The nested loop approach to the O(n) model on random maps

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Quantum Gravity in Paris, Orsay
23 March 2013

Plan

Introduction

Consider (Euclidean) 2D quantum gravity coupled to a CFT of central charge $c \leq 1$. Its lattice regularization is obtained by considering a critical statistical physics model defined on a dynamical (annealed) random map.

An unsettled question is the dependence of the intrinsic Hausdorff dimension d_H on c. Several values have been proposed :

- $d_H(c) = \frac{\sqrt{25-c}}{\sqrt{1-c}} - 1$
- $d_H(c) = 2\frac{\sqrt{25-c}+\sqrt{49-c}}{\sqrt{25-c}+\sqrt{1-c}}$
- $d_H(c) = 4$

It is believed that the second formula is correct for $c \leq 0$ and the third is correct for $c \geq 0$. (All fit the known value $d_H(c=0) = 4$.)

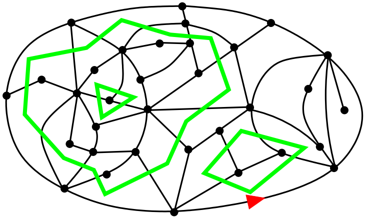
See for instance [Duplantier, arXiv:1108.3327] and references therein.

Introduction

Many statistical physics models can be reformulated in terms of “loop gases” : polymers, self-avoiding walks, percolation, Ising/Potts... and of course the O(n) model where n plays the role of a loop fugacity. For $n \in [-2, 2]$, the O(n) model has critical points of central charge :

$$c = 1 - \frac{6e^2}{1-e}, \qquad n = 2\cos \pi e.$$

This model is naturally defined on random maps (aka dynamical random lattices). On triangulations, the model was solved via matrix integral techniques [Kostov, Staudacher, Eynard, Zinn-Justin, Kristjansen...].

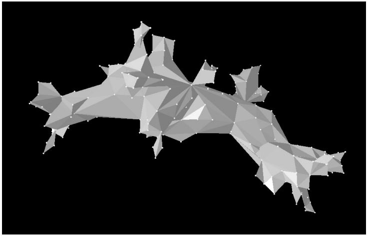


This solution consists in the computation of the partition function and other “global” quantities, but little is known on the “local” geometry...

Introduction

In contrast, the geometry of random maps without loops is now better understood.

In a generic random map, the typical graph distance between vertices is of order $m^{1/4}$, where m is the map size. The scaling limit is the Brownian map [Le Gall, Miermont...].

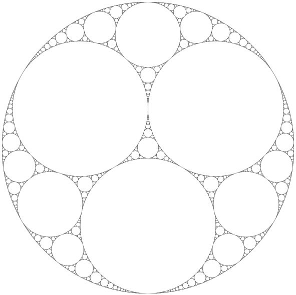


picture by G. Chapuy

It is unclear how to extend this construction to models with matter (bijections exist but do not properly encode the distances).

Introduction

Following a different approach, Le Gall and Miermont introduced models of maps with large faces, or gaskets, whose scaling limits differ from the Brownian map (which has spherical topology, hence no holes).



As we shall see, gaskets naturally arise in the O(n) loop model.

Plan

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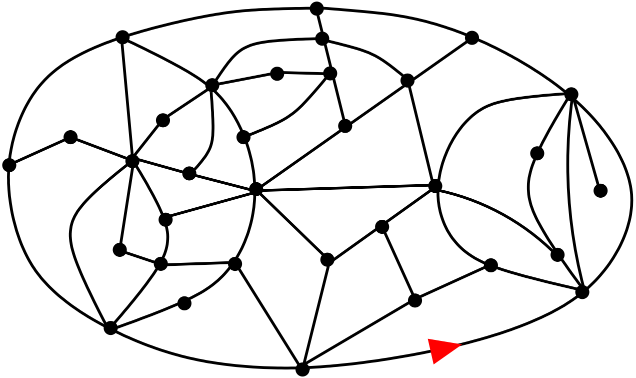
A rooted planar map is a graph embedded in the plane, considered up to continuous deformation, with a distinguished root edge incident to the outer face.



A quadrangulation with a boundary (each inner face has degree 4)

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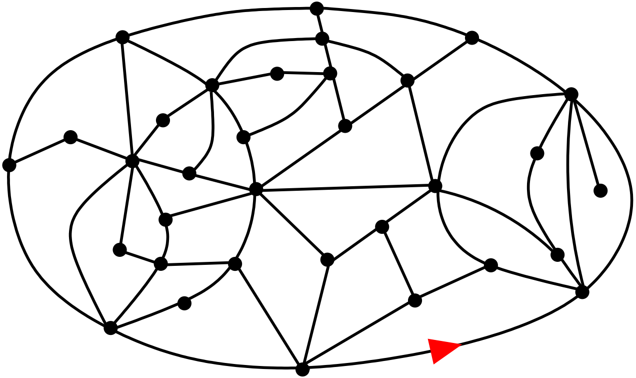

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Natural probability measures over maps : uniform distribution over maps with m edges, over triangulations with m triangles, over quadrangulations with m squares...

Boltzmann ensemble of maps with controlled face degrees

(related to the Hermitian one-matrix model)

Choose a sequence of face weights $g_1, g_2, g_3, \ldots$.

Partition function

$$\mathcal{F}_p(g_1, g_2, \ldots) = \sum_{\substack{\text{maps with} \\ \text{outer degree } p}} \prod_{k \geq 1} g_k^{\#\{\text{inner faces of degree } k\}}$$

By convention $\mathcal{F}_0(g_1, g_2, \ldots) = 1$ (vertex-map).

Specializations

- Triangulations : $g_k = g$ if $k = 3$, 0 otherwise.
- Quadrangulations : $g_k = g$ if $k = 4$, 0 otherwise.
- Maps with a controlled number of edges : $g_k = t^{k/2}$

Classification of weight sequences

A non-negative weight sequence $(g_1, g_2, \ldots)$ is either :

- non-admissible : $\mathcal{F}_p(g_1, g_2, \ldots) = \infty$ for some p

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But do non-generic critical weight sequences exist ?

Le Gall-Miermont construction

Pick an arbitrary sequence $(g_1^\circ, g_2^\circ, \ldots)$ such that

$$g_k^\circ \sim_{k\to\infty} k^{-\alpha-1/2}, \qquad \alpha \in (1,2).$$

There exists unique constants A, B such that the weight sequence $g_k := A\,B^k\,g_k^\circ$ is non-generic critical and then

$$\mathbb{P}(\text{degree of a typical face} > k) \sim \text{cst}.k^{-\alpha}$$

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Conditioning the map to have a large number m of vertices, the typical distance between vertices is then of order $m^{1/2\alpha}$ (instead of $m^{1/4}$ for generic critical sequences). This yields a non-generic scaling limit : a "stable" map of Hausdorff dimension 2α, instead of the Brownian map (dimension 4).

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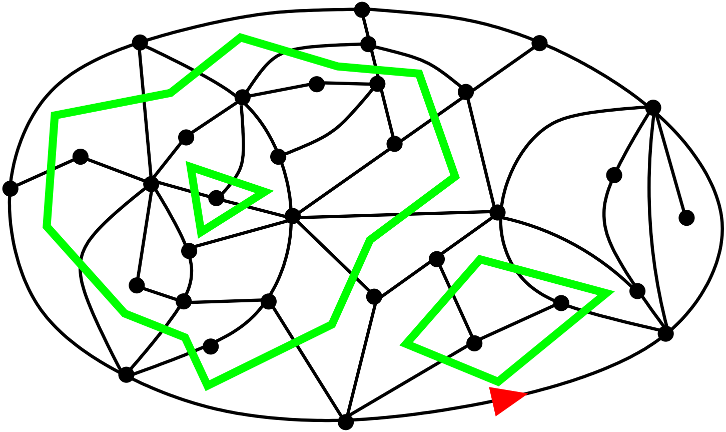
Is this construction related to a "physical" process ?

Loops

We consider self and mutually avoiding loops on the dual map (by convention, the outer face is not visited).



Each face is incident to 0 or 2 covered edges.

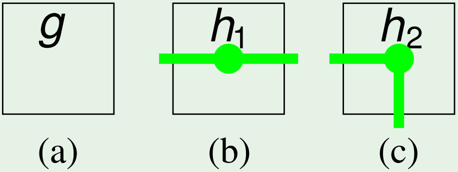
O(n) loop model

Each configuration (map with loops) receives a weight

$$n^{\#\{\text{loops}\}} \times (\text{local weights})$$

Examples

- O(n) loop model on triangulations : weight g per empty triangle, h per visited triangle.
- O(n) loop model on quadrangulations :

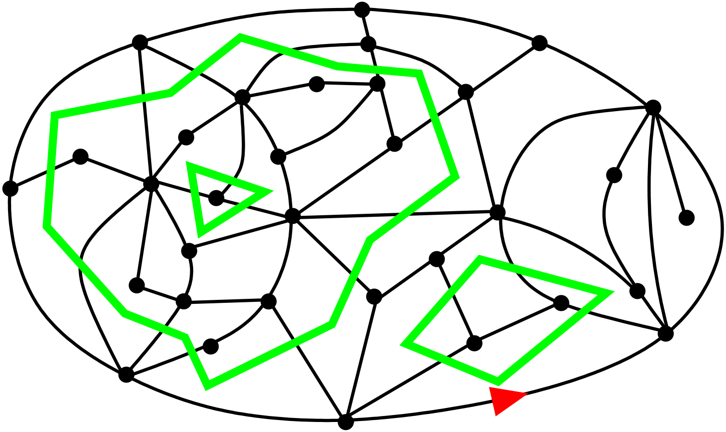


Special cases : rigid case $h_2 = 0$, twisting case $h_1 = 0$.

The partition function of all such models is actually related to the partition function for loopless maps $\mathcal{F}_p(g_1, g_2, \ldots)$.

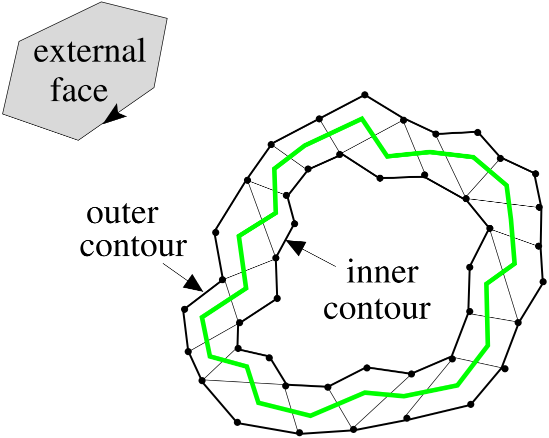
Plan

The gasket decomposition



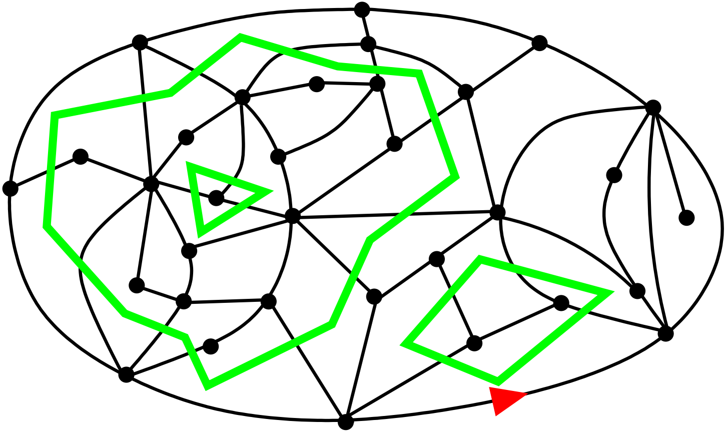
Start with a configuration of the O(n) loop model.

The gasket decomposition



The faces visited by a loop forms a necklace.

The gasket decomposition



Cut along the outer and inner contours of each outermost loop.

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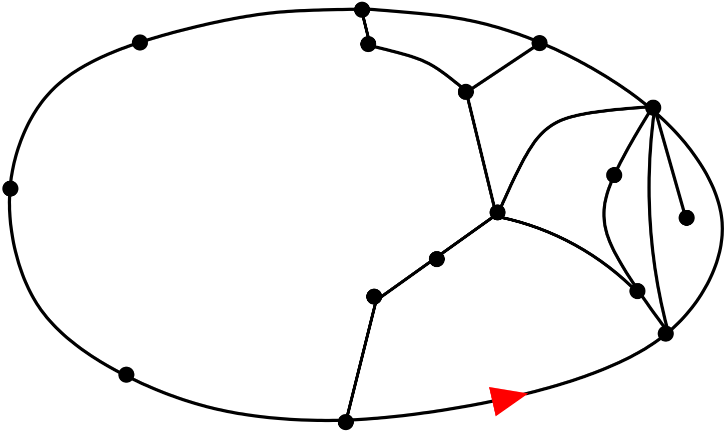

The outer component forms the gasket. It is a map without loops, with the same outer degree as the original map.

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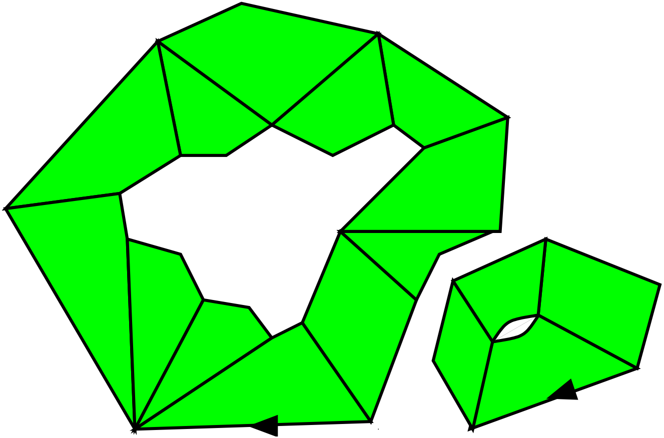
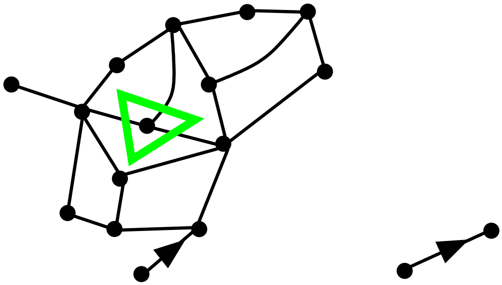


Each outermost loop forms a necklace (cyclic sequence of polygons glued side-by-side).

The gasket decomposition



Each outermost loop contains an internal configuration (of the same nature as our original object).

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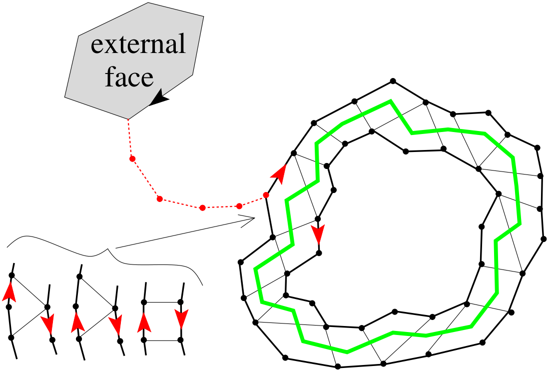

There exists a well-defined rooting procedure :

- necklaces have a distinguished edge on the outer contour,
- internal configurations are rooted.

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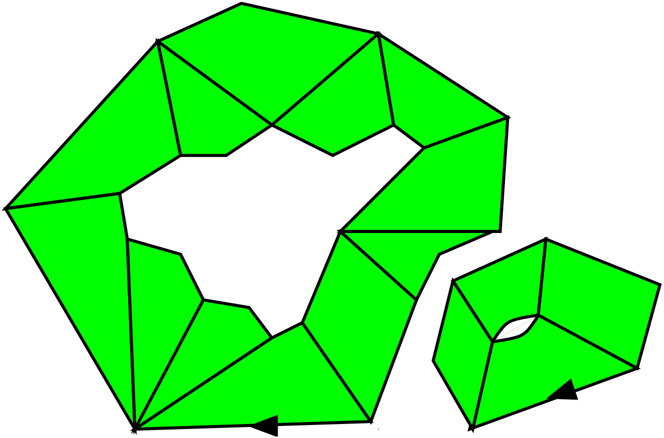

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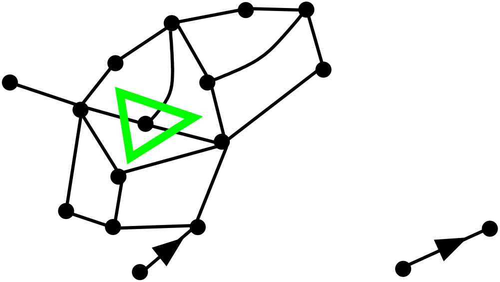

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The gasket decomposition

Bijection

$$\{\text{configurations}\} \simeq \{(\text{gasket, necklaces, internal configurations})\}$$

- A gasket is a map whose faces are either regular faces or holes.
- Each hole of degree $k \geq 1$ is associated with a necklace of outer length k.
- Each necklace of inner length $k' \geq 0$ is associated with an internal configuration of outer degree k'.

The gasket decomposition : consequences

Assumption

Suppose that the weight of a configuration is of the form

$$n^{\#\{\text{loops}\}} \prod_{k \geq 1} \left(g_k^{(0)} \right)^{\#\{\text{empty faces of degree } k\}} \prod_{\text{necklaces}} f(\text{necklace})$$

We denote by

$$F_p = F_p(n; g_1^{(0)}, g_2^{(0)}, \ldots; f)$$

the sum of weights of all configurations with outer degree p. By convention $F_0 = 1$.

Introduce the necklace generating function

$$A(x,y) = \sum_{k \geq 1} \sum_{k' \geq 1} A_{k,k'} x^k y^{k'} := \sum_{\text{necklaces}} f(\text{necklace}) x^{\text{outer length}} y^{\text{inner length}}.$$

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$F_{k'}$

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$$n \sum_{k' \geq 0} A_{k,k'} F_{k'}$$

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$$\mathcal{F}_p(g_1, g_2, \ldots)$$

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} F_{k'}$$

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The gasket decomposition : consequences

Proposition [BBG 2012]

The partition function of our O(n) loop model is obtained from the generating function for maps with controlled face degrees via

$$F_p = \mathcal{F}_p(g_1, g_2, \ldots)$$

where the g_k's satisfy the fixed-point condition

$$g_k = g_k^{(0)} + n \sum_{k' \geq 0} A_{k,k'} \mathcal{F}_{k'}(g_1, g_2, \ldots).$$

Corollary

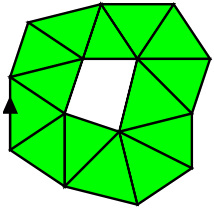
The gasket is distributed according to the Boltzmann measure with face weights $g_1, g_2, \ldots$.

We'll see that critical loop models yield a non-generic weight sequence.

Examples

- O(n) loop model on triangulations

$$A_{k,k'} = \binom{k+k'-1}{k} h^{k+k'}$$



$k = 10,\ k' = 4$

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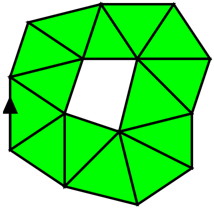

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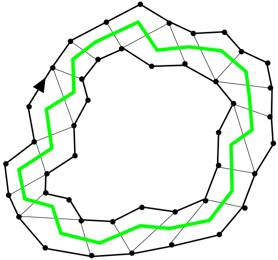
- O(n) loop model on quadrangulations

$$A_{k,k'} = \sum_{j \equiv k \bmod 2} \frac{2k}{k+k'} \binom{\frac{k+k'}{2}}{j,\ \frac{k-j}{2},\ \frac{k'-j}{2}} h_1^j h_2^{\frac{k+k'}{2}-j}$$

(vanishes for $k + k'$ odd)



$k = 24,\ k' = 20,\ j = 8$

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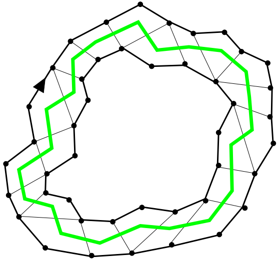

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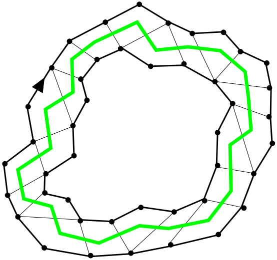

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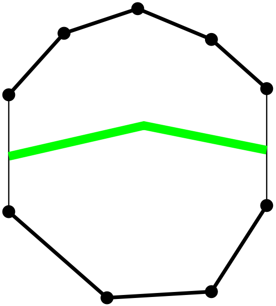
Special cases :

- rigid case $h_2 = 0$: $A_{k,k'} = h_1^k \delta_{k,k'}$
- twisting case $h_1 = 0$: $A_{2k,2k'} = 2\binom{k+k'-1}{k} h_2^{k+k'}$

Examples

- O(n) loop model with general face weights

Attach a weight $h_{\ell,\ell'}$ to each visited face with ℓ (resp. ℓ') edges incident to the outer (resp. inner) contour. In/out symmetry : $h_{\ell,\ell'} = h_{\ell',\ell}$.



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By elementary generatingfunctionology

$$A(x,y) = x \frac{\partial}{\partial x} \log H(x,y)$$

where

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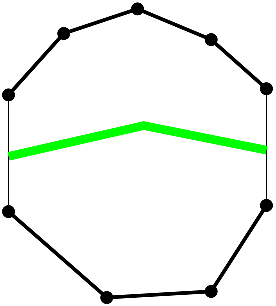

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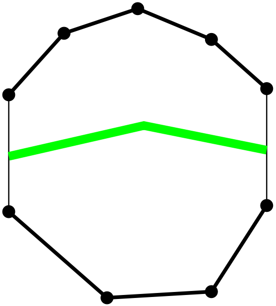
Triangular case : $h_{1,0} = h_{0,1} = h$, all other zero.
Quadrangular case : $h_{1,1} = h_2$, $h_{2,0} = h_{0,2} = h_2$, all other zero.



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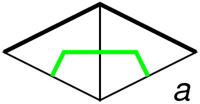
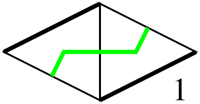
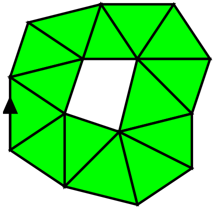
Examples

- O(n) loop model on triangulations with loop bending energy

Introduce an extra weight a whenever a loop makes two successives turns in the same direction.







ring with weight $h^{14} a^6$

Exercise : check that

$$A(x,y) = x \frac{\partial}{\partial x} \log H(x,y)$$

where

$$H(x,y) = \frac{1}{1 - ah(x+y) - (1-a^2)h^2 xy}$$

Plan

The resolvent

$$\mathcal{W}(x) := \sum_{p \geq 0} \frac{\mathcal{F}_p(g_1, g_2, \ldots)}{x^{p+1}} \quad \text{(maps with controlled face degrees)}$$

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One-cut lemma

For any admissible sequence $(g_1, g_2, \ldots)$, $\mathcal{W}$ defines an analytic function on $\mathbb{C} \setminus [\gamma_-, \gamma_+]$ where $|\gamma_-| \leq \gamma_+$. The "spectral density"

$$\rho(x) := \frac{\mathcal{W}(x - i0) - \mathcal{W}(x + i0)}{2i\pi}$$

is positive and continuous on $]\gamma_-, \gamma_+[$ and vanishes for $x \to \gamma_\pm$.

The resolvent

Functional equation for maps with controlled face degrees

The resolvent is determined by

$$\mathcal{W}(x+i0)+\mathcal{W}(x-i0) = x - \sum_{k\geq 1} g_k x^{k-1}, \qquad x \in [\gamma_-,\gamma_+]$$

and the condition $\mathcal{W}(x) \sim 1/x$ for $x \to \infty$.

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The resolvent of the O(n) loop model is obtained by making the g_k's satisfy the fixed-point condition :

$$\begin{aligned} W(x+i0)+W(x-i0) &= x - \sum_{k\geq 1} g_k^{(0)} x^{k-1} - n \sum_{k\geq 1}\sum_{k'\geq 0} A_{k,k'} x^{k-1} F_{k'} \\ &= V_0'(x) - \frac{n}{2i\pi}\oint A(x,y)W(y)dy \end{aligned}$$

Examples

- O(n) loop model on triangulations : $A(x, y) = \frac{hx}{1-h(x+y)}$

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[Kostov, Eynard, Kristjansen...]

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- O(n) loop model on quadrangulations : $A(x,y) = \frac{h_1 xy + 2h_2 x^2}{1 - h_1 xy - h_2(x^2+y^2)}$

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Special cases :

- rigid case : $W(x+i0) + W(x-i0) = V_0'(x) + \frac{n}{x} - \frac{n}{h_1 x^2} W\left(\frac{1}{h_1 x}\right)$

[BBG 2012]

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- twisting case : $\tilde{W}(x+i0) + \tilde{W}(x-i0) = \tilde{V}_0'(x) - 2n\tilde{W}\left(h_2^{-1} - x\right)$ where $W(x) = x\tilde{W}(x^2)$

[BBG 2012]

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- O(n) loop model on triangulations : $A(x,y)=\frac{hx}{1-h(x+y)}$

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[BBG 2012]

- O(n) model with general face weights : many poles...

The one-pole case

Suppose that $A(x, y)$ is rational with a single pole in y at $y = s(x)$

(as in the triangular and rigid quadrangular cases)

- In/out symmetry implies that s is a homographic involution :

$$s(x) = \frac{\alpha - \beta x}{\beta - \delta x}.$$

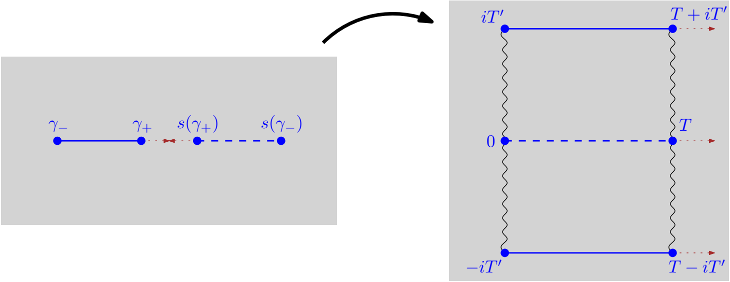
- This situation is generically realized in our model with loop bending energy !
- The functional equation reads

$$W(x + i0) + W(x - i0) - ns'(x)W(s(x)) = V_0'(x) - \frac{ns''(x)}{2s'(x)}$$

whose solution can be explicited using elliptic functions à la Eynard-Kristjansen.

The one-pole case : solution

Introduce a conformal mapping to the torus.



The homogeneous functional equation becomes

$$\omega(v+iT')+\omega(v-iT')=n\,\omega(v)$$

with ω odd and $2T$-periodic.

Non-generic critical points : γ_+ fixed point of s, $T\to\infty$, $T'=\pi$

$$\omega(v)\propto e^{-(2\mp b)v}, \qquad \pi b=\arccos\left(\frac{n}{2}\right), \qquad n\in(0,2)$$

The one-pole case : solution

Returning to the x-plane, this implies that W has a dominant singularity of the form

$$W(x) \propto (x - \gamma_+)^{1 \mp b}, \qquad x \to (\gamma_+)^+$$

hence by transfer

$$\mathbb{P}(\text{degree of a typical gasket face} > k) \sim \text{cst.} k^{-3/2 \pm b}, \qquad b \in (0, 1/2).$$

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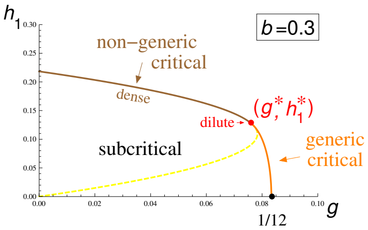
The non-generic critical points forms a "line" in the "phase diagram".

- Unless an extra cancellation occurs, the dense exponent - dominates.
- Only at one point, we obtain the dilute exponent +.
- There is also a generic critical line (as in maps without loops).



Plan

Twofold loop models

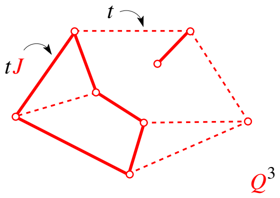
We may consider a variant of our model where we break the symmetry between both sides of the loops (they are viewed as interfaces between domains of two different colors). This is natural in the context of the Potts model.

Recall the Fortuin-Kasteleyn representation of the Q-state Potts model

$$Z_{\mathrm{Potts}}(\mathcal{M}, t, J) = \sum_{S \subset E} t^{|E|} J^{|S|} Q^{c(S)}$$

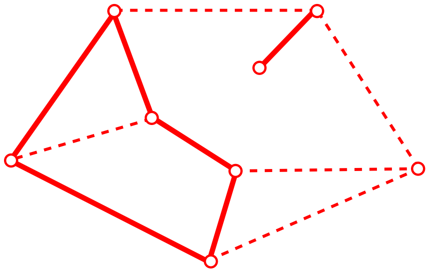
where

- t is a weight per edge,
- $J = e^K - 1$ is a weight per selected edge,
- Q appears a weight per cluster.



Twofold loop models

The Potts model may be reformulated as a twofold loop model on a triangulation :



Weights : tJ per solid edge, t per dashed edge, Q per cluster

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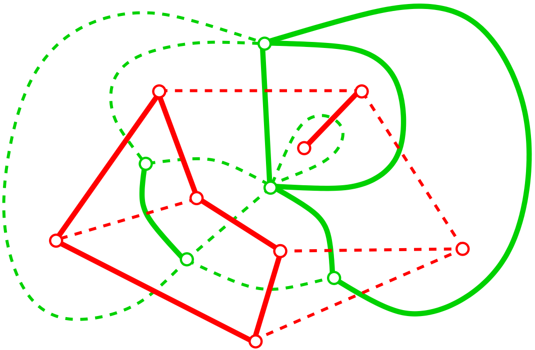

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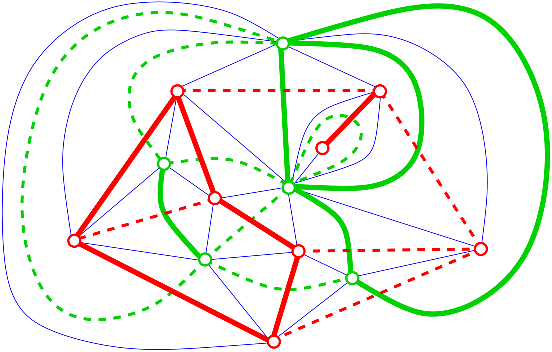

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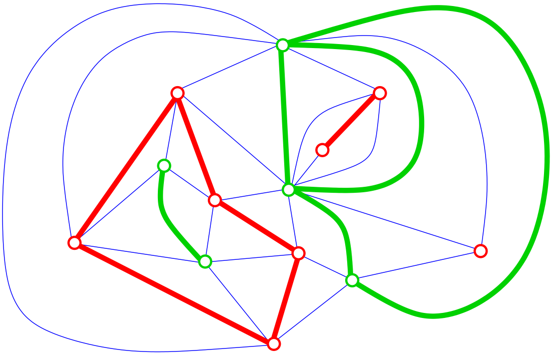

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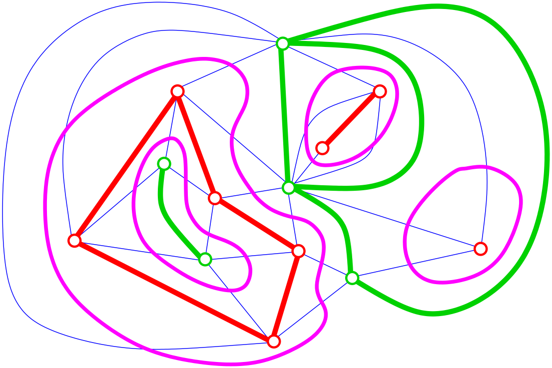

Weights : $tJ/\sqrt{Q}$ per solid edge, t per solid edge, $\sqrt{Q}$ per loop, $\sqrt{Q}$ per vertex

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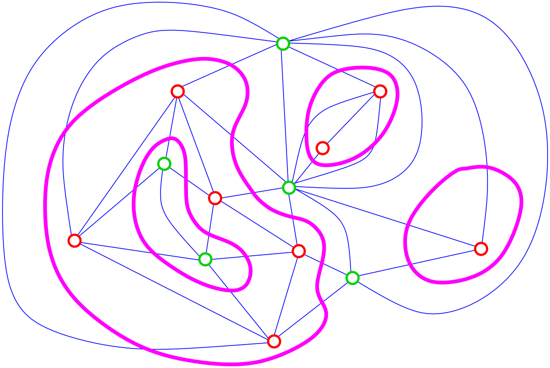

Weights : $(t^2 J^2/Q)^{1/4}$ per RRG triangle, $\sqrt{t}$ per RGG triangle, $\sqrt{Q}$ per loop, $\sqrt{Q}$ per vertex

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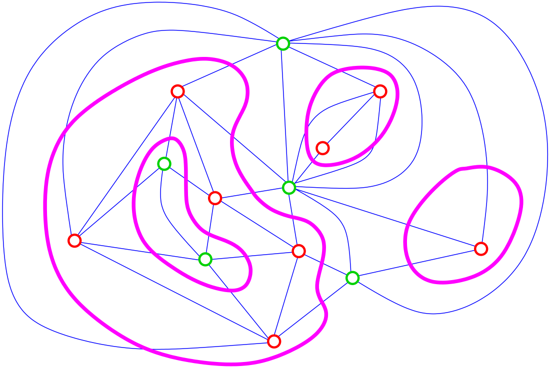

Weights : $h^{(1)}$ per RRG triangle, $h^{(2)}$ per RGG triangle, $\sqrt{Q}$ per loop, $u^{(1)}$ per vertex, $u^{(2)}$ per vertex

Twofold loop models

Caveat

One would naïvely believe that the critical point of the Potts model on random maps is the self-dual point $h^{(1)} = h^{(2)}$ ($J^2 = Q$) but this is not the case because the red-green symmetry is broken by vertex weights : $\sqrt{Q} = u^{(1)} \neq u^{(2)} = 1$!

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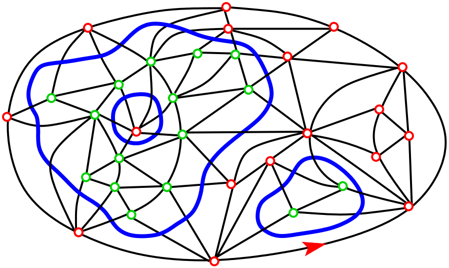
We can also introduce :

- empty faces ($\simeq$ dilute Potts model)
- curvature weight a (Ising-like coupling, leading to a possible spontaneous red-green symmetry breaking even)



Twofold loop models

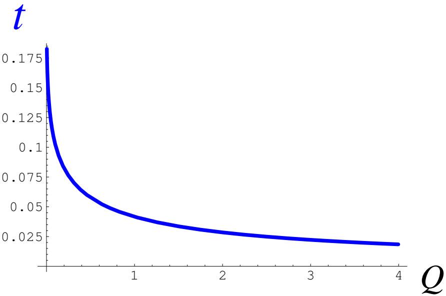
Equations are obtained by a straightforward generalization of the previous approach. Need to introduce two "resolvents" W_R and W_G satisfying

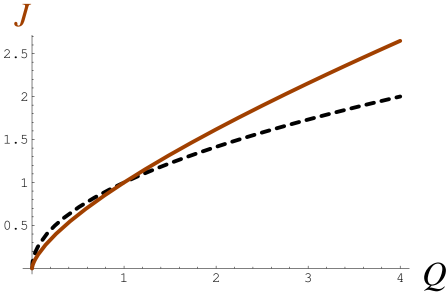
$$W_R(x+i0)+W_R(x-i0)=V_R'(x)-\frac{n}{2i\pi}\oint A_{RG}(x,y)W_G(y)dy$$
$$W_G(y+i0)+W_G(y-i0)=V_G'(y)-\frac{n}{2i\pi}\oint A_{GR}(y,x)W_R(x)dx$$

where V_R', V_G' generate empty faces, and A_{RG}, A_{GR} are necklace generating functions (essentially equal upon exchanging arguments). When loops visit only triangles, these have a single pole and we can solve the model via elliptic functions.
Non-generic critical points form a manifold of codimension 2 in the space of parameters.

Twofold loop models

For instance, for the Q-state Potts model on general random maps, there are two parameters : edge weight t, coupling $J = e^K - 1$. For $Q \leq 4$, there is a unique non-generic critical point $(t(Q), J(Q))$.





$J(Q)$ is distinct from the self-dual value $\sqrt{Q}$!

Conclusion

Main result

We have shown that the gasket of a critical O(n) loop model has a non-generic critical Boltzmann map distribution. Its scaling limit has Hausdorff dimension

$$d_H = 3 \pm \frac{2}{\pi} \arccos\left(\frac{n}{2}\right), \qquad n \in (0, 2).$$

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Caveat

We are only describing the gasket, not the full configuration !

Open questions and directions

- Understand the full scaling limit (not just the gasket), hulls...
- Higher genus, higher dimensions ?
- Extend the nested loop approach to other models : 6-vertex, ADE...