Asymptotic safety on an infinite-dimensional coupling space

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The Asymptotic Safety scenario

UV non-trivial fixed point $\Rightarrow$ Self completion

- The theory can be extended all the way up to arbitrarily high energy

  $\longrightarrow$ the fixed point provides the UV-completion

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Number of relevant directions $< \infty \Rightarrow$ Predictivity

- Continuum limit is characterized by only a finite number of free parameters

  $\rightarrow$ only relevant directions correspond to couplings to be fixed by experiments

Gaussian fixed point

- Beta functions: tree level plus quantum corrections

  In the classical limit couplings are constant
  $\Rightarrow$ dimensionless couplings run in a trivial way at tree level

$$\tilde{u}_i = k^{-d_i} u_i \quad \Rightarrow \quad \beta_i(\tilde{u}) = \underbrace{-d_i \tilde{u}_i}_{\text{tree level}} + \underbrace{O(\tilde{u}^2)}_{\text{quantum corrections}}$$

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- At the trivial FP:

  1. Critical exponents $= d_i$ $\quad$ ($= -$eigenvalues of Jacobian B_{ij} of β_i at FP)

     $\Rightarrow$ Real exponents

  2. Relevant directions = renormalizable interactions

     $\Rightarrow$ Finitely many relevant couplings

Non-trivial fixed points

- For non-trivial FP we need real zeros of a function from its truncated series expansion (usually at best only asymptotic series)

$$\beta(\tilde{u}) = -d\tilde{u} + b_1\tilde{u}^2 + b_2\tilde{u}^3 + \ldots$$

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$\Rightarrow$ Use FRGE

The FRGE [Wilson; Wegner,Houghton; Polchinski; Wetterich; ...]

- A powerful tool for studying the RG flow is the Functional Renormalization Group Equation (FRGE) for the **effective average action** Γ_k

$$k\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta\Phi\delta\Phi} + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k\right]$$

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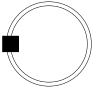
  - Diagrammatically:

$k\partial_k \Gamma_k[\Phi] = \frac{1}{2}$ 

  - Structure of 1-loop equation, but "improved" by presence of full propagator $\Rightarrow$ functional differential equation for Γ_k

Approximation scheme

- Truncate the theory space
  (no small parameter expansion, full k-dependence on rhs):
  - $\Gamma_k[\Phi]$ has the general form

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Truncation: retain only a finite number of terms

$\Rightarrow$ obtain the β-functions by substituting truncation into FRGE and computing the coefficients of the operators $\mathcal{O}_i[\Phi]$

$$k\partial_k \Gamma_k[\Phi]_{\text{Truncated}} \equiv \sum_i^N k\partial_k u_i(k)\, \mathcal{O}_i[\Phi] = \sum_i^{-N-} b_i(u)\, \mathcal{O}_i[\Phi]$$

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  - Obtain $\beta_i(\tilde{u})$: non-polynomial functions of the couplings

Truncations in pure gravity

Einstein-Hilbert truncation

polynomial $f(R)$-truncation

$R^2 + C^2$-truncation

$\vdots$			
R^8	$\dots$		
R^7	$\dots$		
R^6	$\dots$		
R^5	$\dots$		
R^4	$\dots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	$+$ 7 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

[Souma '99; Reuter,Lauscher '01; Litim '03; ... Codello,Percacci,Rahmede '07; Machado,Saueressig '07; Falls,Litim,Nikolakopoulos,Rahmede '13; DB, Machado, Saueressig '09]

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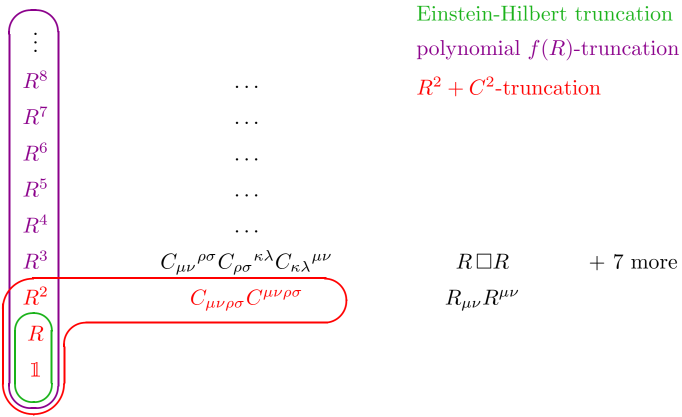

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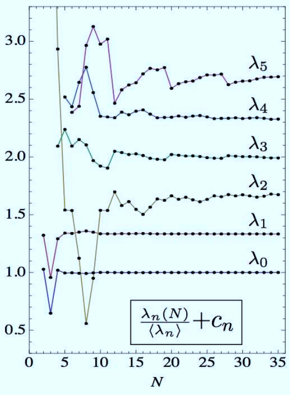
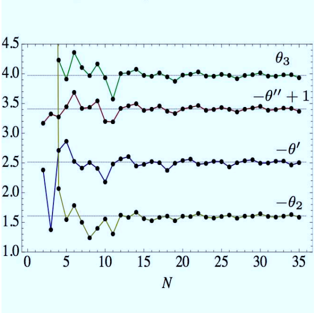
⇒ Always a fixed point with three relevant directions

Polynomial $f(R)$ truncations: $\Gamma_k = \int d^4x \sqrt{g} \sum_{i=0}^{n} u_i R^i$

Latest results: $n = 34$ [Falls,Litim,Nikolakopoulos,Rahmede, 1301.4191]





($\lambda_i = k^{2i-4} u_i$)

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All these questions suggest to look for approximation schemes which retain an infinite number of terms
$\Rightarrow$ restore the word "functional" in the FRGE

A new expansion scheme:
going beyond polynomial truncations

Derivative expansion and local potential approximation

Scalar field theory

- The derivative expansion consists in writing $\Gamma_k[\phi]$ as

$$\Gamma_k[\phi] = \int d^dx \Big[V_k(\phi) + Z_k(\phi)\partial^\mu\phi\partial_\mu\phi + W_k^a(\phi)(\partial^2\phi)^2 + W_k^b(\phi)\partial^\mu\phi\partial_\mu\phi(\phi\partial^2\phi) + W_k^c(\phi)(\partial^\mu\phi\partial_\mu\phi)^2 + O(\partial^6)\Big]$$

with $V_k(\phi)$, $Z_k(\phi)$, etc. generic unknown functions

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- The FRGE results in a partial differential equation for the **dimensionless** potential, e.g. [Morris '94]

$$k\partial_k\tilde{V}_k(\tilde{\phi}) + (\frac{d}{2} - 1)\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) - d\,\tilde{V}_k(\tilde{\phi}) = \ln\big(1 + \tilde{V}''(\tilde{\phi})\big)$$

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- Fixed point: $k\partial_k \tilde{V}_k(\tilde{\phi}) = 0 \Rightarrow$ PDE reduces to ODE

LPA vs truncations

Compare to truncations: $V_k(\phi) = \sum_{n=0}^{N} u_{2n}(k)\phi^{2n}$

	truncations	LPA
flow	ODE	PDE
fixed points	algebraic	ODE

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  [Hasenfratz&Hasenfratz; Felder; Morris; ...]

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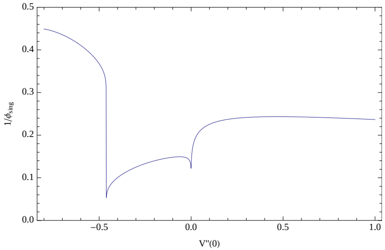
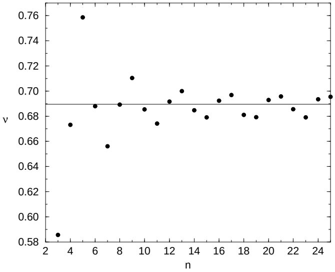


LPA vs truncations

- Derivative expansion typically gives more reliable results than truncations

  In particular, truncations generically stop converging after a certain order: [Morris]



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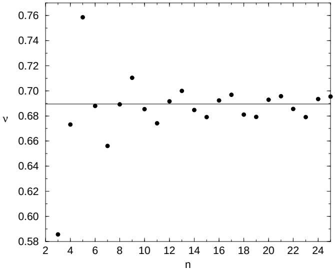

- It provides a criterion for discerning true fixed points from spurious ones

  Increasing order of truncations, more and more fixed points are usually found.
  (In this case we look for FP in similar location as that found in previous truncation)
  On the contrary, global solutions of the FP ODE are usually rare.

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where

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \,, \quad C_{\mu\nu\rho\sigma} = \text{Weyl tensor}$$

- Express action in terms of irreducible components and their derivatives:

$$\Gamma[g] = \int d^d x \sqrt{g} \, \{R + \ldots + R^n + \ldots + R\nabla^2 R + C^3 + S^4 + C^2 S^2 + \ldots\}$$

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- Maximally symmetric spacetime: $\nabla_\mu R = S_{\mu\nu} = C_{\mu\nu\rho\sigma} = 0$

$$\Gamma[g] = \int d^d x \sqrt{g}\, \{f(R) + \text{things which are zero for m.s.s.}\}$$

$\Rightarrow f_k(R) = \Gamma_k[g = \text{max.sym.}]/\text{Volume}$

$f(R)$ functional RG

- The ansatz is

$$\Gamma_k = \int d^d x \sqrt{g} f_k(R) = k^d \int d^d x \sqrt{g} \tilde{f}_k(\tilde{R})$$

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- Plug into FRGE.

(Use, transverse-traceless decomposition, a smart gauge fixing and other standard steps...)

Fixed-point equation

- The FP equation in the $f(R)$ approximation reads

$$\tilde{V}\left(4\tilde{f}_k(\tilde{R}) - 2\tilde{R}\tilde{f}_k'(\tilde{R})\right) = \mathcal{T}_2 + \mathcal{T}_1^{\mathsf{Jac}} + \mathcal{T}_0^{\mathsf{Jac}} + \mathcal{T}_0^{\bar{h}},$$

where $\tilde{V} = k^4 \int d^4x\sqrt{g}$, and ($\Delta_s$ is a Laplacian operator acting on spin s fields)

$$\mathcal{T}_2 = \mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^T}{-f'(R)\Delta_2 - f(R) + \frac{1}{2}Rf'(R) + 2\mathcal{R}_k^T}\right],$$

$$\mathcal{T}_1^{\mathsf{Jac}} = -\frac{1}{2}\mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^V}{\Delta_1 + \mathcal{R}_k^V(\Delta_1 + \alpha_1 R)}\right],$$

$$\mathcal{T}_0^{\mathsf{Jac}} = \frac{1}{2}\mathrm{Tr}\left[\frac{\frac{d}{dt}\mathcal{R}_k^{S_1}}{\Delta_0 + \frac{R}{3} + \mathcal{R}_k^{S_1}}\right] - \mathrm{Tr}\left[\frac{2\frac{d}{dt}\mathcal{R}_k^{S_2}}{(3\Delta_0 + R)\,\Delta_0 + 4\mathcal{R}_k^{S_2}}\right],$$

$$\mathcal{T}_0^{\bar{h}} = \mathrm{Tr}\left[\frac{8\frac{d}{dt}\mathcal{R}_k^{\bar{h}}}{9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + 2f(R) - Rf'(R) + 16\mathcal{R}_k^{\bar{h}}}\right].$$

The role of the cutoff function

- The cutoff function $\mathcal{R}_k(z)$ implements the coarse graining operation, by suppressing the integration over low modes in the path integral

$$\int_\Lambda \mathcal{D}\phi \, \exp\left\{-S_\Lambda[\phi] - \frac{1}{2}\int \phi\, \mathcal{R}_k(-\nabla^2/k^2)\, \phi\right\}$$

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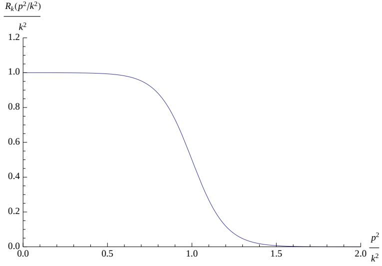
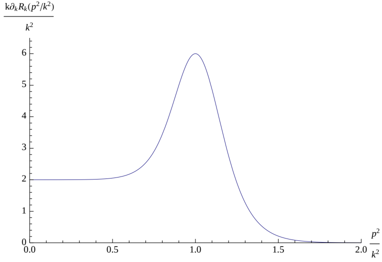
- In practice, it regularizes the FRGE in the UV $\rightarrow$ we can forget about Λ





A good cutoff

- A good cutoff $\mathcal{R}_k(p^2/k^2)$ is one that efficiently implements the coarse-graining picture, or more precisely, one that satisfies:

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5. $\lim_{k \to 0} \mathcal{R}_k(p^2/k^2) = 0$ at fixed p^2
(i.e. removal of cutoff)

A good cutoff

- A good cutoff $\mathcal{R}_k(p^2/k^2)$ is one that efficiently implements the coarse-graining picture, or more precisely, one that satisfies:

1. $\mathcal{R}_k(z) \geq 0$ for all z and $k \geq 0$
(i.e. suppress modes or do nothing)

2. $\mathcal{R}_k(p^2/k^2)$ is monotonically decreasing in p^2, or $\partial_z \mathcal{R}_k(z) \leq 0$
(i.e. low modes suppressed more than high modes)

3. $\lim_{p^2 \to 0} \mathcal{R}_k(p^2/k^2) > 0$, for $k > 0$
(i.e. $\mathcal{R}_k(z) \equiv 0$ is not good)

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Note: Last property implies $\lim_{k \to 0} \Gamma_k = \Gamma$, the full effective action

Good cutoff and large-$\tilde{R}$ behaviour

- Choose the simple cutoff form $\mathcal{R}_k(\Delta_s) = k^m \, r(\frac{\Delta_s + \alpha_s R}{k^2})$, where

  $r(z)$ is a dimensionless profile function, identical for all the fields,
  m is given by the dimension of the Hessian to which it is associated

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- Subleading corrections can be computed once a cutoff is chosen. We find

$$\tilde{f}(\tilde{R}) \sim A\tilde{R}^2 \left(1 + \sum_{n\geq 1} d_n \tilde{R}^{-n}\right)$$

  where $d_n = d_n(A) \Rightarrow$ only one free parameter in asymptotic expansion

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- It takes time...

preliminary results: [DB,Caravelli '12; Demmel,Saueressig,Zanusso (3d) '12; Dietz,Morris '12]
under way: [DB,Dietz,Morris; DB,Codello,Percacci]

However we can already deduce some generic properties of fixed points in gravity

Fixed-point action

"Theorem 1"

If a global fixed-point solution $\tilde{f}^*(\tilde{R})$ exists $\Rightarrow \Gamma^* = \Gamma^*_{k=0} = A^* \int d^4x \sqrt{g} R^2$

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  $\tilde{f}^*(\tilde{R})$ has no k-dependence $\Rightarrow$ limit $k \to 0$ at constant R corresponds to $\tilde{R} \to \infty$:

$$\Gamma^*_k = k^4 \int d^4x\sqrt{g}\,\tilde{f}^*(R/k^2) \simeq k^4 \int d^4x\sqrt{g}\,(A^*\frac{R^2}{k^4} + O(\frac{1}{k^2}))$$

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- Advantage of functional truncation: impossible to see this from polynomial truncations (which give non-trivial FP for $\tilde{R}^3$, $\tilde{R}^4$ and so on)
- Agreement with [Bonanno [1203.1962]; Hindmarsh, Saltas [1203.3957]]

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- Agreement with [Bonanno [1203.1962]; Hindmarsh, Saltas [1203.3957]]
- Physically not surprising: scale invariance $\to R^2$

  $\Rightarrow$ no anomalous scaling within $f(R)$ approximation ($\sim$ LPA)

  (reminder: in $d = 3$ scalar potential at the Wilson-Fisher FP is $V(\phi) = A^*\phi^{\frac{6}{1+\eta}}$
  and $\eta = 0$ in LPA, but $\eta \sim .03$ at higher orders)

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- If we also believe in
  "scale invariance $\Rightarrow$ conformal invariance"
  we are forced to interpret R^2 as Gauss-Bonnet topological term

$$E = C^2 - 6S^2 - \frac{1}{6}R^2; \qquad \int d^4x\sqrt{g}E = 32\pi^2\chi$$

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- We would also expect C^2 in a conformal invariant action
  $\Rightarrow$ open question about unitarity

Eigen-perturbations

- Linearization in the neighborhood of the fixed point ($t = \ln(k/\Lambda)$):

$$\tilde{f}_k(\tilde{R}) \sim \tilde{f}^*(\tilde{R}) + \epsilon\, v(\tilde{R}) e^{-\theta t}\,,$$

and expand the FRGE to linear order in ϵ

$\Rightarrow$ eigenvalue equation:

$$-a_2(\tilde{R}) v''(\tilde{R}) + a_1(\tilde{R}) v'(\tilde{R}) + a_0(\tilde{R}) v(\tilde{R}) = (4-\theta)\, v(\tilde{R})\,.$$

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- For $\tilde{R} \to \infty$, a_0, $a_2 \to 0$ faster than power-law, while $a_1 \sim 2\tilde{R}$

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Renormalized trajectories

In first approximation, a general relevant trajectory leads (for $k \to 0$) to the action

$$\Gamma_k \to \int d^4x \sqrt{g} \{ AR^2 + \sum_{i \in \text{rel.dir.}} m_i^{\theta_i} R^{2-\theta_i/2} \}$$

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- Idea:
  $\tilde{f}_k(\tilde{R}) \sim \tilde{f}^*(\tilde{R}) + \epsilon\, v(\tilde{R})e^{-\theta t}$ $\Rightarrow$ infinitesimal ϵ ensures that at $t = 0$, i.e. at the initial scale $k = \Lambda$, $\tilde{f}_k(\tilde{R})$ is very close to the fixed point solution.
  Integrating towards $k = 0$ (in linear approximation, same proof as for FP)

  $$\Gamma_k \to \int d^4x\sqrt{g}\{AR^2 + \sum_i \epsilon_i\, \Lambda^{\theta_i} R^{2-\theta_i/2}\}$$

  Take $\Lambda \to \infty$ while keeping the action finite:
  positive $\theta \Rightarrow$ take $\epsilon \sim (m_\theta/\Lambda)^\theta$, for some finite mass parameter m_θ
  negative $\theta \Rightarrow$ the perturbations are automatically small in the large-Λ limit
  marginal perturbations with $\theta = 0 \Rightarrow$ go beyond the linear expansion

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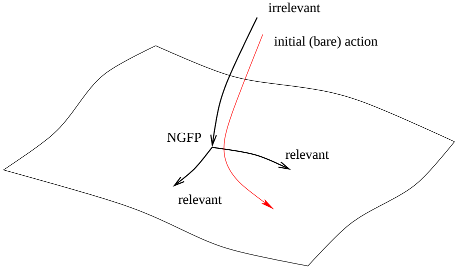

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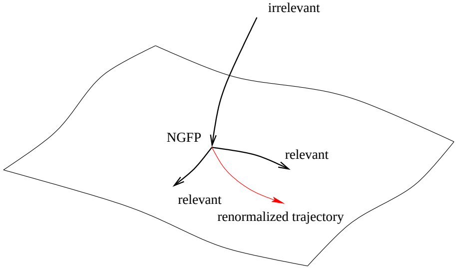

On the number of relevant directions

"Theorem 2"

Under the condition that a "good cutoff" is used,
If a global fixed-point solution $\tilde{f}^*(\tilde{R})$ exists,

$\Rightarrow$ the critical exponents are real,
they form a discrete spectrum,
and the number of relevant directions is finite

Proof

$$\hat{L}\, v \equiv -a_2(\tilde{R}) v''(\tilde{R}) + a_1(\tilde{R}) v'(\tilde{R}) + a_0(\tilde{R}) v(\tilde{R}) = (4-\theta)\, v(\tilde{R})$$

- Because $r(z) > 0$ and $r'(z) = dr(z)/dz < 0$,

$$a_2 = \frac{144}{\tilde{V}}\,\mathrm{Tr}\left[\frac{\Delta_0^2\,(2\,r - \Delta_0\, r')}{(9f^{*\prime\prime}(R)\Delta_0^2 + 3f^{*\prime}(R)\Delta_0 + E^*(R) + 16c_{\bar{h}} r)^2}\right] > 0$$

- a_0, a_1 and a_2 are non singular (singularity would imply singularity of $\tilde{f}^*(\tilde{R})$)

  $\Rightarrow$ "kinetic term" is written with the right sign, and equation is not singular

Proof

$$\hat{L}\,v \equiv -a_2(\tilde{R})v''(\tilde{R}) + a_1(\tilde{R})v'(\tilde{R}) + a_0(\tilde{R})v(\tilde{R}) = (4-\theta)\,v(\tilde{R})$$

- Boundary conditions are provided by the requirement that the asymptotic behavior be power-law. This is equivalent to requiring square integrable solutions with respect to the weight function $w(\tilde{R}) = a_2^{-1}\exp(-\int^{\tilde{R}}\frac{a_1}{a_2})$.
  Note: we can rewrite the operator as

$$\hat{L}v = -\frac{1}{w(\tilde{R})}\frac{d}{d\tilde{R}}\left(p(\tilde{R})\frac{d}{d\tilde{R}}v(\tilde{R})\right) + q(\tilde{R})v(\tilde{R})$$

$\Rightarrow$ Sturm-Liouville operator $\hat{L}$ is self-adjoint, hence its spectrum is real.

$$\begin{aligned}(u,\hat{L}v) &= \int_{-\infty}^{+\infty} dx\, w(x)u(x)\hat{L}v(x)\\ &= (u'pv - upv')\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} dx\, w(x)(\hat{L}u(x))v(x)\\ &= (\hat{L}u, v)\end{aligned}$$

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- Liouville transformation: new variable $x = \int^{\tilde{R}} 1/\sqrt{a_2}$ (with $\int^{+\infty} 1/\sqrt{a_2} = +\infty$), and $y = a_2^{1/4} w^{1/2} v$
  $\Rightarrow$ Schrödinger eigenvalue equation $-y''(x) + U(x)y(x) = \lambda\, y(x)$, with potential

$$U(x) = a_0 + \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2'\left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2}\right) - \frac{a_2''}{4}\,.$$

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  No singularities at finite x (from $a_2 > 0$ and of the absence of singularities in a_0 and a_1), and asymptotic behavior such that for $x \to \pm\infty$ the second term dominates, and $U(x) \to +\infty$.

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  $\Rightarrow$ The spectrum is discrete, bounded from below, and the only accumulation point is at $\lambda \equiv 4 - \theta = +\infty$.

  $\Rightarrow$ Finite number of eigen-perturbations with $\theta > 0$.

Conclusions and outlook

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- Polynomial truncations have provided substantial evidence in favour of the asymptotic safety scenario for gravity
- We just started moving on to a higher level of complexity: functional truncations
- Some important generic results can be inferred by an analysis of the $f(R)$-approximation
- Numerical integration is under way

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Some future directions:

- Existence of global solutions of the differential FP equation
- Bound number of relevant directions, and explain near-Gaussianity
- Next order of functional expansion: e.g. Einstein spacetime: $\nabla_\mu R = S_{\mu\nu} = 0$, $C_{\mu\nu\rho\sigma} \neq 0$:

$$\Gamma[g] = \int d^d x \sqrt{g} \left\{ f(R) + f_1(R) C^2 + O(C^3) \right\}$$

- Nonlocal actions and IR applications