization of diffeomorphically inequivalent spaces, nor that the spatial metric is the most suitable variable to employ in describing such equivalence classes.

- Abhay Ashtekar and M. Pierri, *Probing quantum gravity through exactly soluble midi-superspaces I*, Journal of Mathematical Physics **37** (1996), 6250–6270.

- Peter G. Bergmann, *Non-linear field theories*, Physical Review **75** (1949), 680 – 685.

- ______, *Hamilton-Jacobi and Schrödinger theory in theories with first-class Hamiltonian constraints*, Physical Review **144** (1966), no. 4, 1078–1080.

- ______, *Hamilton-Jacob1 theory with mixed constraints*, Annals of the New York Academy of Sciences **172** (1971), 571–579.

- Peter G. Bergmann and Arthur B. Komar, *Poisson brackets between locally defined observables in general relativity*, Physical Review Letters **4** (1960), no. 8, 432–433.

- Bryce S. DeWitt, *The quantum and gravity: The Wheeler-DeWitt equation*, Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories (Tsvi Piran and Remo Ruffini, eds.), World Scientific Publishers, 1999.
- ______, *Quantum gravity yesterday and today*, General Relativity and Gravitation **41** (2009), 413–419.
- P. A. M. Dirac, *Generalized Hamiltonian dynamics*, Canadian Journal of Mathematics **2** (1950), 129 – 148.
- ______, *The Hamiltonian form of field dynamics*, Canadian Journal of Mathematics **3** (1951), 1 – 23.
- B. Dittrich, *Partial and complete observables for canonical general relativity*, Classical and Quantum Gravity **23** (2006), 6155–6184.

- Arthur B. Komar, *Hamilton-jacobi quantization of general relativity*, Physical Review **153** (1967), no. 5, 1385–1387.
- ______, *Hamilton-Jacobi version of general relativity*, Physical Review **170** (1968), no. 5, 1195–1200.
- ______, *General-relativistic observables via Hamilton-Jacobi functionals*, Physical Review D **4** (1971), no. 4, 923–927.
- Karel V. Kuchař, *Canonical quantization of cylindrical gravitational waves*, Physical Review D **4** (1971), no. 4, 955–986.
- ______, *A bubble-time canonical formalism for geometrodynamics*, Journal of Mathematical Physics **13** (1972), no. 5, 768–781.
- A. Peres, *On Cauchy's problem in general relativity - II.*, Il Nuovo Cimento **26** (1962), 53–62.

Josep Pons and Donald Salisbury, *The issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity*, Physical Review D **71** (2005), 124012.

Josep Pons, Donald Salisbury, and Lawrence Shepley, *Gauge transformations in the Lagrangian and Hamiltonian formalisms of generally covariant theories*, Physical Review D **55** (1997), 658–668.

Josep Pons, Donald Salisbury, and Kurt Sundermeyer, *Gravitational observables, intrinsic coordinates, and canonical maps*, Modern Physics Letters A **24** (2009), 725–732.

———, *Revisiting observables in generally covariant theories in light of gauge fixing methods*, Physical Review D **80** (2009), 084015–1–084015–23.

- Léon Rosenfeld, *Zur Quantelung der Wellenfelder*, Annalen der Physik **5** (1930), 113 – 152.
- Carlo Rovelli, *Quantum mechanics without time: a model*, Physical Review D **42** (1990), no. 8, 2638–2646.
- Donald Salisbury, *Translation and commentary of Léon Rosenfeld's "Zur Quantelung der Wellenfelder", Annalen der Physik 397, 113 (1930), Max Planck Institute for the History of Science Preprint 381, http://www.mpiwg-berlin.mpg.de/en/resources/preprints.html*, November 2009.
- Paul Weiss, *On quantization of a theory arising from a variational principle for multiple integrals with applications to Born's electrodynamics*, Proceedings of the Royal Society of London **156** (1936), no. 887, 192–220.

———, *On the Hamilton-Jacobi theory and quantization of a dynamical continuum*, Proceedings of the Royal Society of London **169** (1938), no. 936, 102–119.