Random tensors

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Orsay, Quantum gravity in Paris

Introduction to random tensors

- Tensor $T_{a_1 \cdots a_d}$ is an array of complex numbers of size $N \times N \times \cdots N = N^d$
- **Goal**: Find the **typical** properties of tensors at large N.

Example

Wigner and the spectrum of heavy nuclei
Consider Hamiltonian as random Hermitian matrix
What is the eigenvalue density?

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Example

Disordered systems, like spin glasses
Spins S_a on lattice sites a
Take d spins interacting together, $H = -\sum_{\text{sites}} T_{a_1 \cdots a_d} S_{a_1} \cdots S_{a_d}$
VB, Gurau, Smerlak, arXiv:1206.5539

Summing over lattices

- Family of 'lattices' (e.g. triangulations) $\{G\}$, with amplitude $A(G)$

$$\text{Generating function} \qquad \sum_{\{G\}} \lambda^{\#\text{vertices}(G)} \; A(G)$$

- Interpret this as a Feynman expansion of some integral

2d quantum gravity

Sum over quadrangulated surfaces = integral over (hermitian) matrices

$$\int [dM_{N\times N}] \; e^{-N(\text{tr}\, M^2 + \lambda\, \text{tr}\, M^4)} \underset{\text{large } N}{=} \sum_{\substack{\text{quadrangulations} \\ \text{of the sphere}}} (-\lambda)^{V(G)} \; \frac{1}{s(G)}$$

- Invariant quantities $\text{tr}\, M^p$: loop observables

$$\langle \text{tr}\, M^p \rangle = \text{disc amplitude with } p \text{ boundary segments}$$

- Exists for more generic potentials than M^4

Matrix models

A combinatorial tool, surprisingly powerful

- Add matter degrees of freedom to the amplitude
- Generating functions of discretized surfaces with matter
- Exactly solve enumerative problem in stat. mech.
- Get 2d quantum gravity coupled to matter (= non-critical strings)
  Developed by David, Kazakov and many others in the 80s-90s
- Related to conformal field theories in several ways
- Schwinger-Dyson equations exhibit a Virasoro algebra
- Get integrable hierarchies (Volterra, KdV)

Methods

- Integral over eigenvalues
- Schwinger-Dyson equations
- Pure combinatorics
  Modern developments with Schaeffer's bijection, integrable combinatorics

Tensors generalize matrices

The historical way: random triangulations of manifolds

- Sum triangulations in d dimensions using rank d tensors
  Generalize matrix M_{ab} to $T_{a_1,\cdots a_d}$
  Suggested by Ambjorn, Gross, Sasakura in '91 independently.
- But no way to get analytical results, until 2011 !
- Numerical simulations (fixed topology): Euclidean Dynamical Triangulations [and better: Renate Loll's talk]
- In group field theories (Boulatov, Ooguri, '90s), one sums over cell complexes too, with different amplitudes
  [Daniele Oriti's talk]
- Control sum over triangulations via Feynman rules only
- No eigenvalues
- Solely rely on Schwinger-Dyson eqs and combinatorics

Objective I

- Extract an exactly solvable sector of dynamical triangulations
- That works in arbitrary dimensions

Reproduce the basic results of matrix models

- Show that the large N limit exists
- Show that it projects onto a summable family of triangulations
  The melonic family
- Extract singularities of the generating functions
  Continuum limit and multicritical behaviors
- Prove universality of continuum limits
  - irrespective of the microscopic details
    (triangulations, quadrangulations, etc.)
  - find how to change the the microscopic details from tensors!

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Program started in 2011, Done

References

- Critical behavior of colored tensor models in the large N limit
  VB, R. Gurau, A. Riello, V. Rivasseau, Nucl. Phys. B (2011)
- The Ising Model on Random Lattices in Arbitrary Dimensions
  VB, R. Gurau, V. Rivasseau, Phys. Lett. B (2011)
- Multicritical tensor models and hard dimers on spherical random lattices, *VB*, Phys. Lett. A (2012)
- Random tensor models in the large N limit: Uncoloring the colored tensor models, *VB, R. Gurau, V. Rivasseau*, Phys. Rev. D (2012)
- Hard dimers coupled to random tensors, *VB, H. Erbin*, JSTAT (2012)
- Counting Line-Colored D-ary Trees, *VB, R. Gurau*, (2012)
- Phase Transition in Dually Weighted Colored Tensor Models
  R. Gurau, D. Benedetti, Nucl. Phys. B (2012)
- Melons are branched polymers
  R. Gurau, J. Ryan, (2013)

Objective II

- Melonic geometry: random trees [Come to J. Ryan's talk!]

Further explore the richness of tensor models

- More invariant objects = boundary triangulations, than in matrix models!
- Landscape of large N behaviors
  New 1/N expansions in random tensor models, VB, arXiv:1211.1657
- Universality theorem vs non-universal models
- Study the Schwinger-Dyson equations
  Revisiting random tensor models at large N via the Schwinger-Dyson equations, VB, JHEP
- Complicated set of equations labeled by boundary triangulations
- Relations to combinatorics
  (meanders aka polymer folding [to appear])
- Form an infinite-dimensional Lie algebra!
  Reduces to Virasoro in the matrix case
  Conformal aspects, integrability?

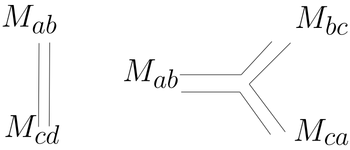
Generating graphs with integrals

$$Z(\lambda, \alpha) = \sum_{\text{Graphs } \{G\}} \lambda^V \; Z_{G_n}(\alpha)$$

Interpret it as the Feynman expansion of some integrals
Choice of variables depends on the family of graphs under consideration.

Example: Matrix integrals generate ribbon graphs

$$[dM] \; \underbrace{e^{-N \operatorname{tr} M^2}}_{\text{Gaussian}} \; \underbrace{e^{-Ng \operatorname{tr} M^4}}_{\text{perturbation}}$$



Stranded graphs and tensors

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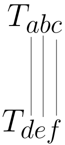
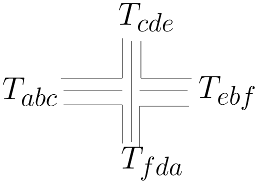
Choice of variables depends on the family of graphs under consideration.

Example: Tensor integrals generate stranded graphs

Propagator of a Gaussian measure
Interaction





But no way to control these models analytically!

Tensor Ensemble

External tensor product of $U(N)$ acts like

$$T'_{a_1\cdots a_d} = \sum_{b_1,\ldots,b_d} U^{(1)}_{a_1 b_1} \cdots U^{(d)}_{a_d b_d} \; T_{b_1\cdots b_d}$$

and on c.c. $\bar{T}$.

Invariant monomials

Contract index in position i of T with an index in position i in a $\bar{T}$.

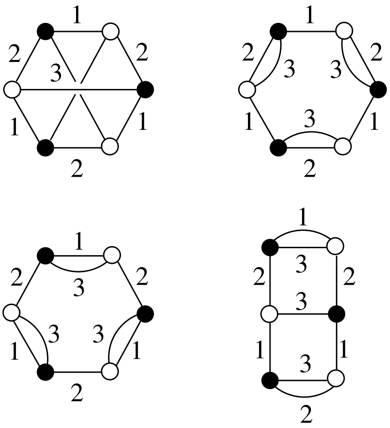
Equivalence with colored graphs

- a T $\rightarrow$ white vertex
- a $\bar{T}$ $\rightarrow$ black vertex
- Index $\rightarrow$ half-line on a vertex
- Position of index $\rightarrow$ color label on half-line
- Index contraction $\rightarrow$ connect half-lines with same color

Colored graphs

d-colored, connected graphs

- Bipartite graphs: black and white vertices
- Colored lines with *d* possible colors
- *d*-valent nodes with distinct colors



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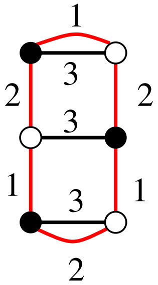

Faces are closed cycles with only two colors.

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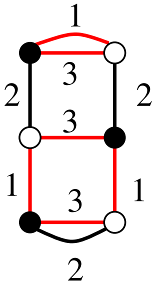

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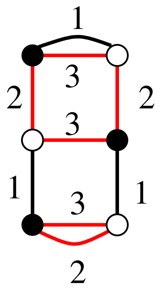

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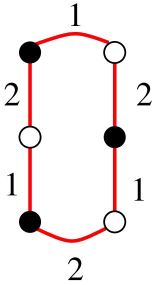

$d = 2$ reduces to loops with 1 face and $2p$ vertices, i.e. $\mathrm{tr}(TT^\dagger)^p$

Richer structure with tensors than with matrices

Colored graphs and unitary-invariant tensor ensemble

Colored Graphs and Triangulations

Tensor models and their large N expansion

Schwinger-Dyson equations and Universality

Relation between graphs and topology
Graphs encode topology of triangulations

- Triangulation: gluing of simplices (triangles, tetrahedra, pentachora, d-simplices), defined by attaching maps
- Δ-colored graph $\leftrightarrow$ $(\Delta - 1)$-triangulation

Graph dual to the triangulation

$$\begin{aligned} \text{vertex} &\rightarrow (\Delta - 1)\text{-simplex} \\ \text{line} &\rightarrow (\Delta - 2)\text{-simplex} \\ \text{face} &\rightarrow (\Delta - 3)\text{-simplex} \\ k\text{-bubble} &\rightarrow (\Delta - k - 1)\text{-simplex} \end{aligned}$$

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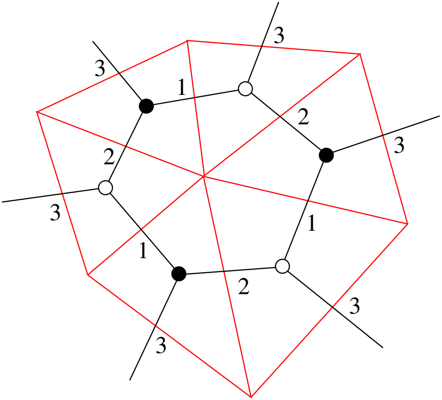
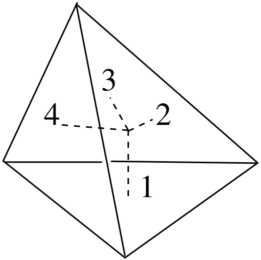


Triangulations from colored graphs

A Δ-colored vertex is dual to a $(\Delta - 1)$-simplex. The legs of the vertex are dual to the boundary $(\Delta - 2)$-simplices.

- Boundary triangles labeled by a color $c = 1, \ldots, \Delta$



Colors identify all sub-simplices

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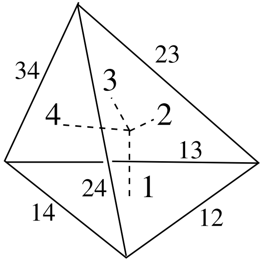
- Induced colorings
- Edges labeled by pair of colors



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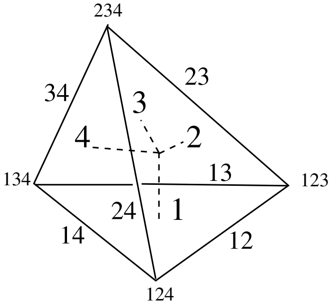
- Boundary triangles labeled by a color $c = 1, \dots, \Delta$
- Induced colorings
- Edges labeled by pair of colors
- Nodes labeled by three colors



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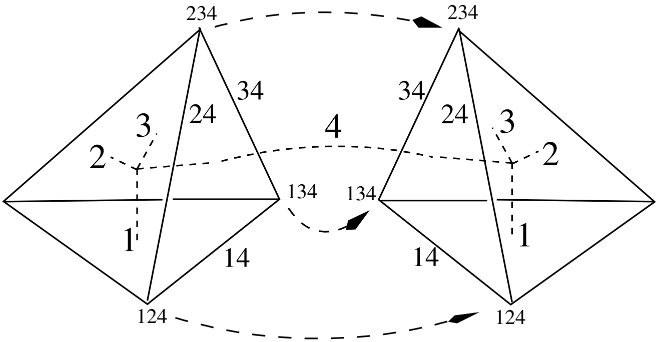
Triangulations from colored graphs

A propagator glues two tetrahedra along a triangle, respecting the induced coloring of all sub-simplices



Theory of crystallization and GEMs (graph-encoding manifolds): Δ-colored graphs are dual to triangulations of pseudo-manifolds of dimension $\Delta - 1$ [Pezzana, Ferri, Cagliardi, Lins, Caravelli].

Tensor Models

- $(B_i)_{i\in I}$ finite set of bubbles. Set action to

$$S(T,\bar{T}) = T\cdot\bar{T} + \sum_{i\in I} t_i\ B_i(T,\bar{T})$$

- Free energy F

$$\exp -F(\{t_i\}) = Z = \int [dT\, d\bar{T}]\ \exp -\frac{N^{\alpha}}{\lambda}\, S(T,\bar{T})$$

- Bubble expectation value

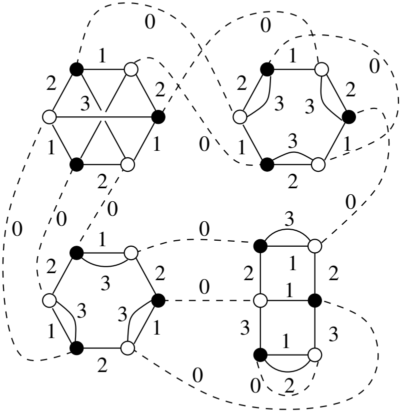
$$\langle B\rangle = \frac{1}{Z}\ \int [dT\, d\bar{T}]\ B(T,\bar{T})\ \exp -\frac{N^{\alpha}}{\lambda}\, S(T,\bar{T})$$

- In particular $\langle B_i(T,\bar{T})\rangle \sim \dfrac{\partial F}{\partial t_i}$

Feynman expansion and colored graphs

Feynman expansion of F

- Connect bubbles B_i with lines (the covariance)
- Each vertex receives such a line
- Give it color 0
- Generate $(d+1)$-colored, connected graphs
- Dual to d-triangulations



$$F = \sum_{(d+1)\text{-colored graphs } G} N^{\text{power}(G)} \; A_G(\lambda, \{t_i\})$$

1/N expansion

Connected colored graph G: f faces, l lines of color 0, b bubbles

$$A(G) \sim N^{f-\alpha(l-b)} \lambda^{l-b}$$

Value of α fixed by balance between f and $(l-b)$ so that

- $A(G) \leq K\, N^{\text{fixed exponent}}$
  If α too small, no bound
- If α too large, then suppression at large number of vertices, hence no infinite family, no continuum limit

Counting of faces in colored graphs

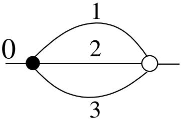
$$f - (d-1)(l-b) = d - \underbrace{\omega(G)}_{\substack{\text{positive integer} \\ \text{the degree}}} \leq d$$

$d = 2$, the degree is the genus
Set $\alpha = d - 1$

Dominant graphs

$$f - (d-1)(l-b) \leq d$$

- Leading order: maximize the number of faces
- $d = 2$, get planar graphs
- $d \geq 3$, get melonic graphs!
- Movements which maximize f
  Add only two vertices, create $\frac{d(d-1)}{2}$ faces



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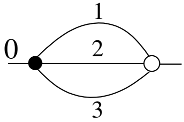
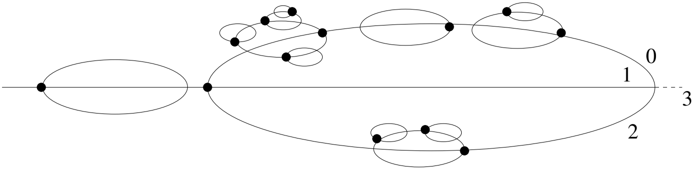
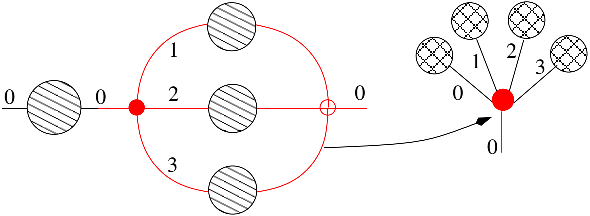




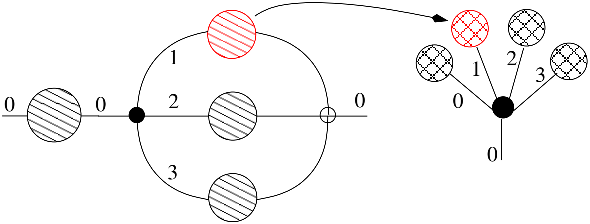
Melonic graphs have melonic bubbles
Only them are relevant in the action

Mapping melons to trees

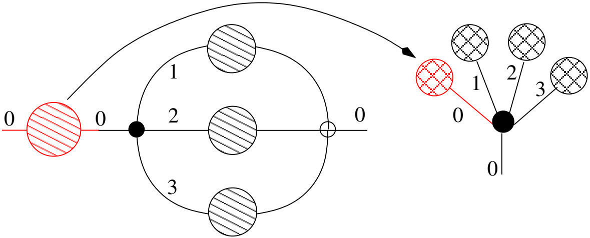


The black and white vertices of external legs are mapped to a tree vertex. The lines connected to the black vertex are mapped to tree lines which inherit colors.

Mapping melons to trees

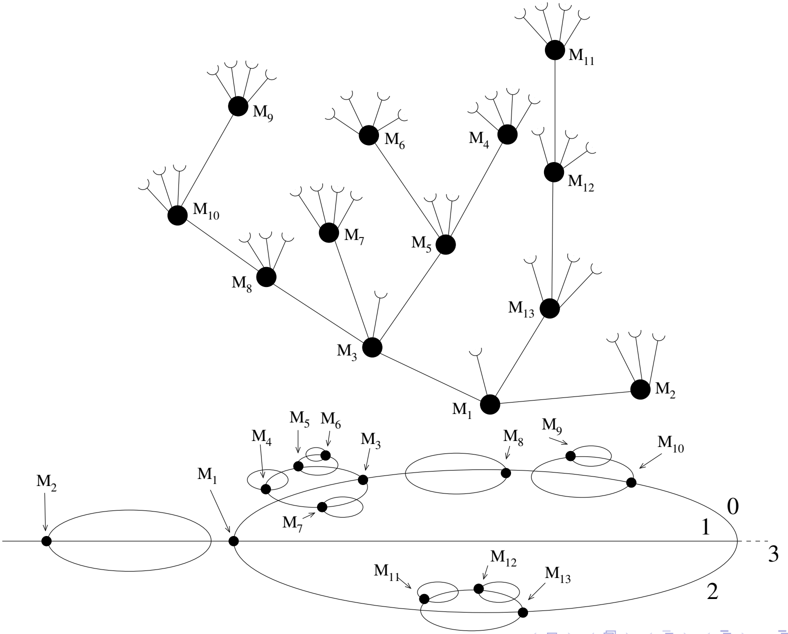


Mapping melons to trees



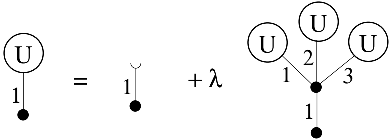
Get rooted $(d+1)$-ary trees, a species of Branched Polymers (BP).

Mapping melons to trees



Counting melons

- Generating functions for d-colored melonic graphs with two external lines, $U(\lambda) = \sum_p \lambda^p C_p^{(d)}$



$$U = 1 + \lambda U^d$$

$$\Rightarrow \quad C_p^{(d)} = \frac{1}{d\,p+1}\binom{dp+1}{p}$$

Generalized Catalan numbers

- Trees with exactly p_i lines of color i

$$C^{(d)}_{p_1 \cdots p_d} = \frac{1}{1+\sum_{i=1}^d p_i} \prod_{j=1}^{d} \binom{1+\sum_{i=1}^d p_i}{p_j}$$

Generalized Narayana numbers

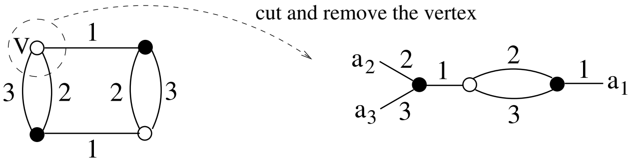
Counting line-colored D-ary trees, VB and R. Gurau, arXiv:1206.4203 [math-ph]

SDEs: Quantum equations of motion

- To solve a generic model, need systematic eqs

$$\int [dTd\bar{T}] \, \frac{\partial}{\partial \bar{T}_{a_1 \cdots a_d}} \left(\mathcal{O}(T, \bar{T}) \; e^{-N^{d-1} S(T,\bar{T})} \right) = 0$$

Insertions: Open bubbles

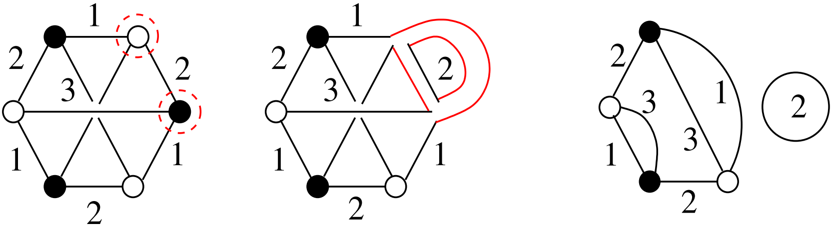


$$\sum_{a_1,\ldots,a_d} \int [dTd\bar{T}] \, \frac{\partial}{\partial \bar{T}_{a_1 \cdots a_d}} \left((B \setminus v)_{a_1 \cdots a_d} \; e^{-N^{d-1} S(T,\bar{T})} \right) = 0$$

Operations on bubbles

Contraction

Cut 2 vertices $v, \bar{v}$ and connect the lines pairwise

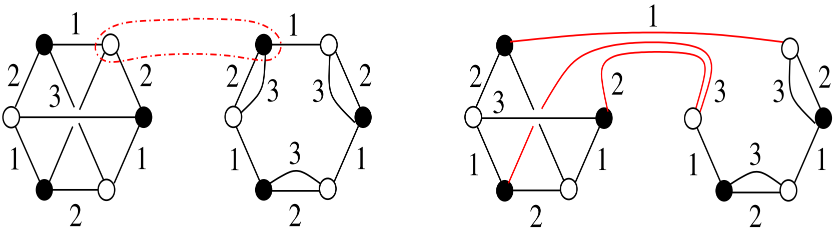


$$\sum_{a_1,\dots,a_d} \frac{\partial (B \setminus v)_{a_1\cdots a_d}}{\partial \bar{T}_{a_1\cdots a_d}} = \sum_{\bar{v}\in B} B \setminus (v,\bar{v})$$

Operations on bubbles

Gluing

Take 2 open bubbles and glue them



$$\sum_{a_1,\dots,a_d} (B_1 \setminus v_1)_{a_1\cdots a_d} (B_2 \setminus \bar{v}_2)_{a_1\cdots a_d} = (B_1 \setminus v_1) \cdot (B_2 \setminus \bar{v}_2)$$

Schwinger-Dyson Equations

For any open bubble $B \setminus v$, at all orders

$$\frac{1}{N^{d-1}} \sum_{\bar{v} \in B} \underbrace{\langle B \setminus (v, \bar{v}) \rangle}_{\text{contraction of } B} - \langle B \rangle - \sum_{i \in I} t_i \sum_{\bar{v} \in B_i} \underbrace{\langle (B \setminus v) \cdot (B_i \setminus \bar{v}_i) \rangle}_{\text{gluing with } B_i} = 0$$

- $\langle B \rangle = \partial_{t_B} \ln Z$
  Translate SDEs as constraints $L_{(B \setminus v)} Z = 0$ with differential operators $L_{(B \setminus v)}$
- Infinite-dimensional Lie algebra

$$[L_{B_1 \setminus v_1}, L_{B_2 \setminus v_2}] = \sum_{\bar{v}_1 \in B_1} L_{((B_1 \setminus \bar{v}_1) \cdot (B_2 \setminus v_2)) \setminus v_1} - \sum_{\bar{v}_2 \in B_2} L_{((B_1 \setminus v_1) \cdot (B_2 \setminus \bar{v}_2)) \setminus v_2}$$

- $d = 2$, all bubbles are loops labeled by number of vertices p,

$$[L_{p_1}, L_{p_2}] = (p_1 - p_2) L_{p_1 + p_2}$$

Gaussian models

- For a real variable ϕ, covariance λ,

$$\langle \phi^{2p} \rangle = \sum_{\text{Wick pairings}} \lambda^p$$

- For matrix M at large N, not all pairing have the same scaling!

$$\langle \text{tr}\, M^{2p} \rangle = \sum_{\text{Wick pairings}} N^{2-2h}\, \lambda^p = \sum_{\text{planar pairings}} N\, \lambda^p = N\, \lambda^p\ c_p$$

where c_p Catalan number.

- For tensor at large N, bubble with $2p$ vertices, only one melonic contraction

$$\langle B_{2p} \rangle = \sum_{\text{Wick pairings}} N^{d-\omega}\, \lambda^p = \sum_{\text{melonic pairings}} N\, \lambda^p = N\, \lambda^p$$

Solution and Universality at large N

Revisiting random tensor models at large N via the Schwinger-Dyson equations,
V. Bonzom, JHEP (2012)

Schwinger-Dyson equations have a single perturbative solution

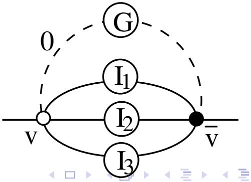
- For any two melonic bubbles with $2p$ vertices, $\langle B\rangle = \langle B'\rangle \equiv N\, G_p$
- Physical solution

$$G_p = G^p, \quad \text{for} \quad 1 - G - \sum_{i\in I} p_i\, t_i\, G^{p_i} = 0$$

  Algebraic equation, and take branch $G = 1$ when $t_i = 0$
- $p = 1$ gives the full 2-point function $\langle T \cdot \bar{T}\rangle = N\ G$

Models are Gaussian at large N

- I_1, I_2, I_3: melonic insertions
- G: insertion of full 2-point function

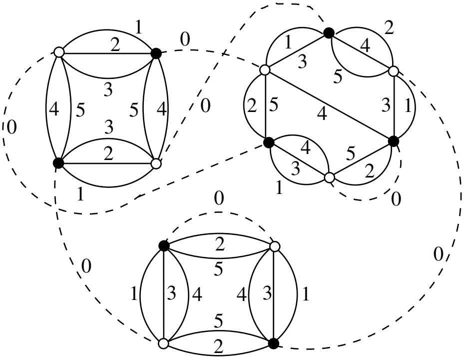


Is it possible to retain more than melonic graphs?

New 1/N expansions, V. Bonzom, JHEP (2012)

Probe sub-graphs

- Split the set of colors into two sets $\{1, \ldots, k\}$, $\{k+1, \ldots, d\}$
- Compare degree of Feynman graphs with degrees of subgraphs with colors $\{0, 1, \ldots, k\}$ and $\{0, k+1, \ldots, d\}$



Is it possible to retain more than melonic graphs?

New 1/N expansions, V. Bonzom, JHEP (2012)

Probe sub-graphs

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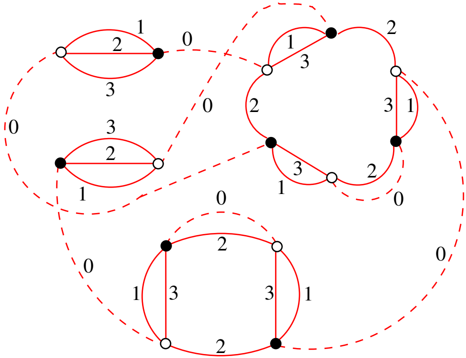

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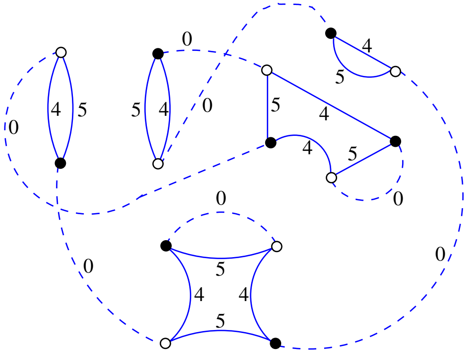


$$\omega(G) = \omega_{01\dots k} + \omega_{0k+1\dots d} + (l - b)$$

- Amplitude of a graph

$$A(G) \sim N^{-\omega(G)} \; \lambda^{l-b} = N^{-\omega_{01\dots k} - \omega_{0k+1\dots d}} \; \left(\frac{\lambda}{N}\right)^{l-b}$$

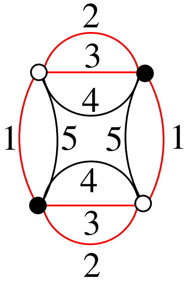
- Suggests to scale $\lambda = N\,\kappa$

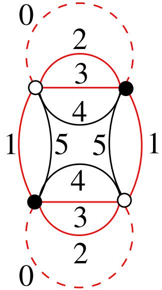
$$\int [dTd\bar{T}] \;\exp - \frac{N^{d-2}}{\kappa} S(T, \bar{T}) \qquad \text{i.e. } \alpha = d - 2$$

- Danger: melonic graphs become unbounded
  Rescaling melonic bubbles is sufficient
- It re-shuffles the $1/N$ expansion
  Add new large N contributions

Large N: Allows more than melonic bubbles!

- sub-bubbles $\{1\ldots,k\}$ melonic
- sub-bubbles $\{k+1\ldots,d\}$ melonic
- subgraphs $\{0,1\ldots,k\}$ melonic
- subgraphs $\{0,k+1,\ldots,d\}$ melonic





- Still Gaussian model

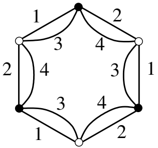
What else?

Matrix models are non-Gaussian!

- Introduce *fat indices*

  $A = (a_1, a_3),\ \ B = (a_2, a_4)$

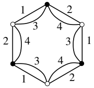
  $T_{a_1 a_2 a_3 a_4} = M_{AB}$



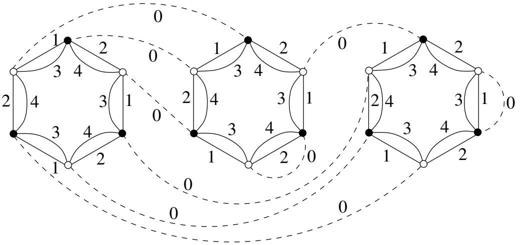
$= \mathrm{tr}(MM^\dagger)^3$

- A model that generates triangulations of the 4-sphere at large N

$$\int [dT\, d\bar{T}]\ \exp -N^2 \left(T \cdot \bar{T} + \right)$$

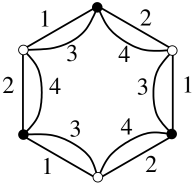


$\underset{\text{large } N}{=} \sum_{\substack{\text{Triangulations} \\ \text{4-sphere}}}$



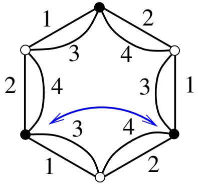
Extend the properties of that model

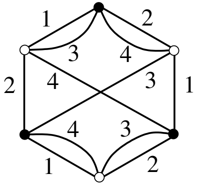
- Consider bubbles with a single loop $(1, 2)$ and a single loop $(3, 4)$



Extend the properties of that model

- Consider bubbles with a single loop $(1, 2)$ and a single loop $(3, 4)$
- Draw the loop $(3, 4)$ after a permutation on the black vertices





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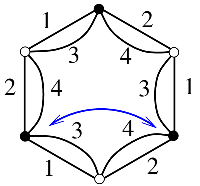
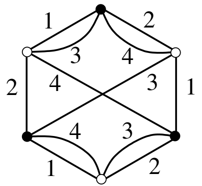
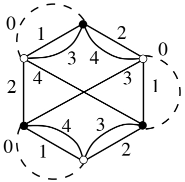
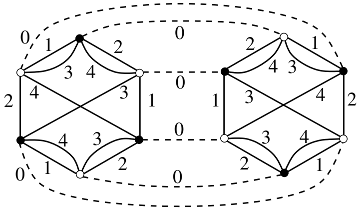




Non-Gaussian contributions





Conclusion

Tensor models provides control on random triangulations

- Generate random colored triangulations from bubbles
- Large N: maximize number of faces
- $d = 2$, planar graphs ; $d \geq 3$, melonic or have melonic subgraphs
- In probability language: Gaussianity
- However non-Gaussian models do exist, in dimension 4!

Further explore

- Today talk: ways to probe post-melonic world
- Another way: double scaling
- Rich combinatorics
- There is a closed system of equations: Schwinger-Dyson, but gets more complicated
- Preliminary investigation: study families of bubbles