Towards the continuum limit in spin foams: renormalization flow in background independent models

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[BD, 1205.6127, New J. Phys. '12]
[Bahr, BD, Hellmann, Kaminski, 1208.3388]

[BD, Laurie v. Massenbach, Martin-Benito, w.i.p.]

QG in Paris, March 2013

Overview

A. Spin (holonomy) foams

B. The method: tensor network renormalization

C. Coarse graining S-3 spin nets: first results!

D. Summary and wishful thinking

thanks to collaborations and discussions with and help from:

Benjamin Bahr, Valentin Bonzom, Frank Eckert, Frank Hellmann, Wojchiech Kaminski, Felix Laurie v. Massenbach, Etera Livine, Mercedes Martin-Benito, Arnau Riera, Erik Schnetter, Sebastian Steinhaus, Guifre Vidal, ...

What are spin foams?

- path integral approach related to loop quantum gravity

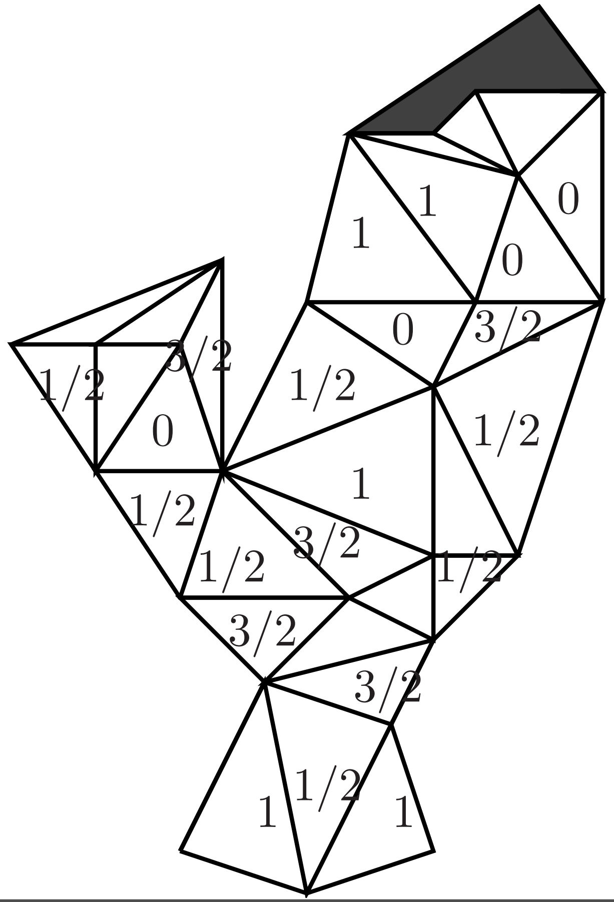
View point here:
Spin foams are a class of (quantum statistical) models, generalizing lattice gauge theories.

Nice result: Semi-classical limit with large building blocks reproduces Regge action.
[Barrett and many others for different models] [Recently: issues pointed out by Hellmann, Kaminski '12]

Open question: Continuum limit?
Do we obtain smooth 4D manifold on large scales?
Do we regain diffeomorphism symmetry (Lorentz invariance)?

What are spin foams?

- path integral approach to quantum gravity:
    - sum over geometric data associated to triangulations
    - first order action: geometry encoded in connection (group) variables
    - dual variables: spin (SU(2) representation) labels
    - models can be understood as generalized lattice gauge theory



3d and 4d actions

$$\text{geom} \sim \text{metric} \sim (\text{n-bein}\, e,\ \text{connection}\, A)$$

Plebanski action in 4d

curvature of A

$$S_{4d} = \int B \wedge F + \phi B \wedge B \quad , \quad B \sim \star(e \wedge e)$$

(Lie algebra valued) d-2 form

Lagrange multiplier

simplicity constraints

first order action in 3d

$$S_{3d} = \int B \wedge F \quad , \quad B \sim e$$

BF theory

$$S_{BF} = \int B \wedge F$$

$$F = 0 \quad , \quad D_A B = 0$$

topological field theory

Yang Mills in first order

$$S_{YM} = \int B \wedge F + g^2 B \wedge \star B$$

uses background metric

Lattice gauge theory

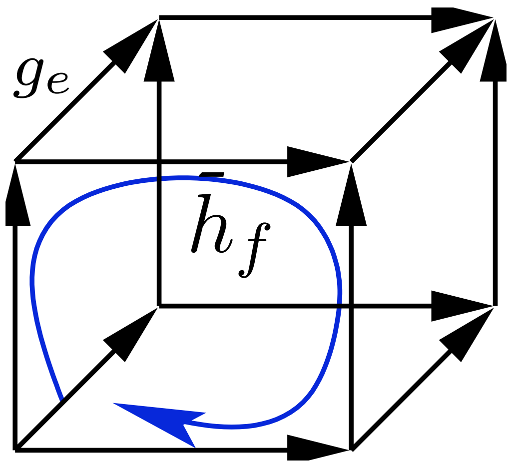
lattice gauge theory:

$$Z \sim \sum_{g_e} \prod_f w_f(h_f)$$

G-group variables at edges

face weight (class function)

face holonomy



standard lattice gauge theory: dynamics encoded in face weights

$$w_f(h) = \begin{cases} \delta_G(h) \\ \exp(-S_{YM}(h)) \\ \text{const.} \end{cases}$$

zero coupling
BF (topological) theory

needs lattice metric for construction (for instance heat kernel action)

strong coupling limit (degenerate phase /no geometry vacuum)

We have to introduce another principle to allow for non-flat face holonomies.

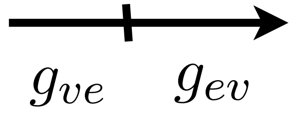
Spin foams as generalized lattice gauge theory

[Bahr, BD, Hellmann, Kaminski '12]

$$Z = \sum_{g_e} \prod_f \omega_f(g_e g_{e'} \cdots)$$

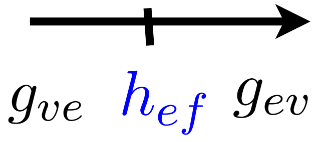
double variables

$$Z = \sum_{g_{ve}} \prod_f \omega_f(g_{ve} g_{ev'} g_{v'e'} g_{e'v''} \cdots)$$



insertions

$$Z' = \sum_{g_{ve}} \sum_{h_{ef}} \prod_{(ef)} E(h_{ef}) \prod_f \omega_f(g_{ve} h_{ef} g_{ev'} g_{v'e'} h_{e'f} g_{e'v''} \cdots)$$



Allows face holonomy $h_f = g_{ve} g_{ev'} g_{v'e'} g_{e'v''} \cdots$ to be non–flat even for $\omega_f = \delta_G$.

Choice of E-function determines the model (dynamics). Implements simplicity constraints.

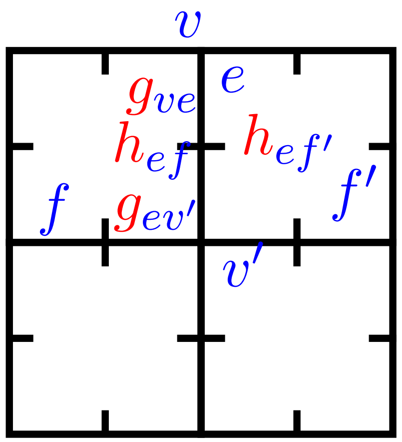
Almost all current spin foam models can be expressed in this way.

Dimensional reduction

4D lattice gauge theories ⇦⇨ 2D Ising like models
(gauge symmetry ⇨ global symmetry)

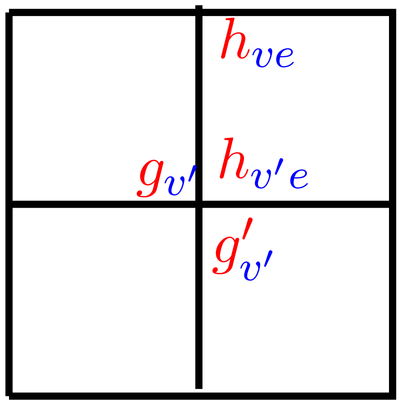
Spin foams → Spin nets

[BD, Eckert, Martin-Benito ’11]



→

generalizes correspondence between
4D lattice gauge theories and
2D Ising like models



associate
- to every edge e
  two group elements $g_{ve}, g_{ev'}$
- to every edge–face pair ef
  a group element h_{ef}

associate
- to every vertex v
  two group elements g_v, g'_v
- to every vertex–edge pair ve
  a group element h_{ve}

$$Z = \sum_{h_{ef}} \sum_{g_{ev}} \prod_{(ef)} \underset{\uparrow}{E(h_{ef})} \prod_{f} \delta(g_{ve} h_{ef} g_{ev'} \cdots)$$

simplicity constraints

$$Z = \sum_{h_{ve}} \sum_{g_{ev}} \prod_{(ve)} \underset{\uparrow}{E(h_{ve})} \prod_{e} \delta(g_v h_{ve} g'_v g_{v'} h_{v'e} g'_{v'})$$

simplicity constraints

Analogue model building

to make numerical simulations feasible

[Bahr, BD, Ryan '11, BD, Eckert, Martin-Benito '11]

Spin foams = generalized lattice gauge theories

dimensional reduction
hope: statistical properties similar

Spin nets = generalized Ising like models

replace rotation group with finite group

future:
with quantum groups

Spin net state sum is now a finite sum.

Despite simplifications we can still aim to understand influence of simplicity constraints on dynamics.

Main question

- lattice gauge theory phases:

weak coupling (deconfining) ⟶ strong coupling (confining)

↓ ↓

'topological' phase (perturbation around BF) degenerate geometry phase

- spin foams: generalization of standard lattice gauge theories:
 higher dimensional phase space parametrized by (simplicity constraint function) E-function

Are there additional phases in spin foams? Phase transitions?

- hoping that 4D lattice gauge theory - 2D edge model correspondence generalizes:

Are there additional phases in spin nets? Phase transitions?

How far can we go with simulation?

Remarks

- Spin foams are NOT Wick rotated: real time path integral
    - a priori cannot use (standard) Monte Carlo simulations
    - is there a conformal factor problem?

- choice of Euclidean or Lorentzian signature metrics encoded in symmetry (rotation) group

- but: path integral does in general not lead to unitary transfer operators
- rather: projection operators (on simplicity constraints and on diffeo and Hamiltonian constraints)

- (surprising) fact:
    - for some models (Barrett-Crane) and in some representations amplitudes are nevertheless real or even positive
    - Monte Carlo simulations (mostly test face weights) [Baez, Christensen, Khavkine et al ‘00s] revealed fast convergence to either confining or deconfining phase

We are looking for a method applicable for general models.

Real space coarse graining

- gives effective dynamics at different scales (needs to be defined in quantum gravity)

- problem: real space renormalization methods have been very restricted [Migdal-Kadanoff 70's]
    - proliferation of non-local couplings
    - truncations not under control

- in the last years new developments in condensed matter/ quantum information
    - density matrix renormalization [White '92,...]
    - matrix product states [Cirac, Verstraete,... 04+]
    - tensor network renormalization [Levin, Nave '06, Gu, Wen '09]
    - entanglement renormalization [Vidal 07+]

Coarse graining with tensor network methods

[Levin & Nave, Gu & Wen, Vidal ...'06+]

[BD, Eckert, Martin-Benito, New. J. Phys. '11]

[BD, Laurie v. Massenbach, Martin-Benito, w.i.p.]

Coarse graining state sums: splitting the sum

effective amplitude
includes sum over
finer field variables

amplitude function

$$Z = \sum_{\psi} \overset{\downarrow}{a}(\psi) = \sum_{\underset{\uparrow}{\Psi}} \sum_{\psi:\, \underset{\uparrow}{B}(\psi)=\Psi} a(\psi) = \sum_{\Psi} \overset{\downarrow}{a'}(\Psi)$$

coarse field variables

blocking of finer field
variables

- How to block finer variables into coarser ones?

- What is the [finite dimensional] space of models, renormalization flow takes place in?

- How to truncate the flow back to this space?

- How to deal with non-local couplings?

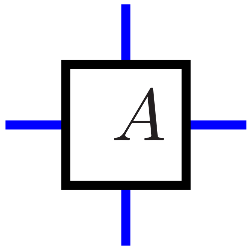
- How to coarse grain the boundary?
 Should we require triangulation independence for the boundary?

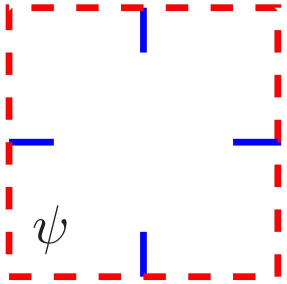
➡tensor network renormalization provides answers

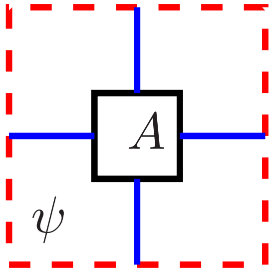
State sums with (generalized) boundaries

State sum models associate amplitudes to space time regions with boundary (data)

[Oeckl 03]







$$A(x_1, x_2, x_3, x_4) = \sum_{x_{\text{bulk}}} a(x_1, x_2, x_3, x_4, x_{\text{bulk}})$$

where x are boundary data

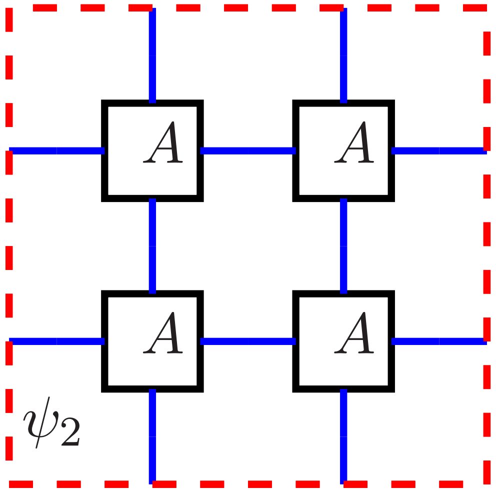
$\psi(x_1, x_2, x_3, x_4)$ is a boundary wave function

A is an (anti-)linear functional on bdry Hilbert space $\mathcal{H}_1$,

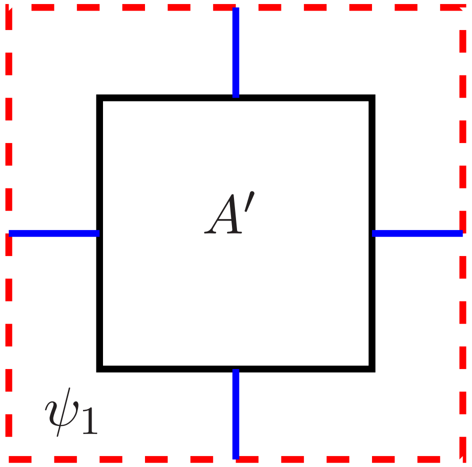
$$A(\psi) = \sum_{x_i} A(x_i)\bar{\psi}(x_i)$$

defines (transition) amplitudes

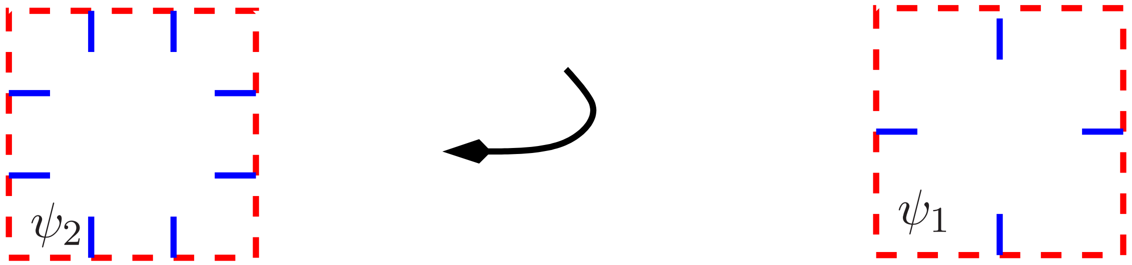
Coarse graining space time regions



Amplitude for a 'larger' region glued from amplitudes of smaller regions, acts on 'refined' bdry Hilbert space $\mathcal{H}_2$

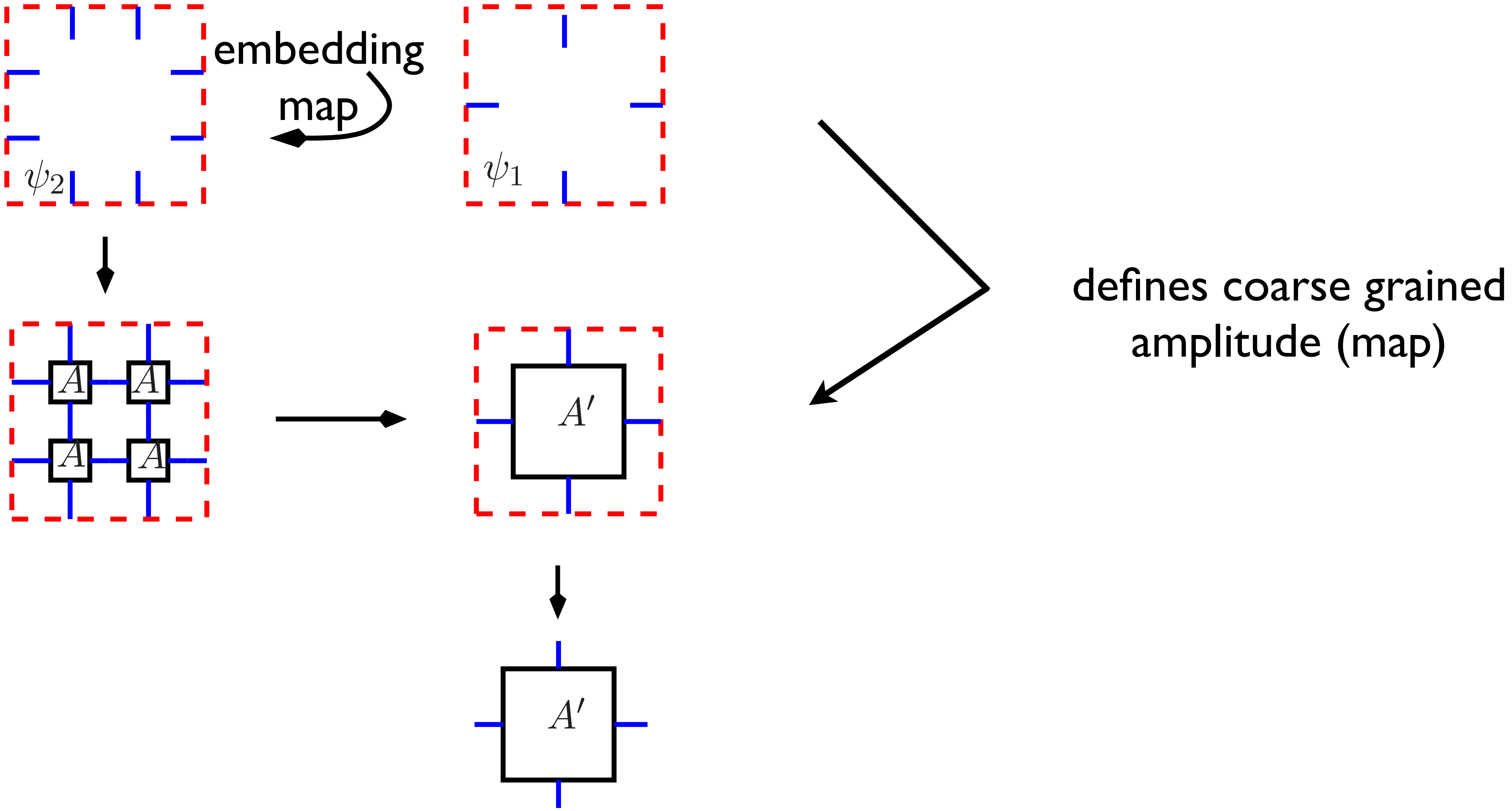


We want to define an effective amplitude acting on coarser boundary Hilbert space $\mathcal{H}_1$



Need to relate coarser and finer bdry Hilbert spaces by embedding maps

Embedding boundaries



embedding map

defines coarse grained amplitude (map)

Via the embedding map we can find the effective amplitude functional A' on $\mathcal{H}_1$.

Take A' as new amplitude functional. Iterate and find fixed point.

Embedding maps

- embed coarser boundary configurations into finer ‘typical’ states
- splitting of boundary Hilbert space into relevant and irrelevant degrees of freedom
- block finer variables into coarser ones
- truncate coarse graining flow

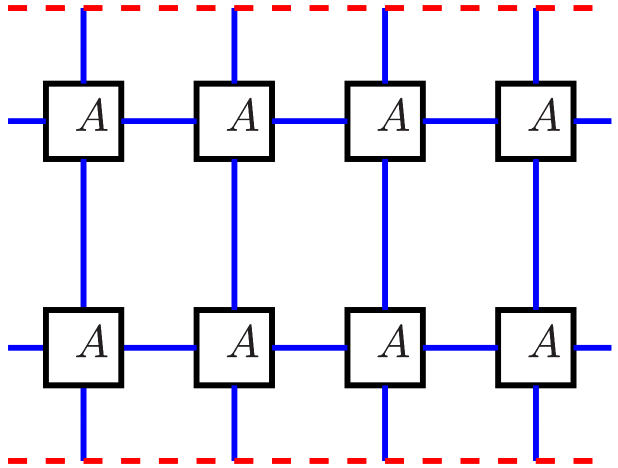
Cylindrical consistent measure:
used in the kinematics of Loop Quantum Gravity to take continuum limit,
defines kinematical (degenerate geometry) vacuum

Embeddings allow to define dynamical cylindrical consistent measure:
defines dynamical vacuum.

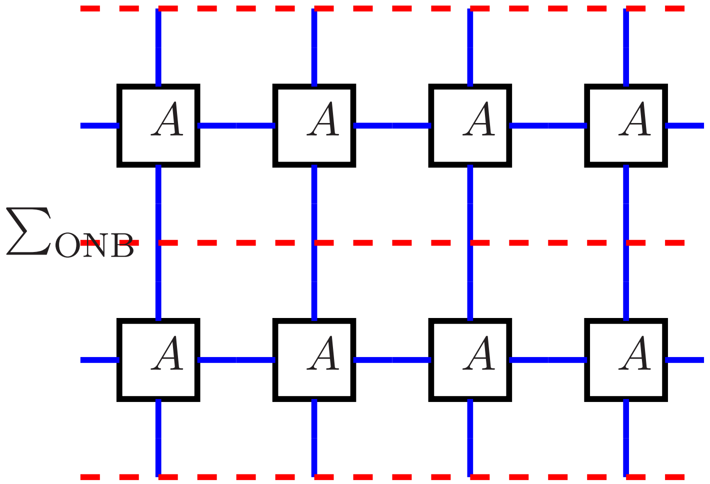
[BD, NJP 2012]

How to choose the embedding maps?

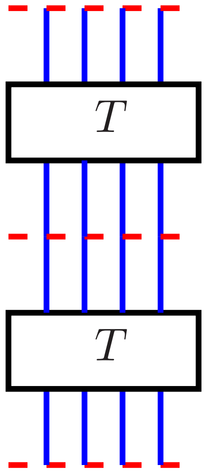
Motivation: transfer operator technique



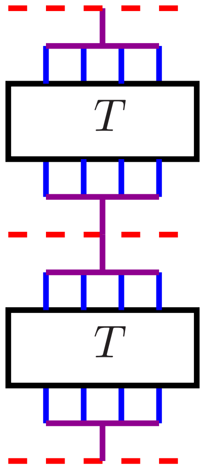
Transition amplitude between two states $\langle\psi_1|\mathcal{A}|\psi_2\rangle$



insert id $= \sum_{\text{ONB}} |\psi\rangle\langle\psi|$



$\mathcal{A} = T^N$

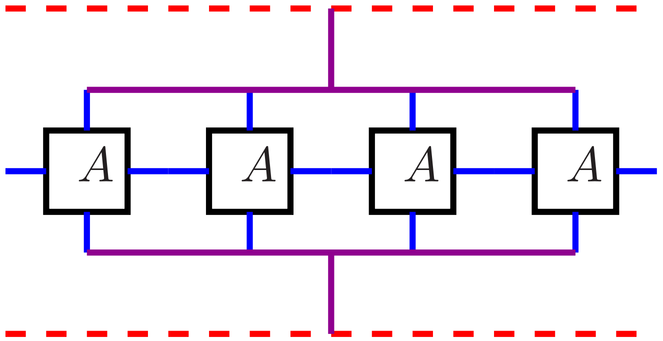


Truncate by restricting $\sum_{\text{ONB}}$ to the eigenvectors of T with the χ largest (in mod) eigenvalues.

Expect good approximation if ψ_1, ψ_2 are in span of these eigenvectors.

But: explicit diagonalization of T difficult.

Dynamically determined embedding maps



Truncate by restricting $\sum_{\mathrm{ONB}}$
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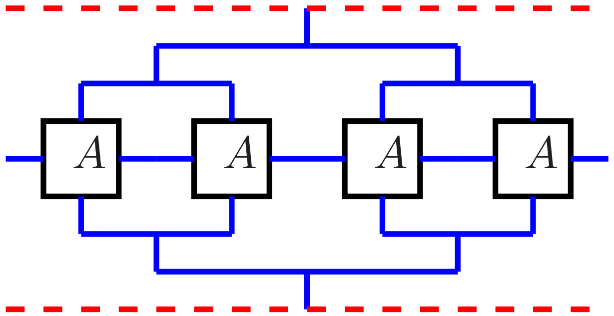
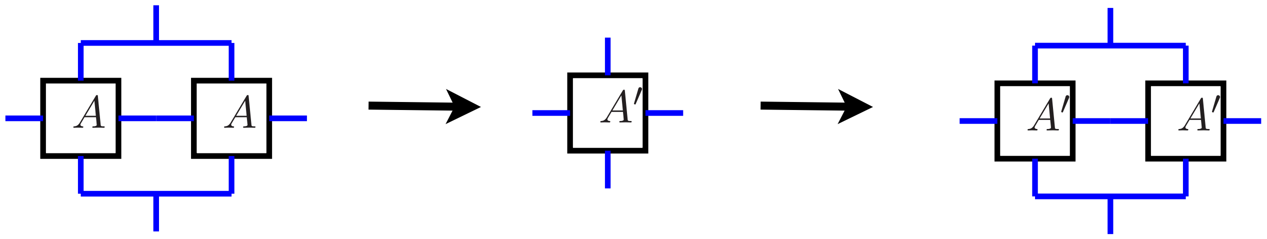
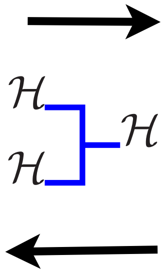


Localize truncations,
diagonalize only subparts
of transfer operator

iteration procedure



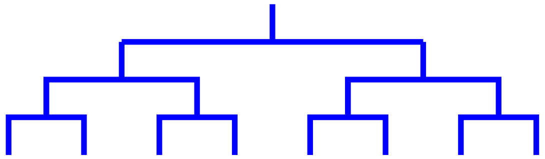
blocking



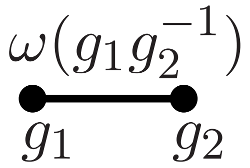
Determined by (generalized
EV-decomposition.

embedding

embedding map after 3 iterations

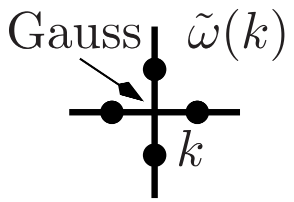


Example: Ising model

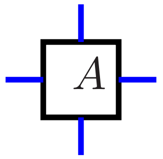


group elements ±1
at vertices,
edge weights ω

$\longrightarrow$ Fourier trafo



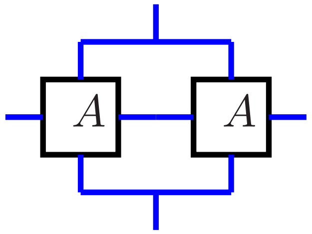
rep labels $k = 0, 1$ at edges
edge weights $\tilde{\omega}(k)$
Gauss constraints at vertices



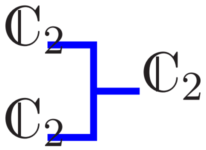
$$A(k_1, \ldots, k_4) = \sqrt{\tilde{\omega}_1} \cdots \sqrt{\tilde{\omega}_4} \, \delta(k_1 + k_2 - k_3 - k_4)$$

Example: Ising model

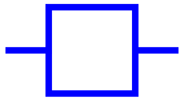
determine embedding maps



embedding maps

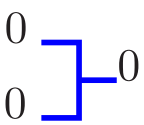


condition on embedding maps

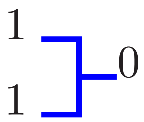


$= \mathrm{id}$

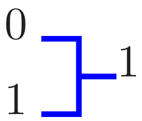
Embedding maps parametrized by:



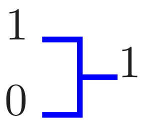
$\cos(\alpha)$



$\sin(\alpha)$



$1/\sqrt{2}$



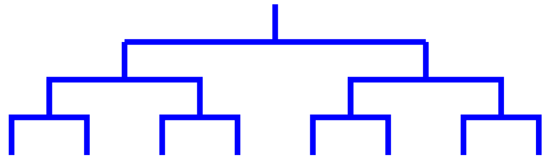
$1/\sqrt{2}$

high temperature: $\cos\alpha = 1,\ \alpha = 0$
(symmetric phase)

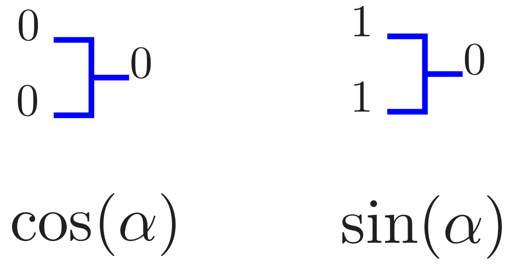
$$\begin{tikzcd}
\tilde{\omega}(1) = 0 \arrow[d] & \alpha = 0 \\
\tilde{\omega}(1) = 1 & \alpha = \frac{\pi}{4}
\end{tikzcd}$$

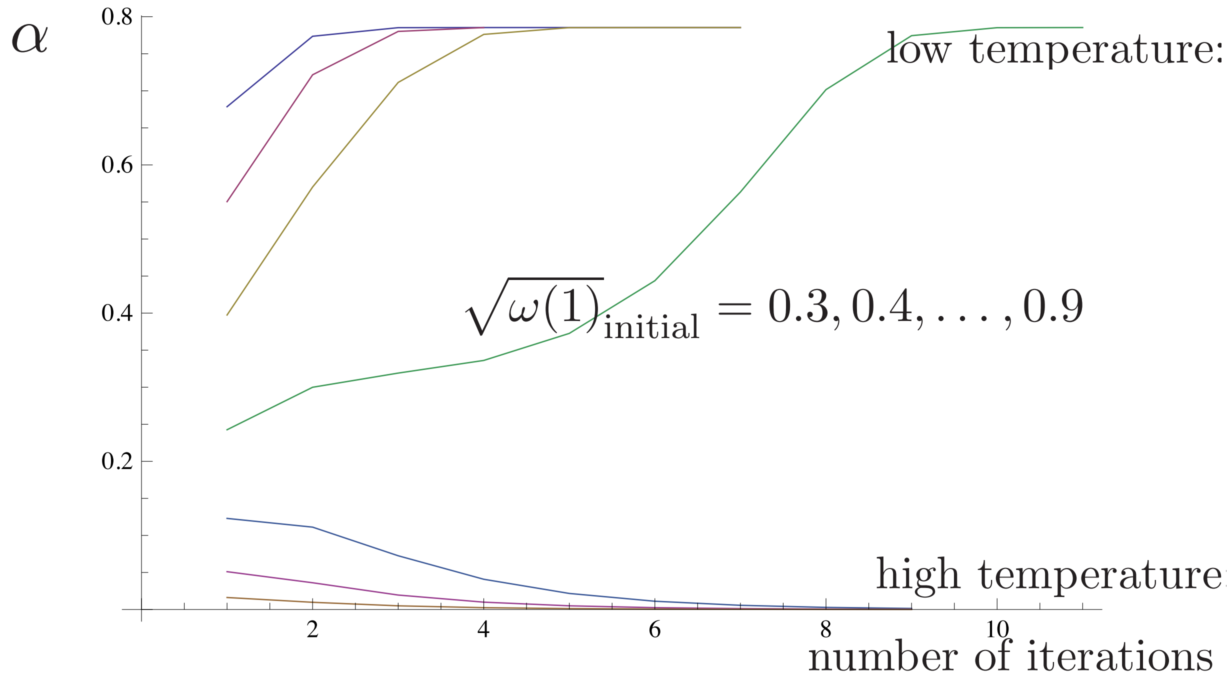
low temperature: $\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}},\ \alpha = \frac{\pi}{4}$
(symmetry broken phase)

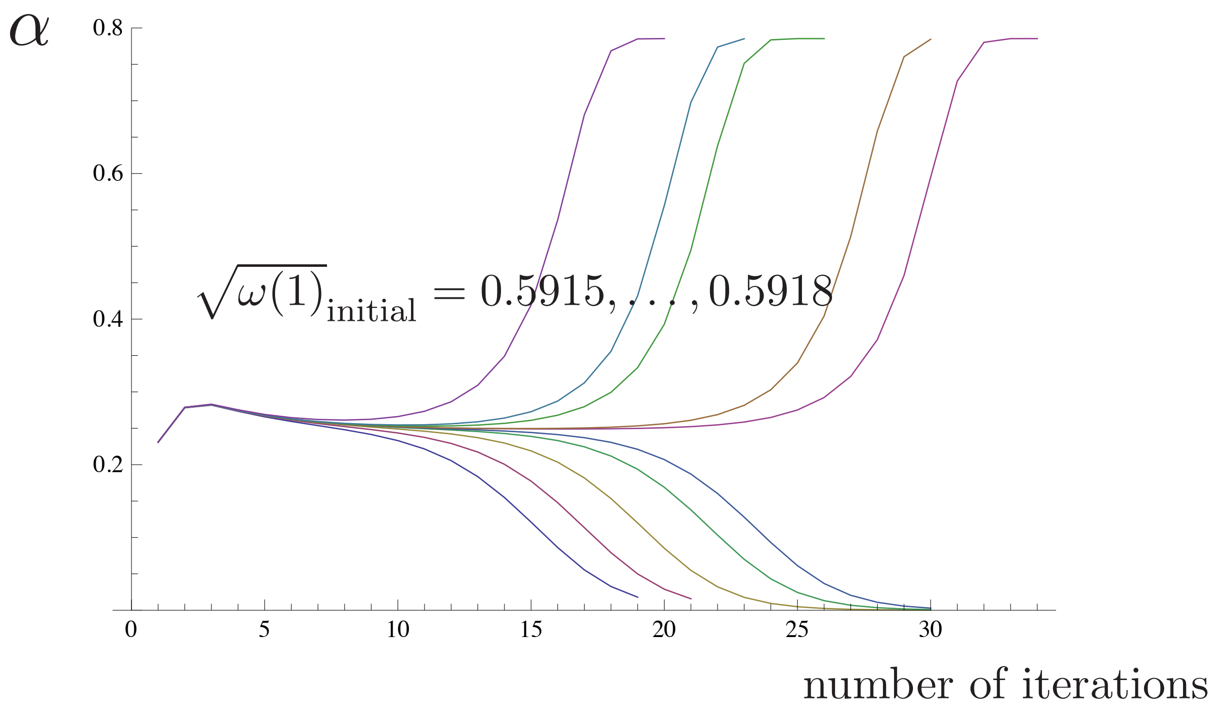
Example: Ising model



Iteration 3
Iteration 2
Iteration 1



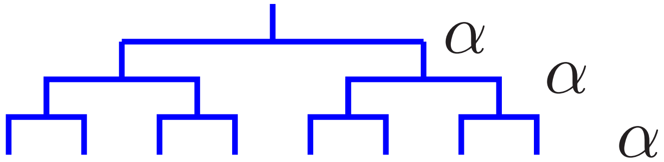




Plateau (scale free dynamics) of almost constant embedding maps around phase transition

Embeddings determined by the dynamics of the system. Represent the physical vacuum for finer degrees of freedom.

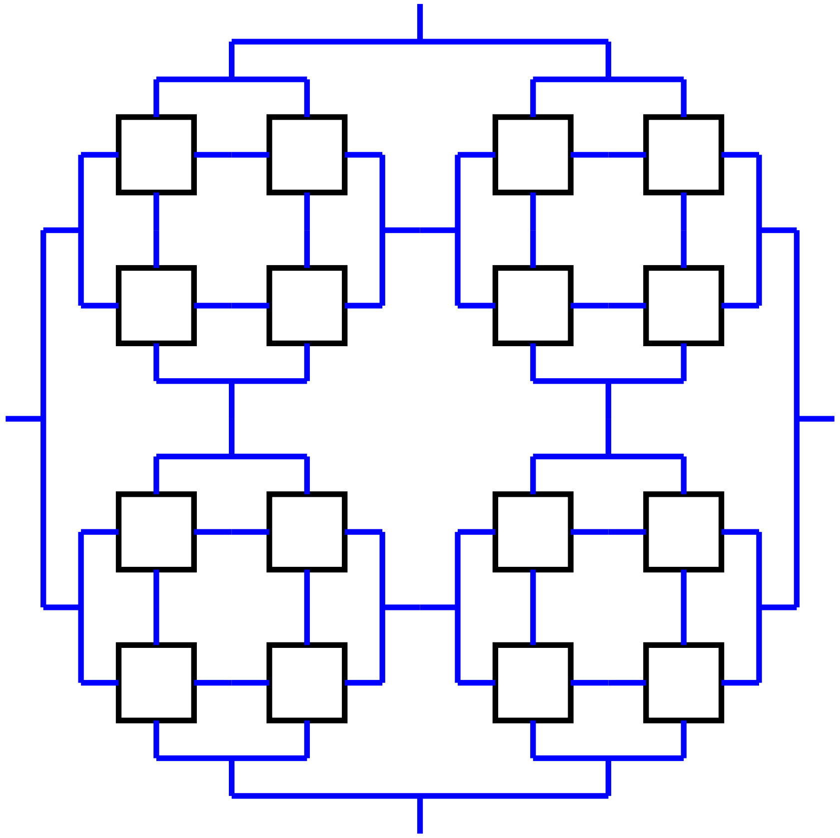
Example: Ising model



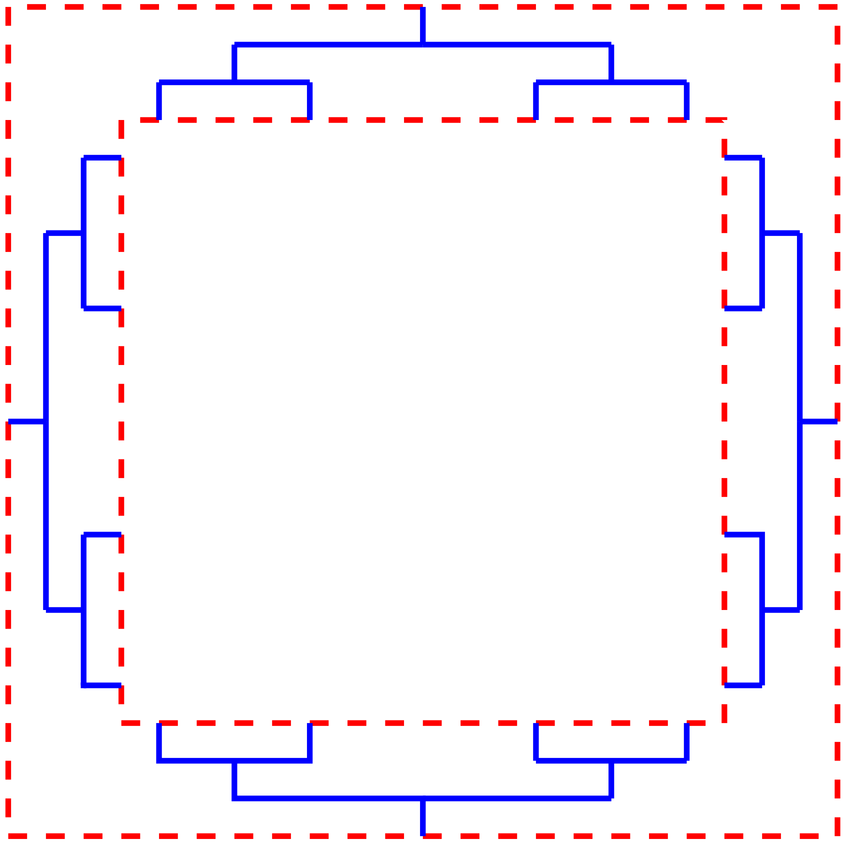
Embedding maps describe structure of vacuum (at given temperature) at finer and finer scales.

Highly excited state (from kinematical vacuum)?

The procedure for 2D state sum

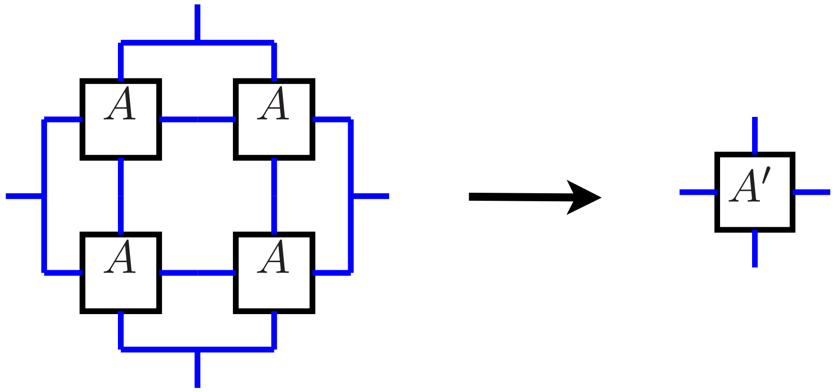


approximation



embedding maps
needed to compare results
for different bond dimensions

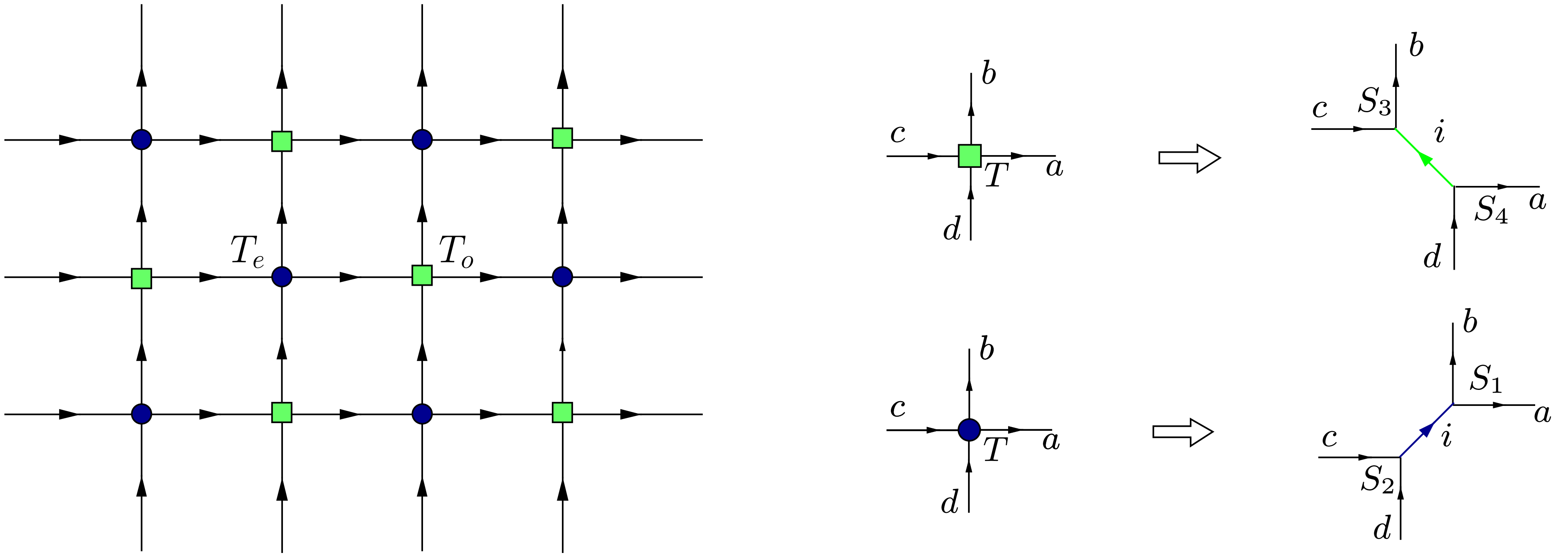
iteration step



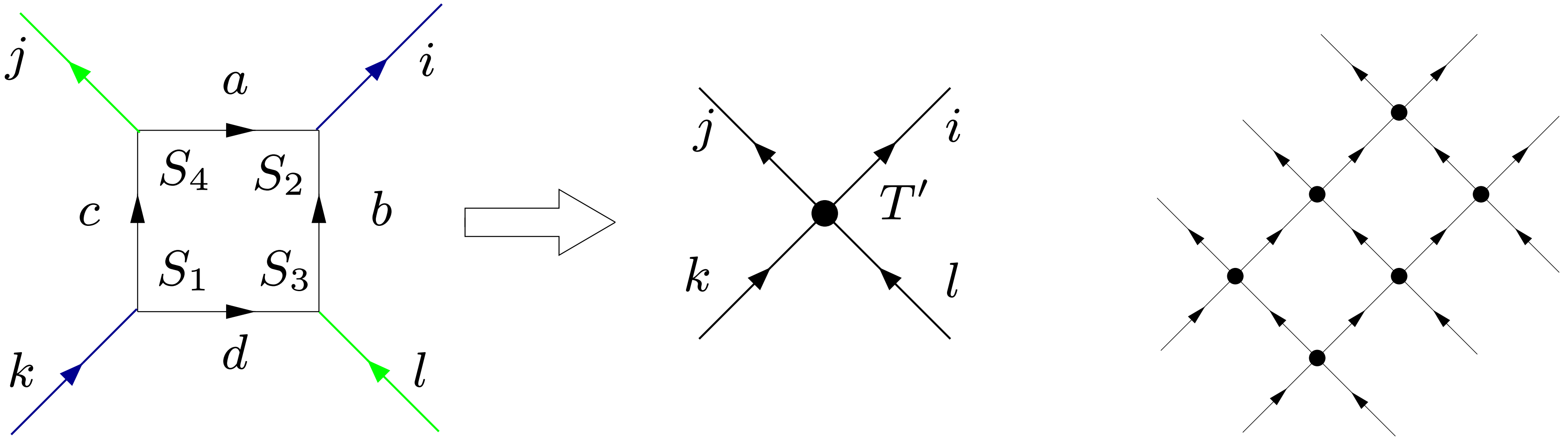
convergence defines continuum limit

The algorithm

[Levin, Nave '07 , Gu, Wen '09]

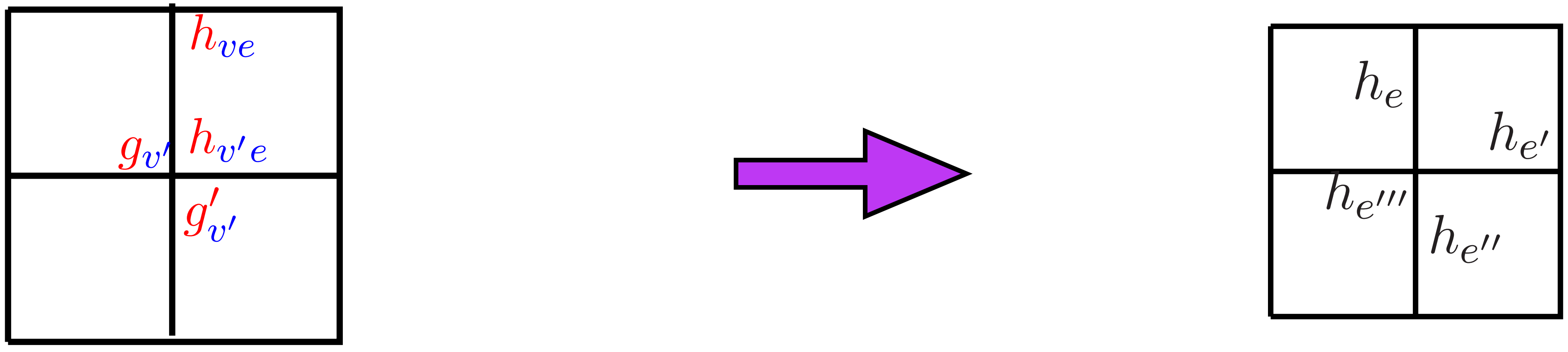


(a) square lattice (b) splitting of vertices



(a) contraction (b) coarse grained lattice

Application to spin nets



$$Z = \sum_{h_{ve}} \sum_{g_{ev}} \prod_{(ve)} E(h_{ve}) \prod_{e} \delta(g_v h_{ve} g'_v g_{v'} h_{v'e} g'_{v'}) \quad \Longrightarrow \quad Z = \sum_{h_e} \prod_{v} C(\{h_e\}_{e \ni v})$$

vertex model
(= tensor network)
in group variables

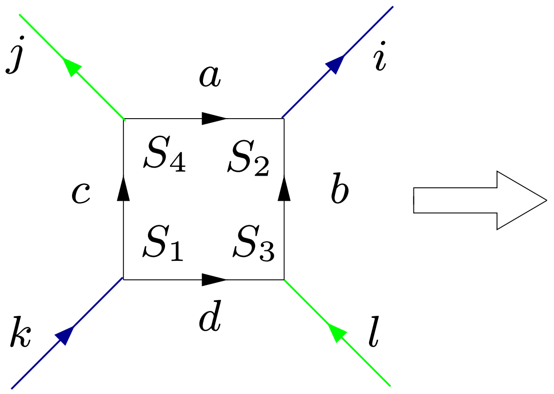
$$\Big\downarrow \ \text{Group Fourier Transform}$$

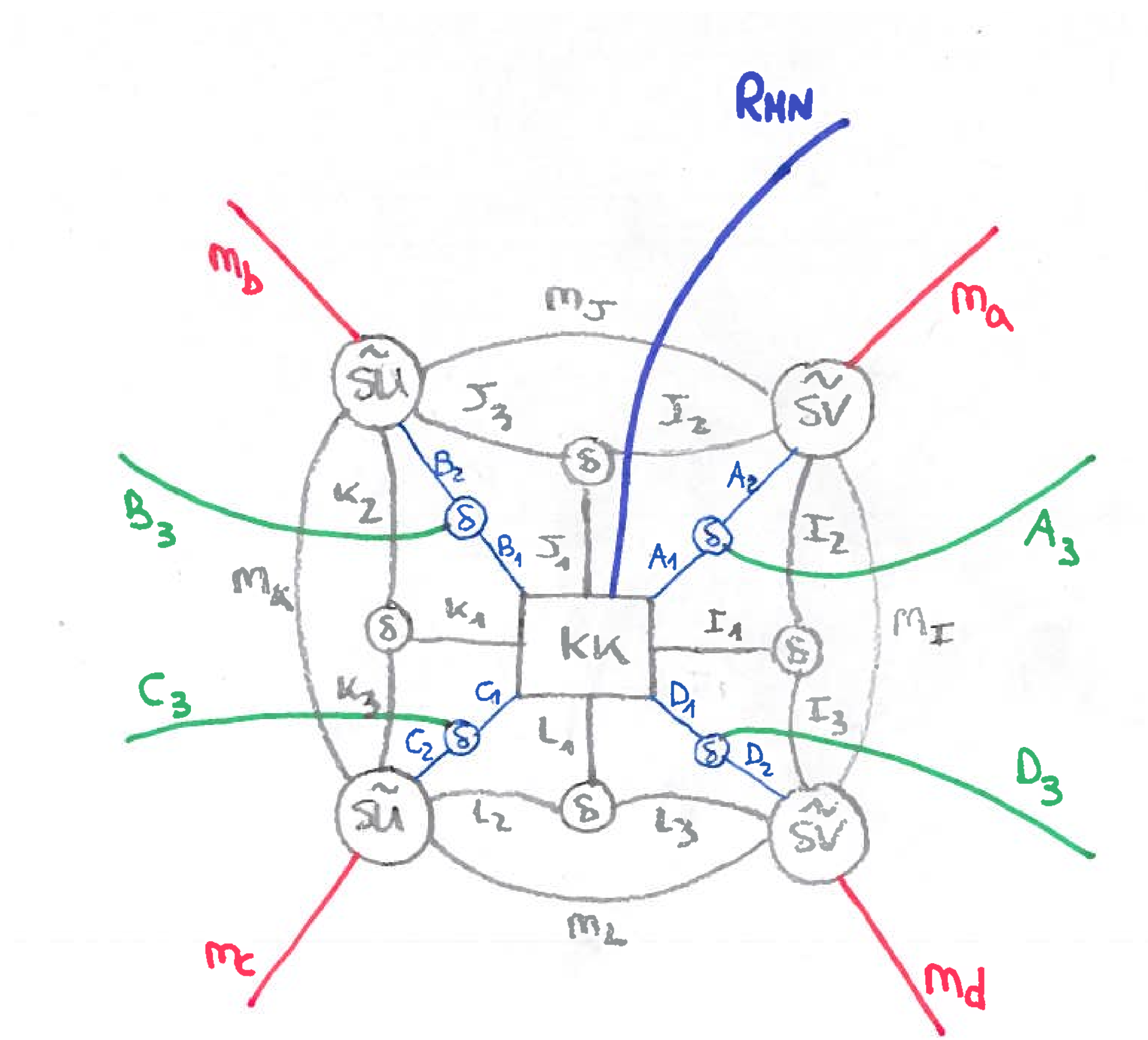
$$Z = \sum_{\rho_e, m_e, n_e} \prod_{v} \widetilde{C}(\{\rho_e, m_e, n_e\}_{e \ni v})$$

vertex model with
representation
labels

The algorithm

- spin nets come with global symmetry group:
  Gauss constraint preserving algorithm in dual (spin) representation
  [BD, Eckert, Martin-Benito, New. J. Phys. '11, BD, Laurie v. Massenbach, Martin-Benito, w.i.p.]





Space of models

[Bahr, BD, Hellmann, Kaminski '12]

S_3, permutation of 3 elements, has 6 elements: unit element, three 2-cycles, two 3-cylces.

E–functions invariant under $\mathbb{Z}_2$ generated by first 2-cycle element:

$$\begin{aligned} E(g) \quad &= \quad \delta(\text{unit}, g) + a\,(\delta(1.\ \text{2-cycle}, g)) + b\,(\delta(2.\ \text{2-cycle}, g)\ +\ \delta(3.\ \text{2-cylce}, g)) + \\ & \qquad c\,(\delta(1.\ \text{3-cylce}, g)\ +\ \delta(2.\ \text{3-cycle}, g)) \end{aligned}$$

$\Rightarrow$ Phase space parametrized by a, b, c.

Space of models

If $a = b$, models can be rewritten into standard 'edge models'.

Obvious fixed points:

- zero temp (BF, weak coupling): $a = b = c = 0$
- BF on quotient group $\mathbb{Z}_2 = S_3/\mathbb{Z}_3$: $a = b = 0,\ c = 1$
- high temp (strong coupling): $a = b = c = 1$

2D subspace
of lattice gauge analogue
models

$a \neq b$

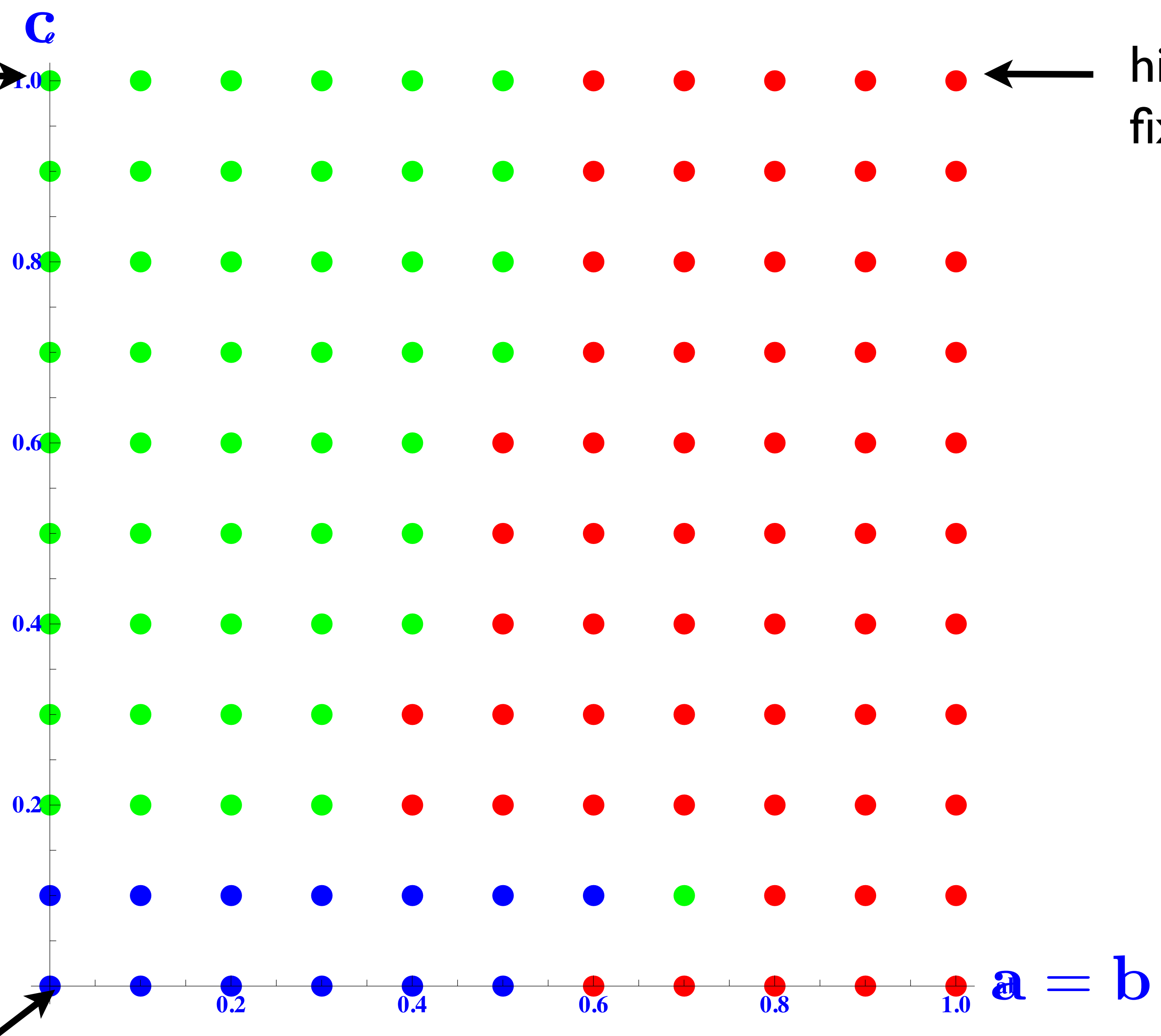
- Barrett Crane analogue model: $a = 1, b = c = 0$ (not a fixed point)

3D space
of spin foam analogue
models

Standard edge models

[BD, Laurie v. Massenbach, Martin-Benito, w.i.p.]

$$a = b$$



quotient group
Z-2
fixed point

high temp
fixed point

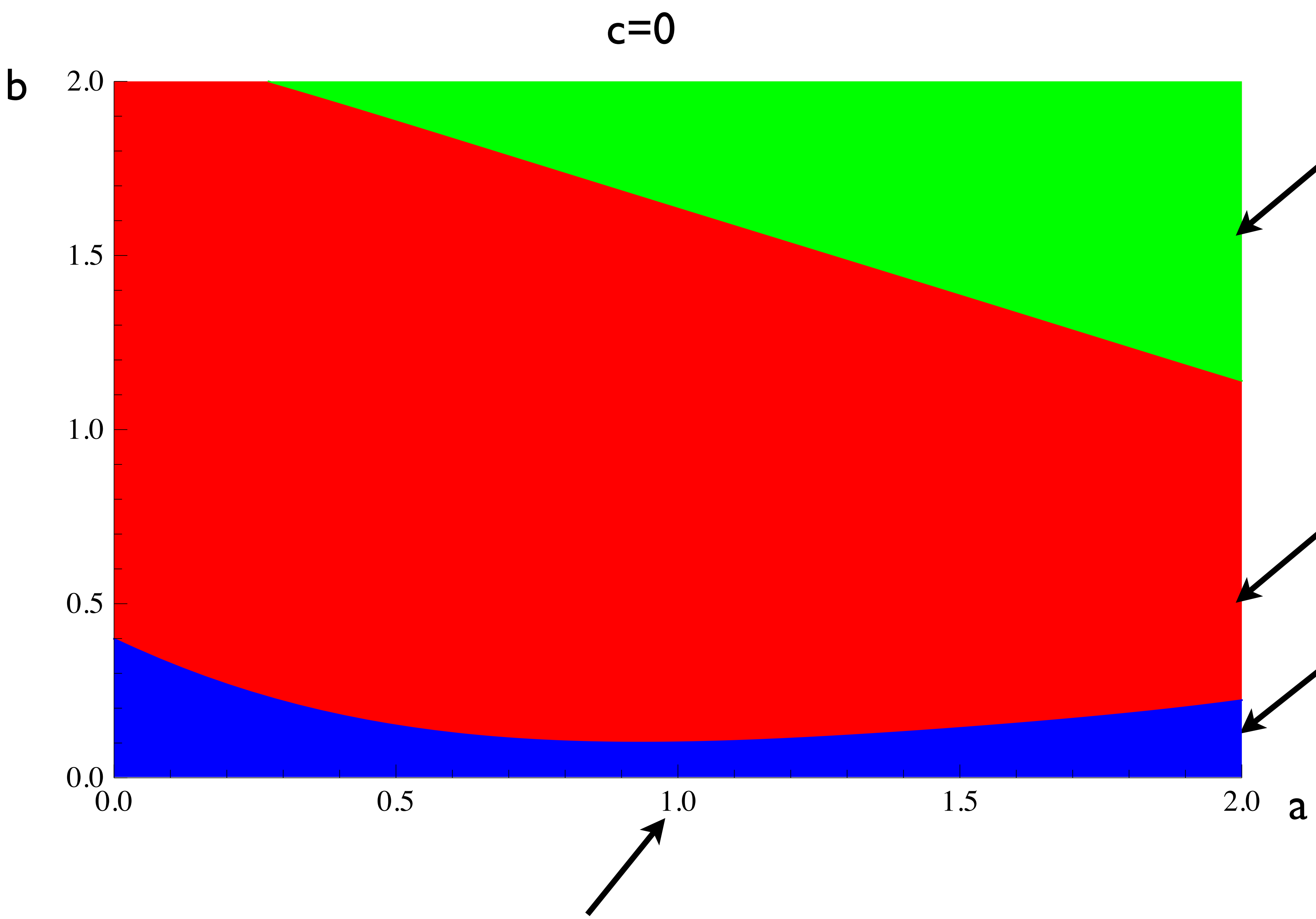
zero temp
fixed point

$a \neq b$

Phase diagram for $a \neq b$?

Model with simplicity constraints

[BD, Laurie v. Massenbach,
Martin-Benito, w.i.p.]



quotient group
Z-2
fixed point

flows to
strong
coupling

flows to zero
coupling

analogue
Barrett-Crane
model

Near the analogue Barrett-Crane model (a=1,b=c=0) phase transition between zero and high temperature.

Will it persist for Lie groups?

Summary

- Presented a strategy to take continuum limit for spin foams: key test for the models
- We are able to test the key dynamical input of spin foams - the implementation of simplicity constraints - in simplified models.
- Lattice gauge theories experience: results might hold in full models.
- Is there a BF phase for SU(2) spin foams as generalized lattice models?
  (confinement conjecture: this is not the case for standard lattice gauge theory models)

Outlook

[w.i.p.]

- methods allow to get insight into dynamics of spin foams models: develop semi-analytical tools in order to go to full models
- analysis of embedding maps will give us information on structure of dynamical vacuum

➡Is the (dynamically determined) blocking geometrically meaningful?

➡Are the simplicity constraints relaxed under coarse graining?

- recently derived structure of transfer operator for spin foams will allow further insight

[BD, Hellmann, Kaminski 1209.4539]

[further in the future:]

- models with quantum groups: have gravity interpretation!
- higher dimensions, spin foams, ...

Stay tuned!

Thanks!