Quantum Gravity in Paris,
Mar. 20, 2013

Generalized CDT as a scaling limit of planar maps

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(collaboration with J. Ambjørn)

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Outline

- Introduction to (generalized) CDT in 2d
- Enumeration using labeled trees
- Continuum limit and two-point function
- Scaling limit of planar maps
- Loop identities

Causal Dynamical Triangulations in 2d

- CDT in 2d is a statistical system with partition function

$$Z_{CDT} = \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} g^{N(\mathcal{T})}$$

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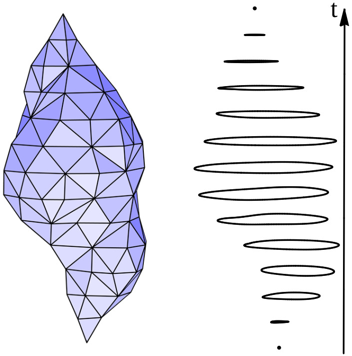

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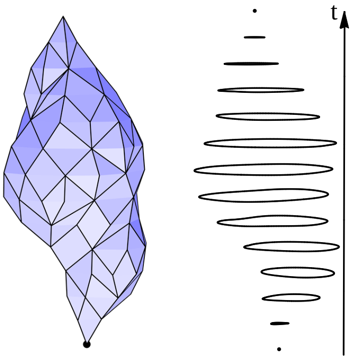

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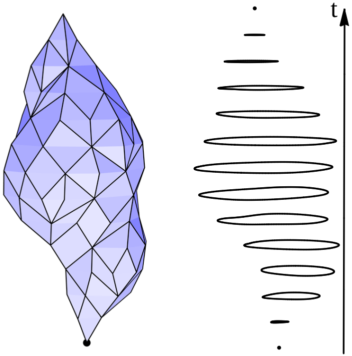
- What if we allow more than one local maximum?



Generalized CDT

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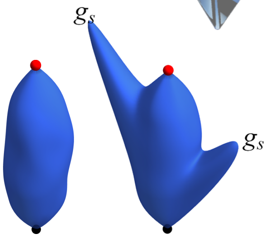

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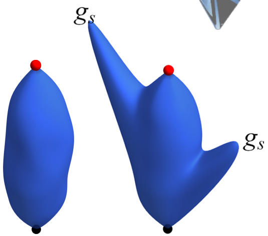

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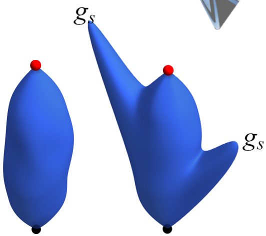

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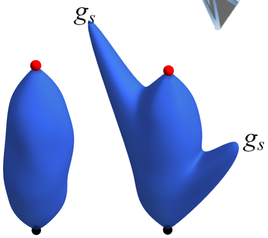
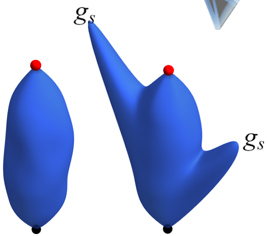


- Generalized CDT partition function

$$Z(g,\mathfrak{g}) = \sum_{\mathcal{Q}} \frac{1}{C_{\mathcal{Q}}} g^{N} \mathfrak{g}^{N_{max}},$$

sum over quadrangulations $\mathcal{Q}$ with N faces, a marked origin, and N_{max} local maxima of the distance to the origin.

Generalized CDT



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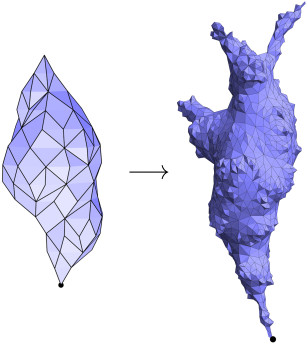
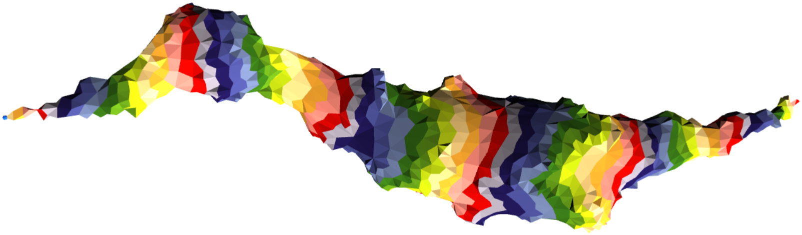
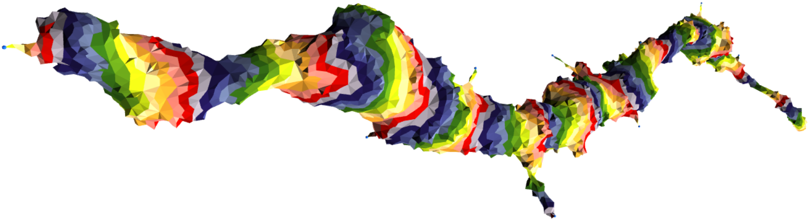


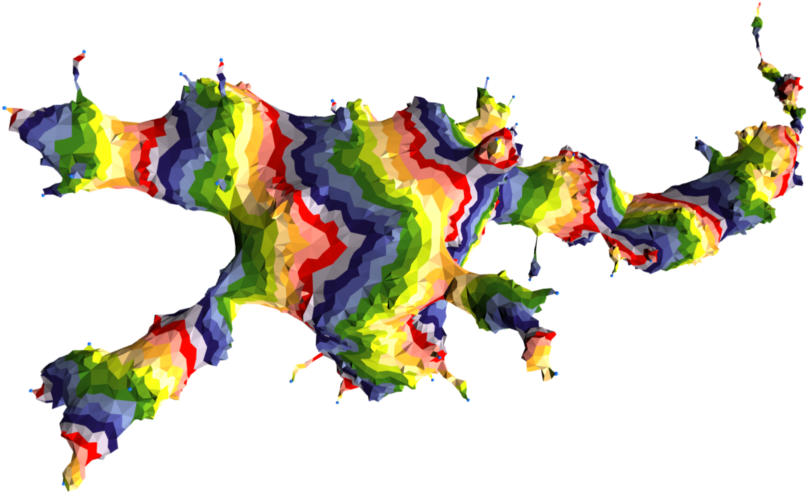
$N = 2000,\ \mathfrak{g} = 0,\ N_{max} = 1$



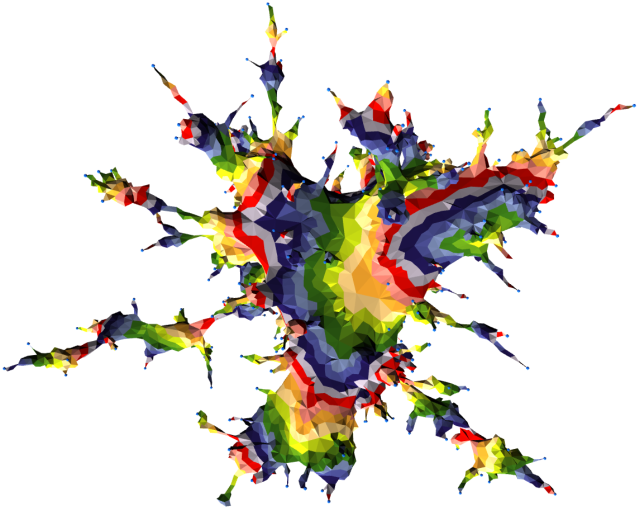
$N = 5000,\ \mathfrak{g} = 0.00007,\ N_{max} = 12$



$N = 7000,\ \mathfrak{g} = 0.0002,\ N_{max} = 38$

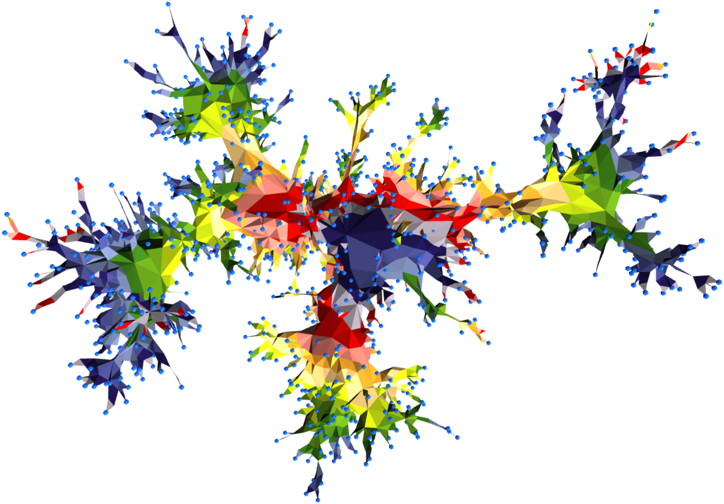


$N = 7000,\ \mathfrak{g} = 0.004,\ N_{max} = 221$

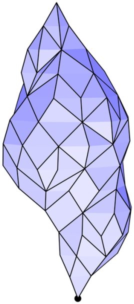


$N = 4000,\ \mathfrak{g} = 0.02,\ N_{max} = 362$

$N = 2500,\ \mathfrak{g} = 1,\ N_{max} = 1216$



Causal triangulations and trees



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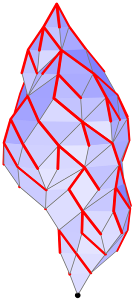

- Union of all left-most geodesics running *away from* the origin.

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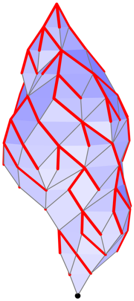
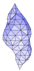


- Union of all left-most geodesics running *away from* the origin.
- Simple enumeration of planar trees:

$\#\Big\{$  $\Big\}_N = C(N), \quad C(N) = \frac{1}{N+1}\binom{2N}{N}$

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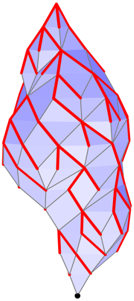

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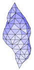
$\# \Big\{$  $\Big\}_N = C(N), \quad C(N) = \frac{1}{N+1}\binom{2N}{N}$

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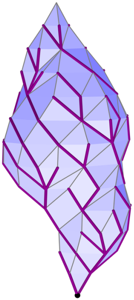

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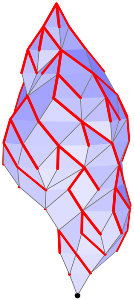

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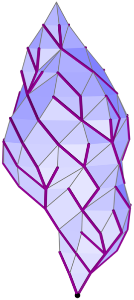
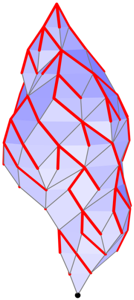
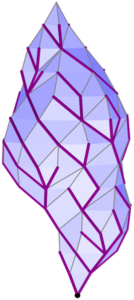


- Union of all left-most geodesics running *towards* the origin.
- Both generalize to generalized CDT leading to different representations.

Causal triangulations and trees





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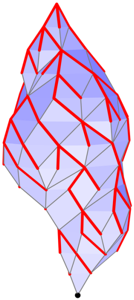
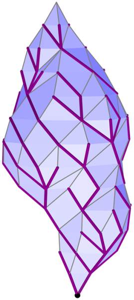
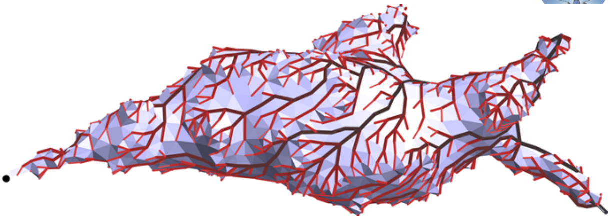
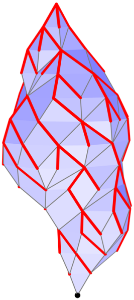


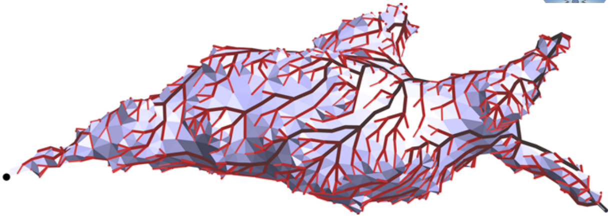




- Labeled planar trees: Schaeffer's bijection.

Causal triangulations and trees





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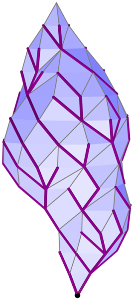
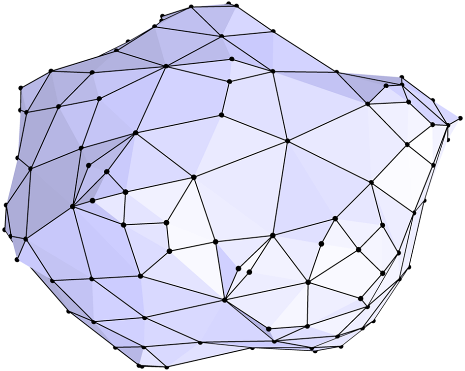


- Unlabeled planar maps (one face per local maximum).

The Cori–Vauquelin–Schaeffer bijection

[Cori, Vauquelin, '81]

[Schaeffer, '98]



- Quadrangulation

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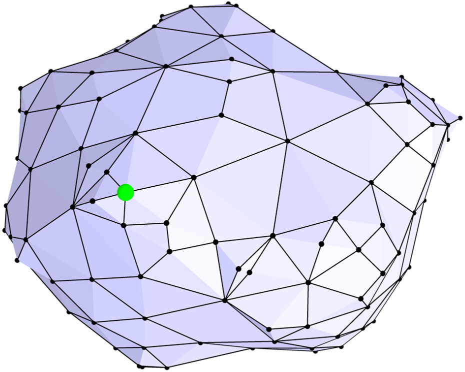

- Quadrangulation → mark a point

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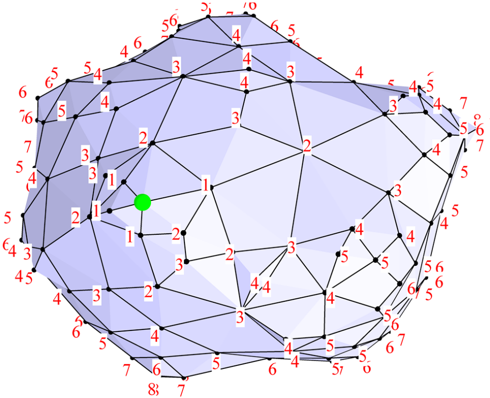

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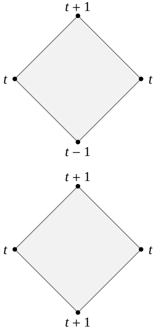

- Quadrangulation → mark a point → distance labeling

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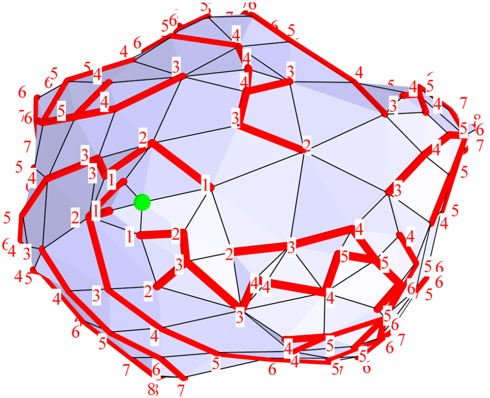

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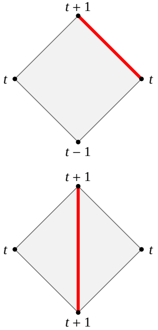
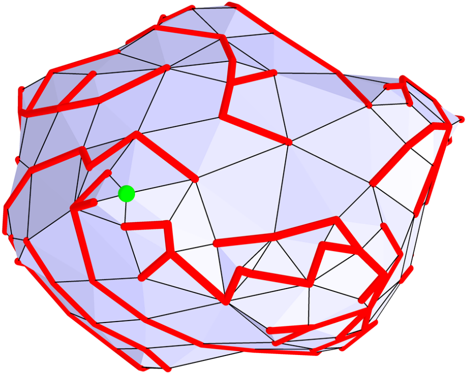


- Quadrangulation → mark a point → distance labeling → apply rules

The Cori–Vauquelin–Schaeffer bijection



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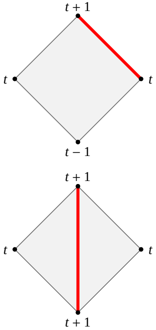

- Quadrangulation $\rightarrow$ mark a point $\rightarrow$ distance labeling $\rightarrow$ apply rules

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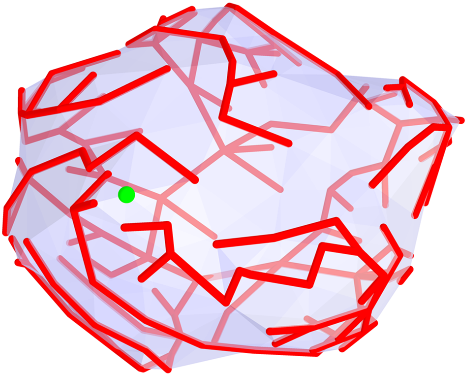

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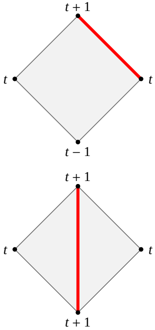

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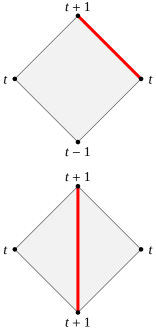

- Quadrangulation $\rightarrow$ mark a point $\rightarrow$ distance labeling $\rightarrow$ apply rules $\rightarrow$ labelled tree.

The Cori–Vauquelin–Schaeffer bijection

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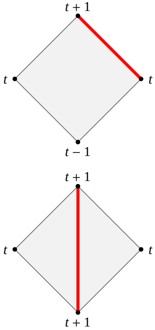

- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.

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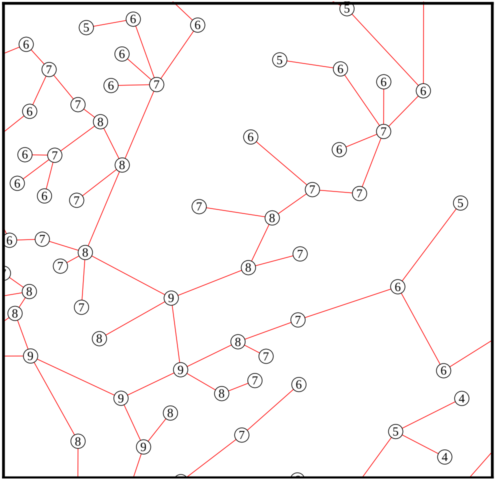

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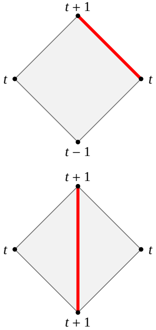

- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.
- Labelled tree

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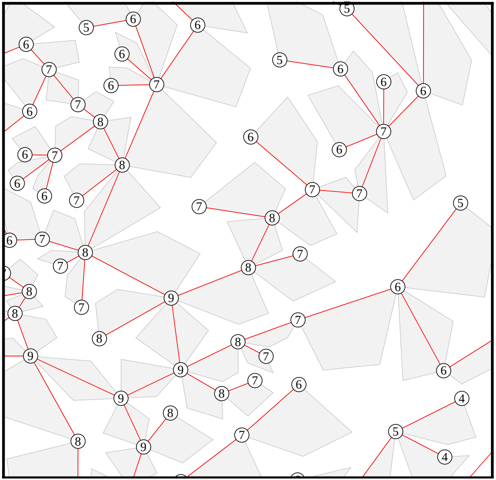

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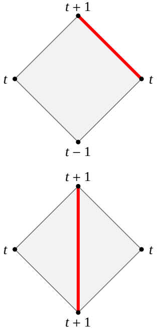
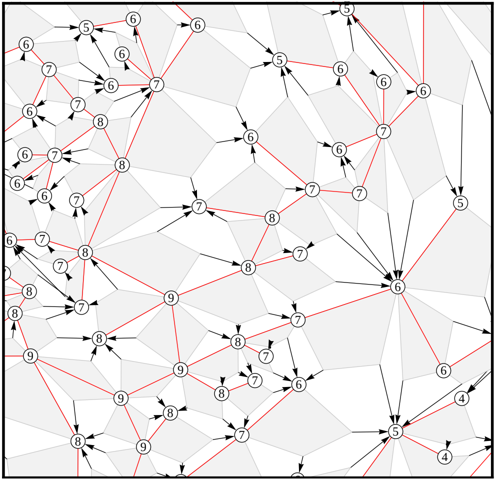


- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.
- Labelled tree → add squares

The Cori–Vauquelin–Schaeffer bijection



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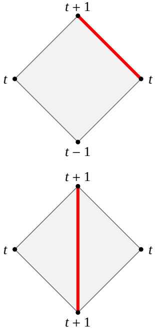
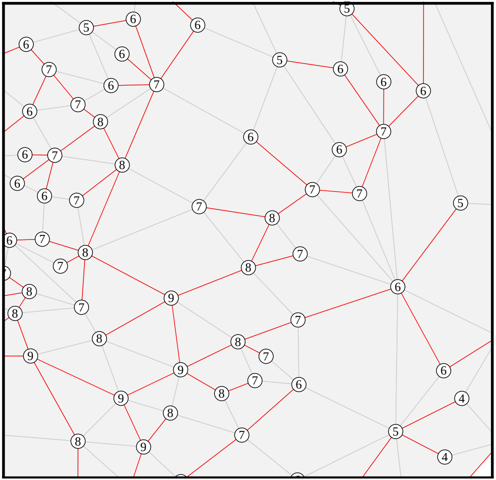


- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.
- Labelled tree → add squares → identify corners

The Cori–Vauquelin–Schaeffer bijection



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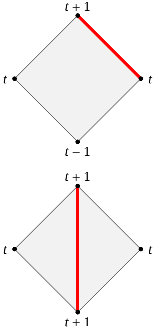

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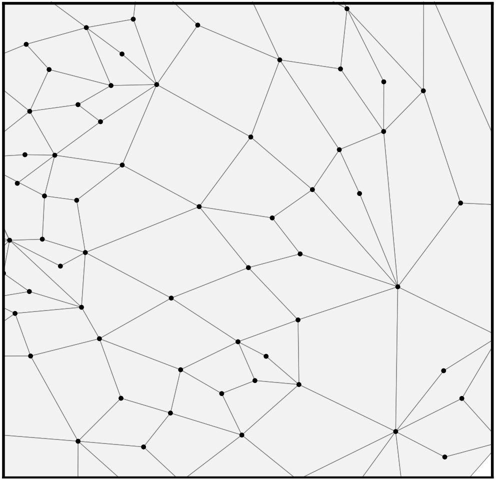

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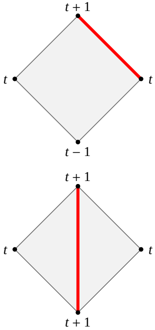
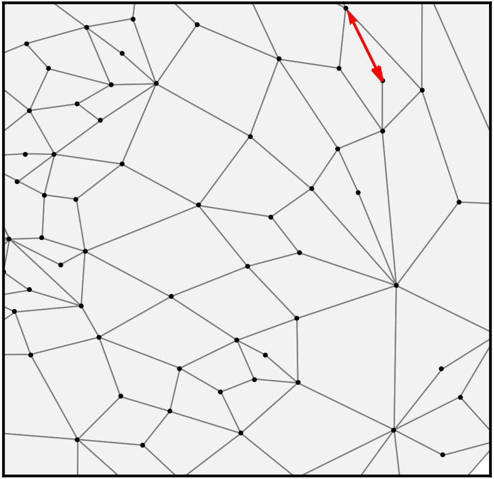
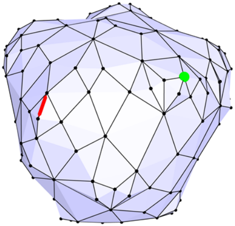


- Quadrangulation → mark a point → distance labeling → apply rules → labelled tree.
- Labelled tree → add squares → identify corners → quadrangulation.

Rooting the tree [e.g. Chassaing, Schaeffer '04]





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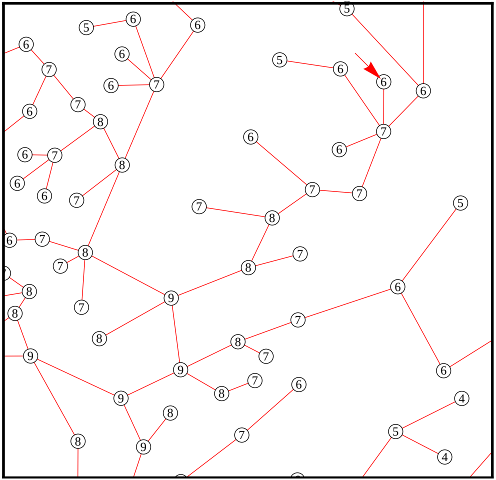

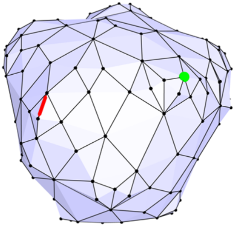

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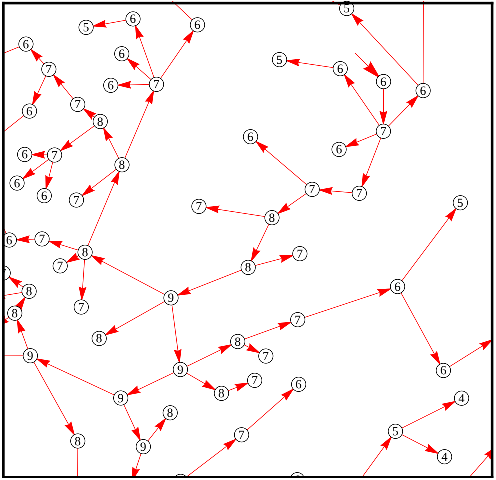

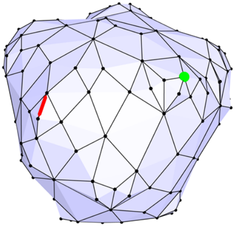

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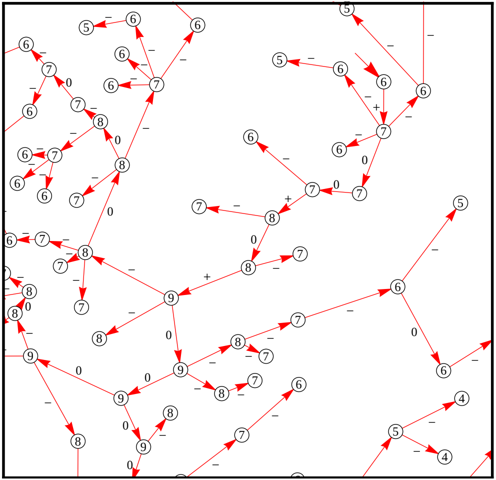

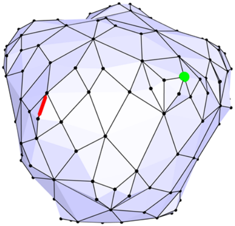

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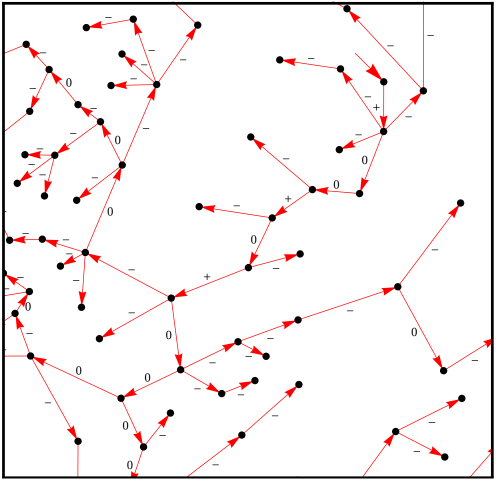

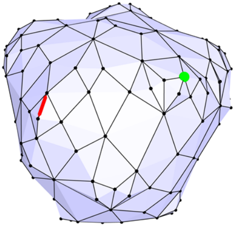

- We will be using the bijection:

$$\left\{ \begin{array}{l} \text{Quadrangulations with origin} \\ \text{and marked edge} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Rooted planar trees} \\ \text{labelled by } +, 0, - \end{array} \right\}$$

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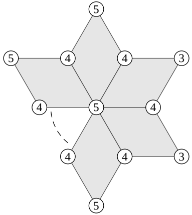

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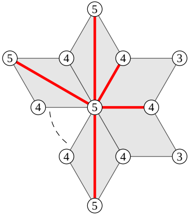

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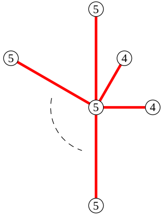

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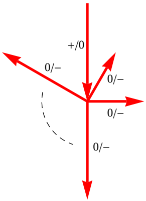

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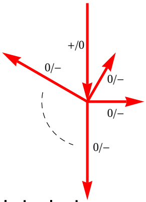

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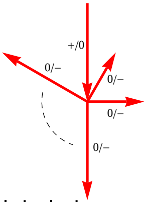

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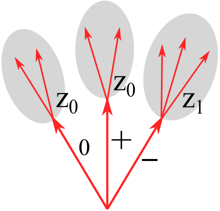
- Similarly $z_1(g, \mathfrak{g})$ but local maximum at the root not counted.
- Satisfy recursion relations:

$$z_1 = \sum_{k=0}^{\infty} (z_1 + z_0 + z_0)^k g^k = (1 - gz_1 - 2gz_0)^{-1}$$

$$z_0 = \sum_{k=0}^{\infty} (z_1 + z_0 + z_0)^k g^k + (\mathfrak{g} - 1) \sum_{k=0}^{\infty} (z_1 + z_0)^k g^k$$

$$= z_1 + (\mathfrak{g} - 1)(1 - gz_1 - gz_0)^{-1}$$



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- Combine into one equation for $z_1(g, \mathfrak{g})$:

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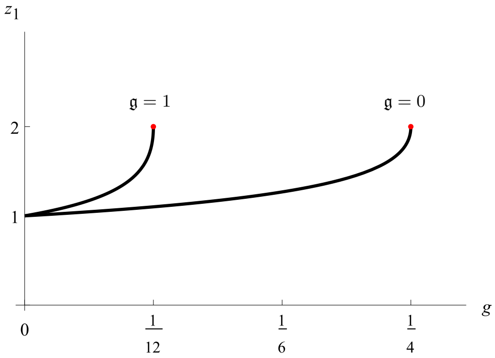

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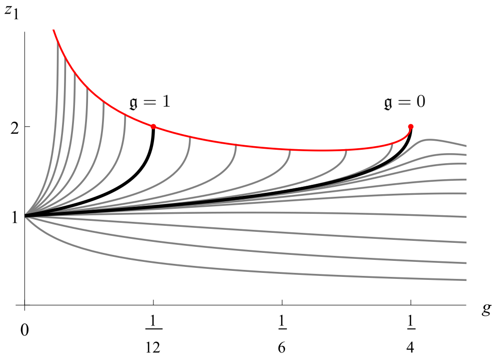

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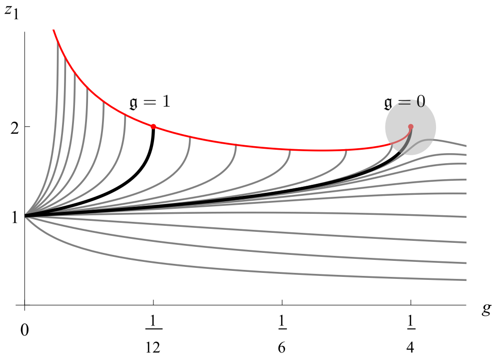

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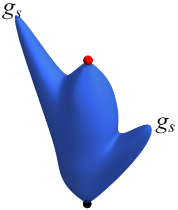

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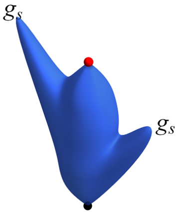
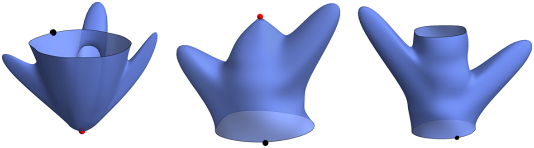
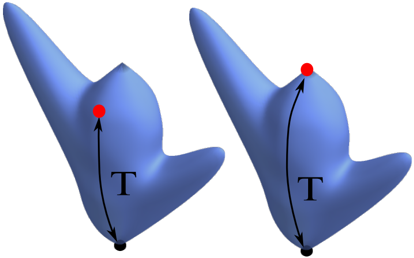


- Can compute:



Two-point function

- Amplitude for having root at distance T from origin.



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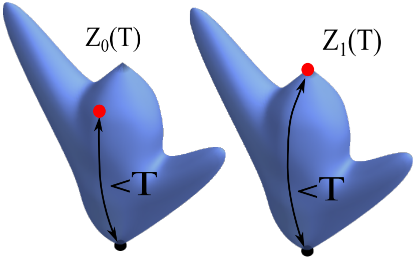

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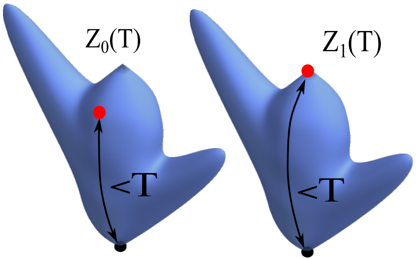

$$z_1(t) = \frac{1}{1 - gz_1(t-1) - gz_0(t) - gz_0(t+1)}$$

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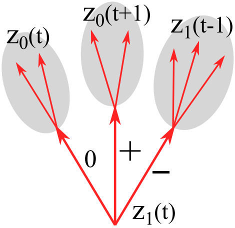

- Solution is (using methods of [Bouttier, Di Francesco, Guitter, '03]):

$$z_1(t) = z_1 \frac{1-\sigma^t}{1-\sigma^{t+1}} \frac{1-(1-\beta)\sigma-\beta\sigma^{t+3}}{1-(1-\beta)\sigma-\beta\sigma^{t+2}},$$

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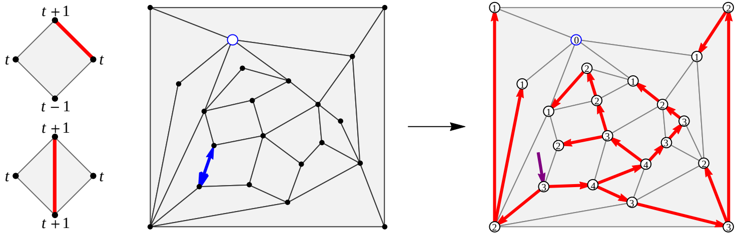
$$\xrightarrow{\mathfrak{g}_s\to\infty} \Lambda^{3/4} \frac{\cosh(\Lambda^{1/4}T')}{\sinh^3(\Lambda^{1/4}T')} \qquad T' = \mathfrak{g}_s^{1/6} T$$

- DT two-point function appears as $\mathfrak{g}_s \to \infty$! [Ambjørn, Watabiki, '95]

Planar maps



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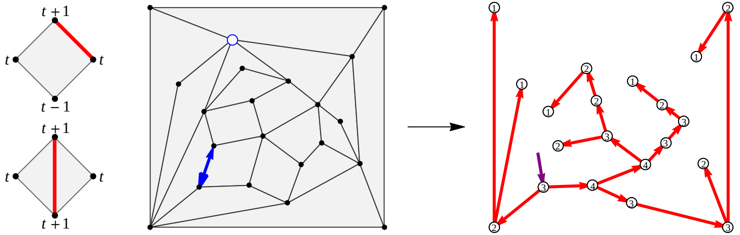

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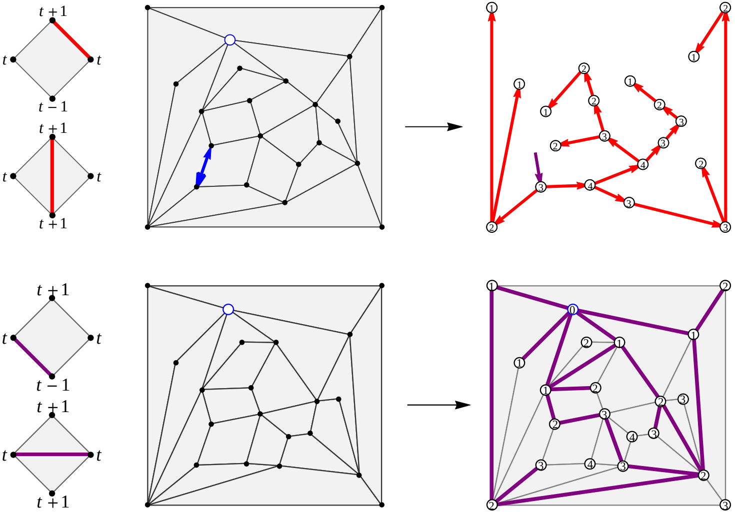

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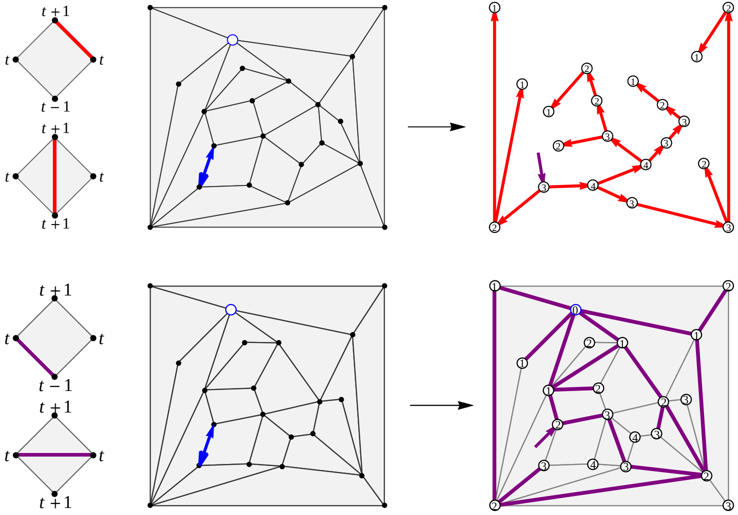

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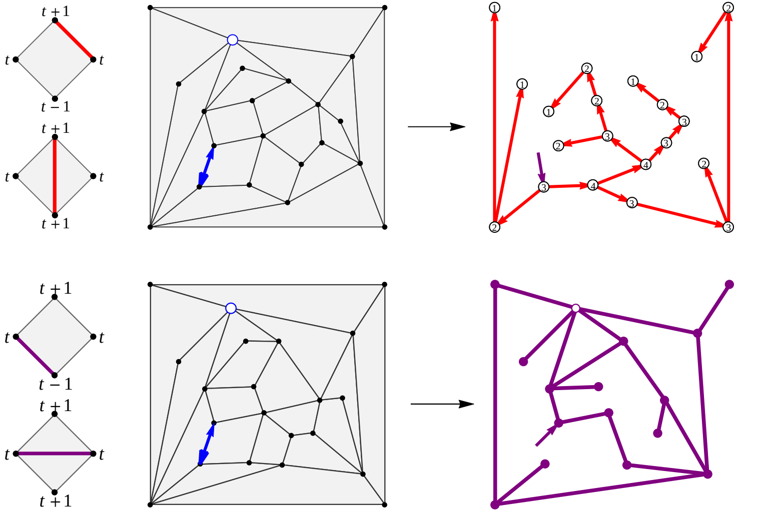

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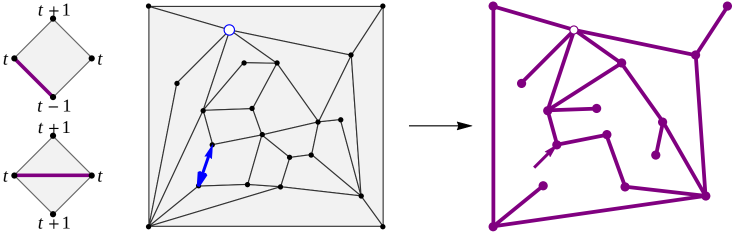

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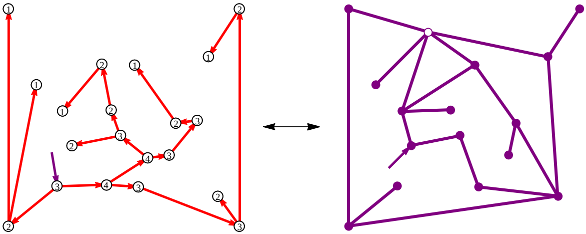
Two-point function for planar maps



- We know generating functions for trees and therefore obtain an explicit generating function

$$z_0(t+1) - z_0(t) = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{N}_t(N,n) g^N \mathfrak{g}^n$$

for the number $\mathcal{N}_t(N,n)$ of planar maps with N edges, n faces, and a marked point at distance t from the root.

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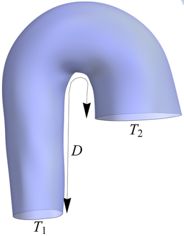
- Consider surfaces with two boundaries separated by a geodesic distance D.



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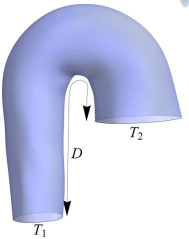

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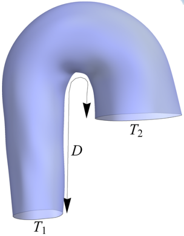

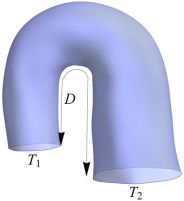

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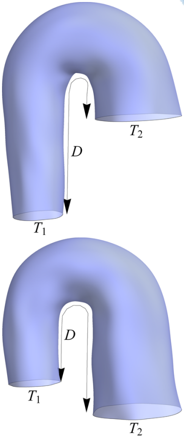

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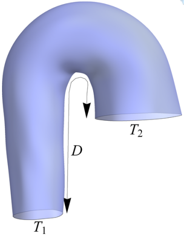

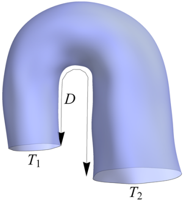

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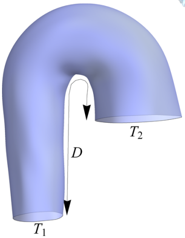

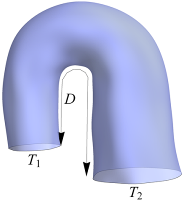

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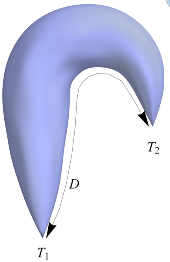
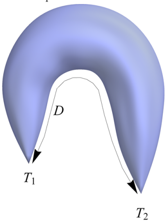
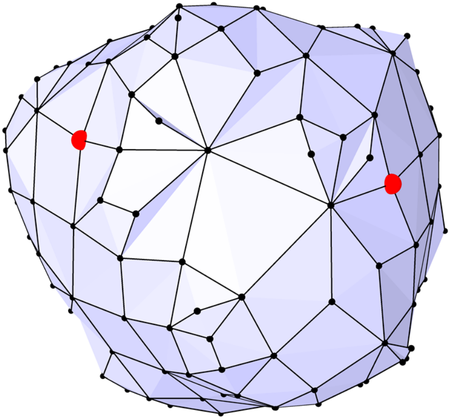
- Can we better understand this symmetry at the discrete level?
- For simplicity set the boundarylengths to zero. Straightforward generalization to finite boundaries.







$T_1 = 0$
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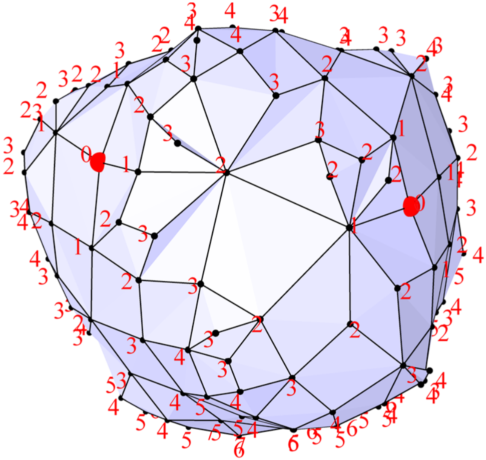
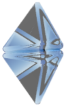
$D = 4$





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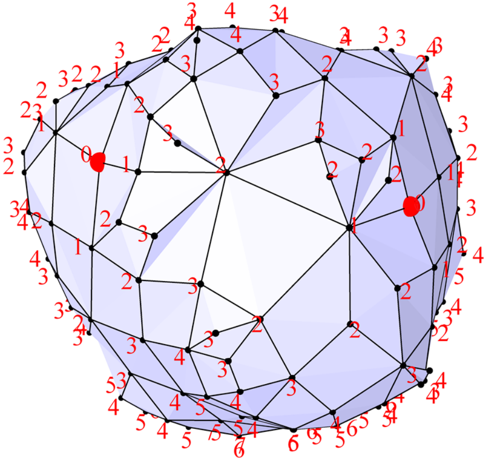

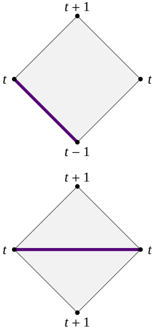

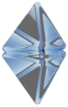

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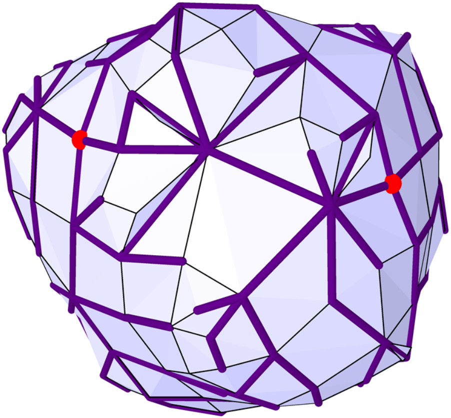

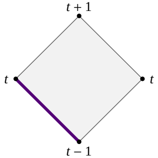

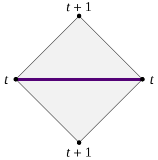

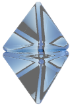

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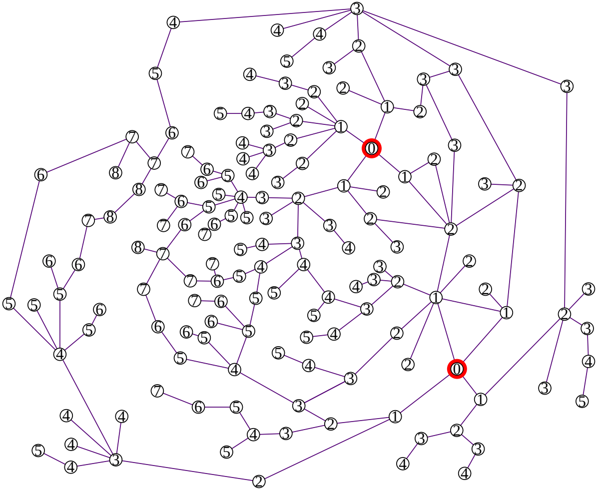

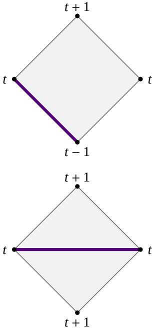

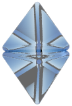

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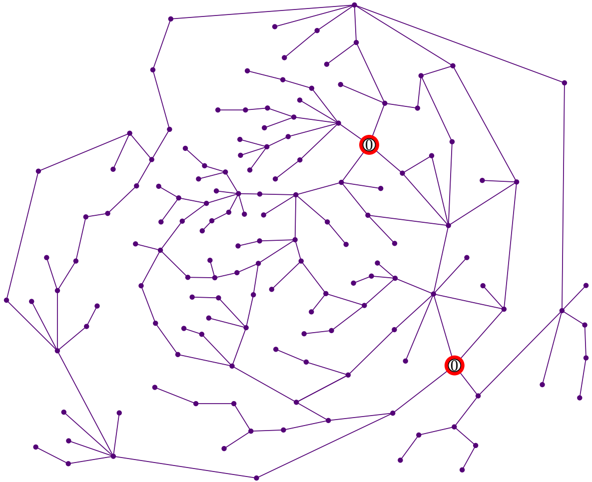

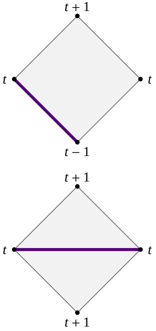

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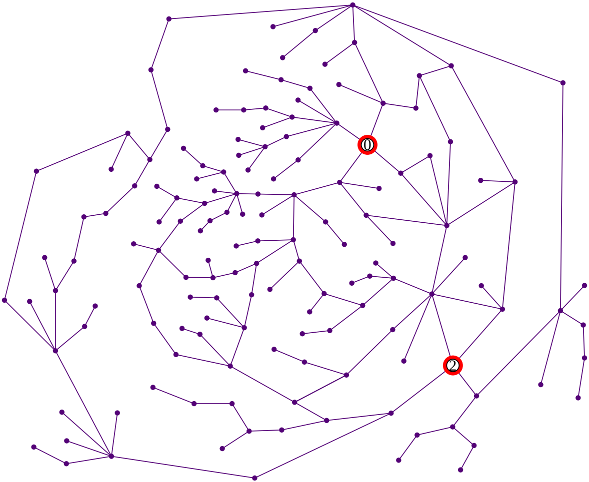

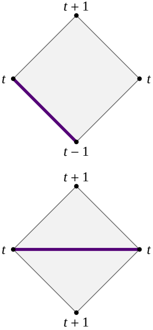

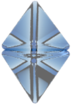

$T_1 = 0$
$T_2 = 2$
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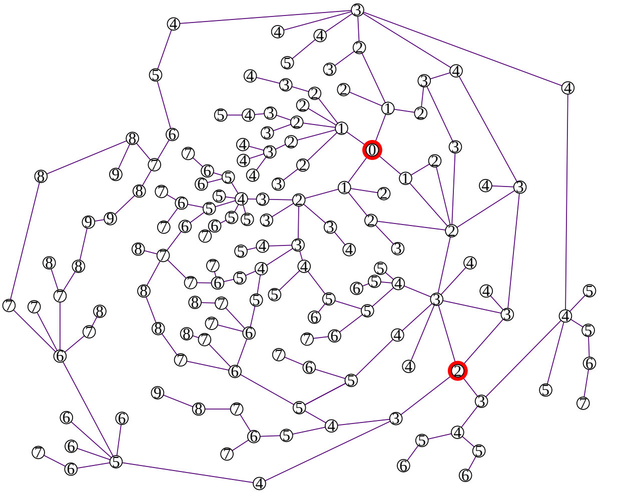

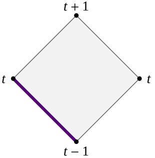

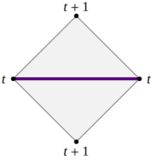

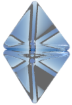

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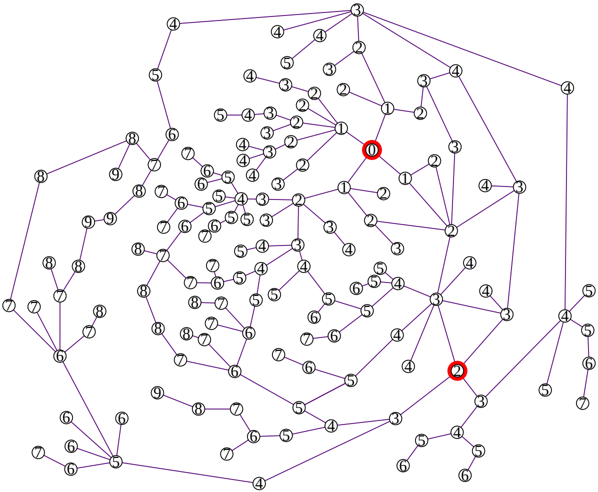

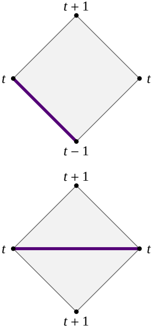

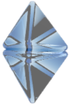

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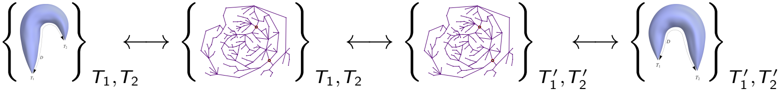
- There exists a bijection preserving the number of local maxima:



Conclusions & Outlook

- Conlusions
  - The Cori–Vauquelin–Schaeffer bijection is ideal for studying "proper-time foliations" of random surfaces.
  - Generalized CDT appears naturally as the scaling limit of random planar maps with a fixed finite number of faces.
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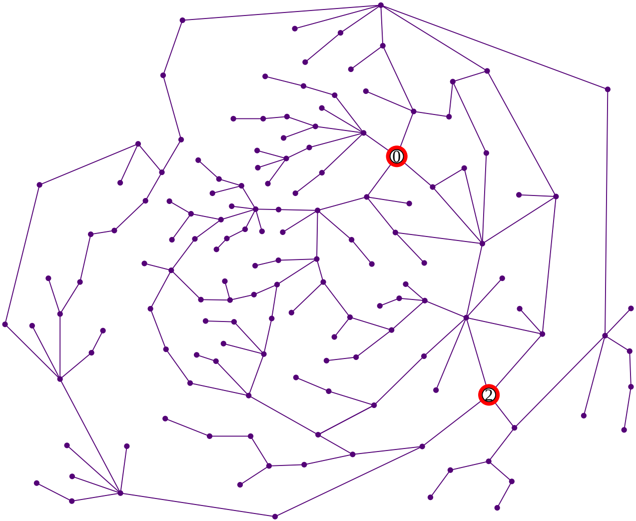
Further reading: arXiv:1302.1763

These slides and more: `http://www.nbi.dk/~budd/`

Questions?

Appendix: canonical labeling



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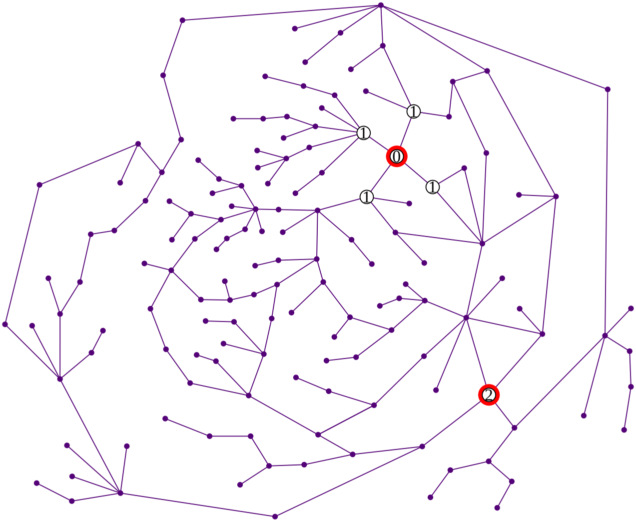

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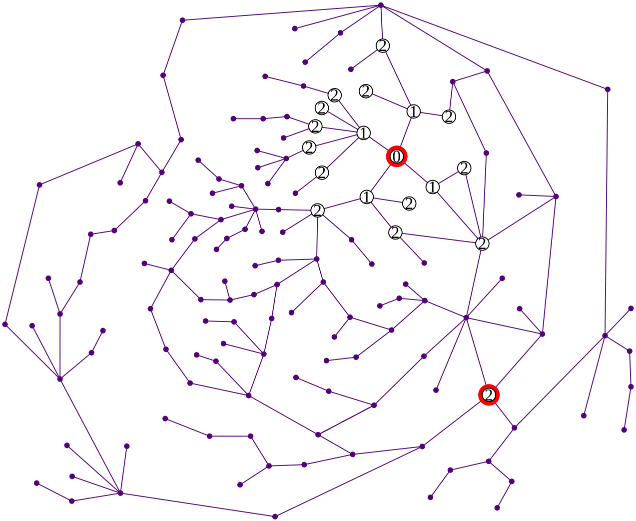

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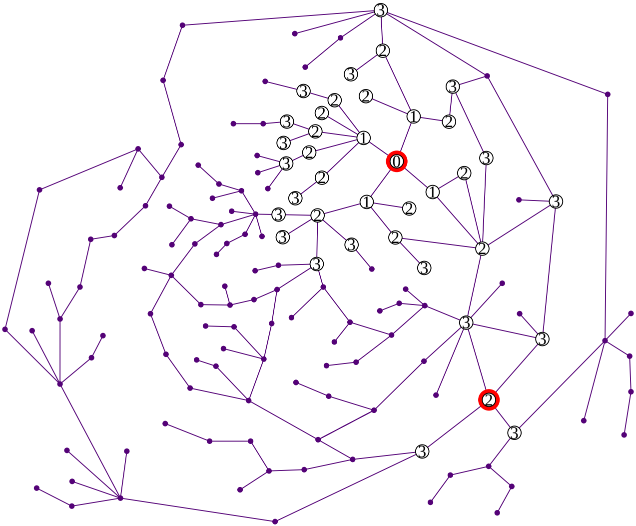

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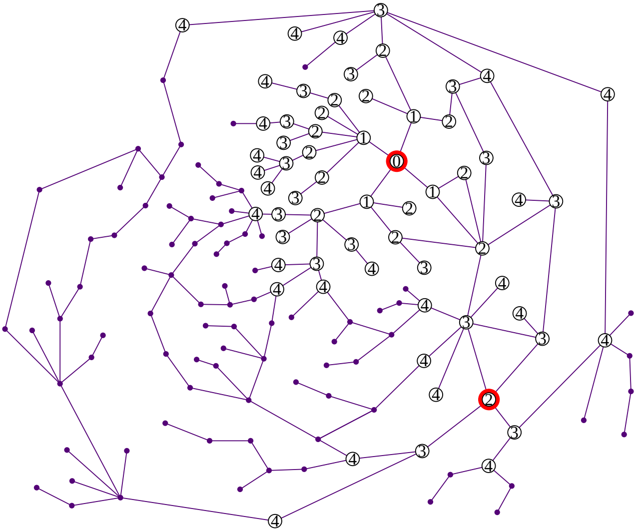

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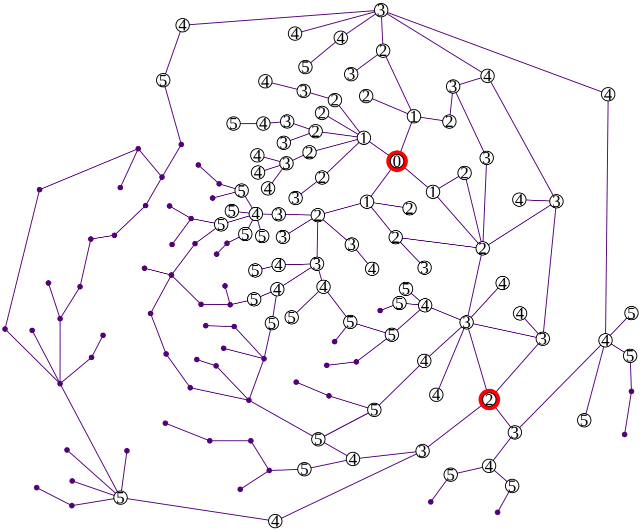

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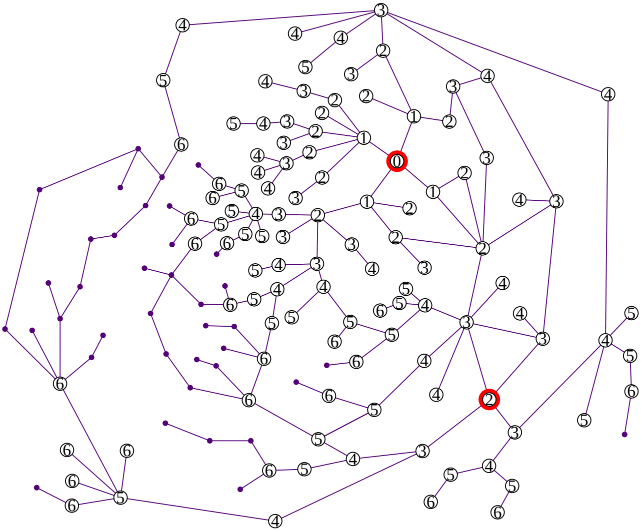

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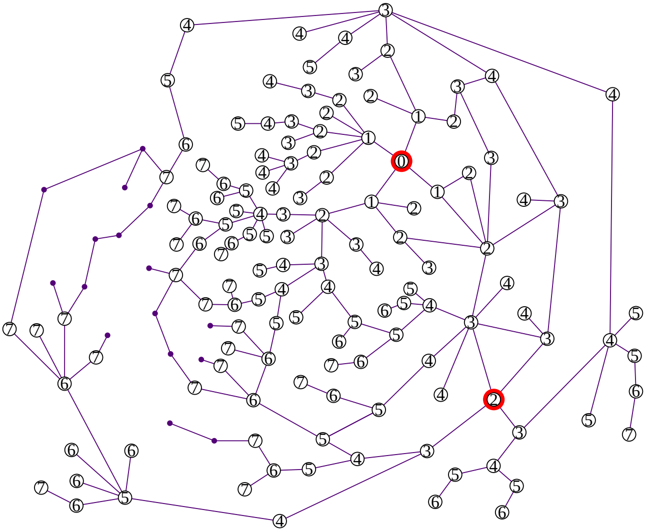

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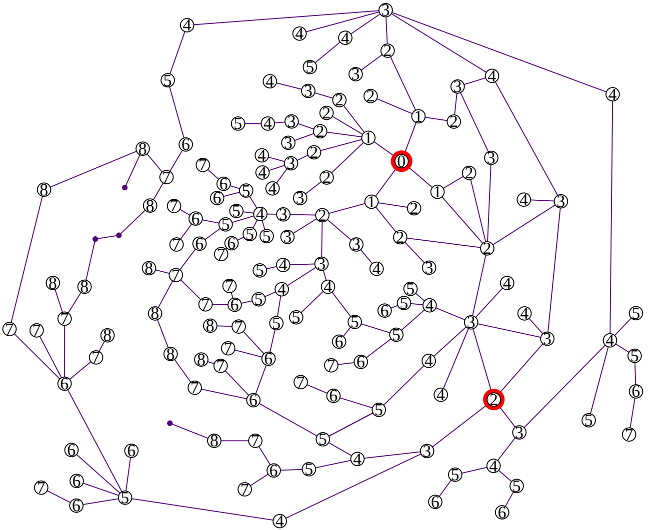

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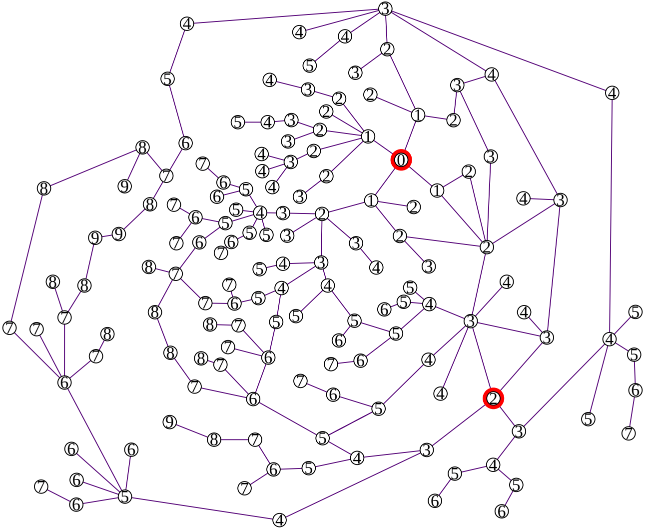
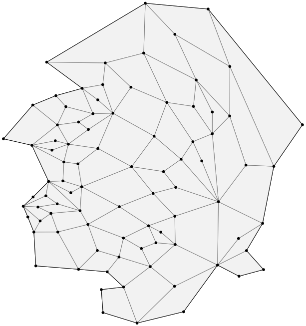


Including boundaries [Bouttier, Guitter '09, Bettinelli '11, Curien, Miermont '12]

- A quadrangulations with boundary length $2l$



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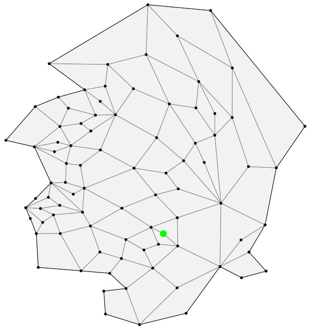
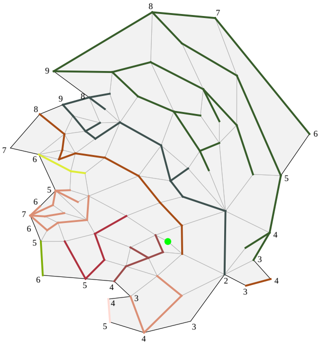


Including boundaries [Bouttier, Guitter '09, Bettinelli '11, Curien, Miermont '12]

- A quadrangulations with boundary length $2l$ and an origin.
- Applying the same prescription we obtain a forest rooted at the boundary.



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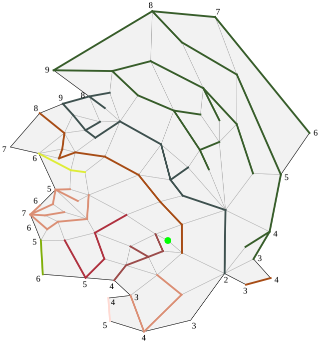

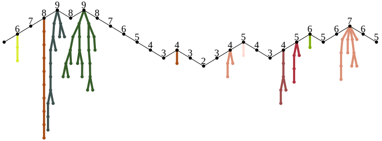

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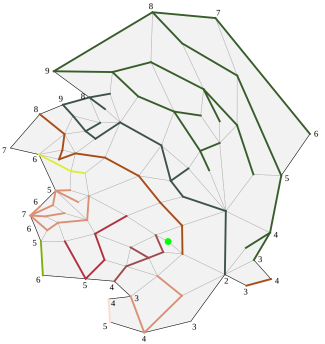

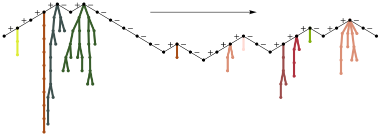

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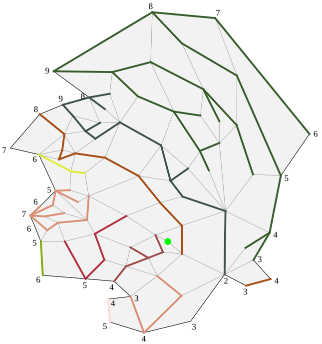

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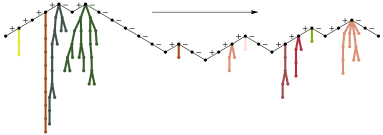
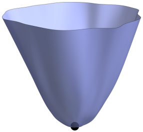
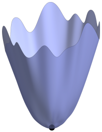
- Applying the same prescription we obtain a forest rooted at the boundary.
- The labels on the boundary arise from a (closed) random walk.
- A (possibly empty) tree grows at the end of every $+$-edge.
- There is a bijection [Bettinelli]



$$\left\{ \begin{array}{l} \text{Quadrangulations with origin} \\ \text{and boundary length } 2l \end{array} \right\} \leftrightarrow \{(+,-)\text{-sequences}\} \times \{\text{tree}\}^l$$

Disk amplitudes





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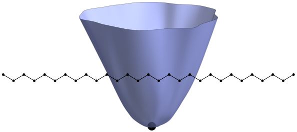

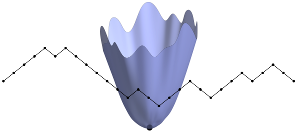

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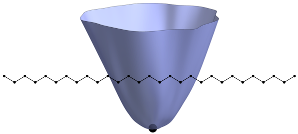
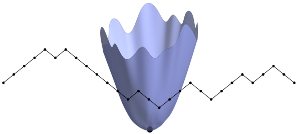


$$w(g,l) = z(g)^l$$

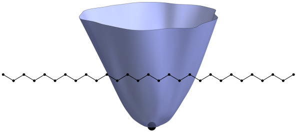
$$w(g,x) = \sum_{l=0}^{\infty} w(g,l)x^l = \frac{1}{1 - z(g)x}$$



$$w(g,l) = \binom{2l}{l} z(g)^l$$

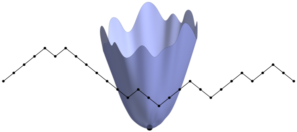
$$w(g,x) = \frac{1}{\sqrt{1 - 4z(g)x}}$$

Disk amplitudes



$$w(g,l) = z(g)^l$$

$$w(g,x) = \sum_{l=0}^{\infty} w(g,l)x^l = \frac{1}{1 - z(g)x}$$



$$w(g,l) = \binom{2l}{l} z(g)^l$$

$$w(g,x) = \frac{1}{\sqrt{1 - 4z(g)x}}$$

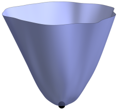
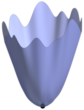
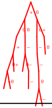
Generating function for unlabeled trees:

$$z(g) = \frac{1 - \sqrt{1 - 4g}}{2g}$$

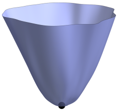
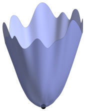
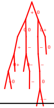
Generating function for labeled trees:

$$z(g) = \frac{1 - \sqrt{1 - 12g}}{6g}$$

Continuum limit

 $w(g,x) = \dfrac{1}{1 - zx}$	 $w(g,x) = \dfrac{1}{\sqrt{1 - 4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

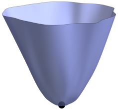
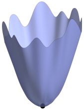
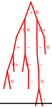
Continuum limit

 $w(g,x) = \frac{1}{1-zx}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1-\Lambda\epsilon^2), \quad z(g) = z_c(1-Z\epsilon), \quad x = x_c(1-X\epsilon)$$

Continuum limit

 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

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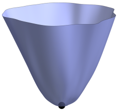
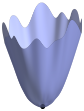
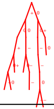
Continuum limit

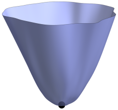
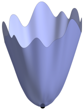
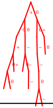
 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1-\Lambda\epsilon^2), \quad z(g) = z_c(1-Z\epsilon), \quad x = x_c(1-X\epsilon)$$

- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$

Continuum limit

 $w(g,x) = \frac{1}{1-zx}$ $W_\Lambda(X) = \frac{1}{X+Z}$	 $w(g,x) = \frac{1}{\sqrt{1-4zx}}$ $W'_\Lambda(X) = \frac{1}{\sqrt{X+Z}}$
 $z(g) = \frac{1-\sqrt{1-4g}}{2g}$ $Z = \sqrt{\Lambda}$	 $z(g) = \frac{1-\sqrt{1-12g}}{6g}$ $Z = \sqrt{\Lambda}$

- Expanding around critical point in terms of "lattice spacing" ϵ:

$$g = g_c(1-\Lambda\epsilon^2), \quad z(g) = z_c(1-Z\epsilon), \quad x = x_c(1-X\epsilon)$$

- CDT disk amplitude: $W_\Lambda(X) = \frac{1}{X+\sqrt{\Lambda}}$
- DT disk amplitude with marked point: $W'_\Lambda(X) = \frac{1}{\sqrt{X+\sqrt{\Lambda}}}$. Integrate w.r.t. Λ to remove mark: $W_\Lambda(X) = \frac{2}{3}(X - \frac{1}{2}\sqrt{\Lambda})\sqrt{X+\sqrt{\Lambda}}$.