Discrete and continuum 2 dimensional quantum gravity

François David & Bertrand Eynard
IPhT Saclay

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1. Continuum and discrete 2D gravity: what remains to be understood?
2. Circle packings and circle patterns
3. Delaunay circle patterns and planar maps
4. A measure over planar triangulations
5. Kähler geometry over triangulation space and 3D hyperbolic geometry
6. Discretized Faddev-Popov operator and Polyakov's 2D gravity
7. Conclusion, open questions

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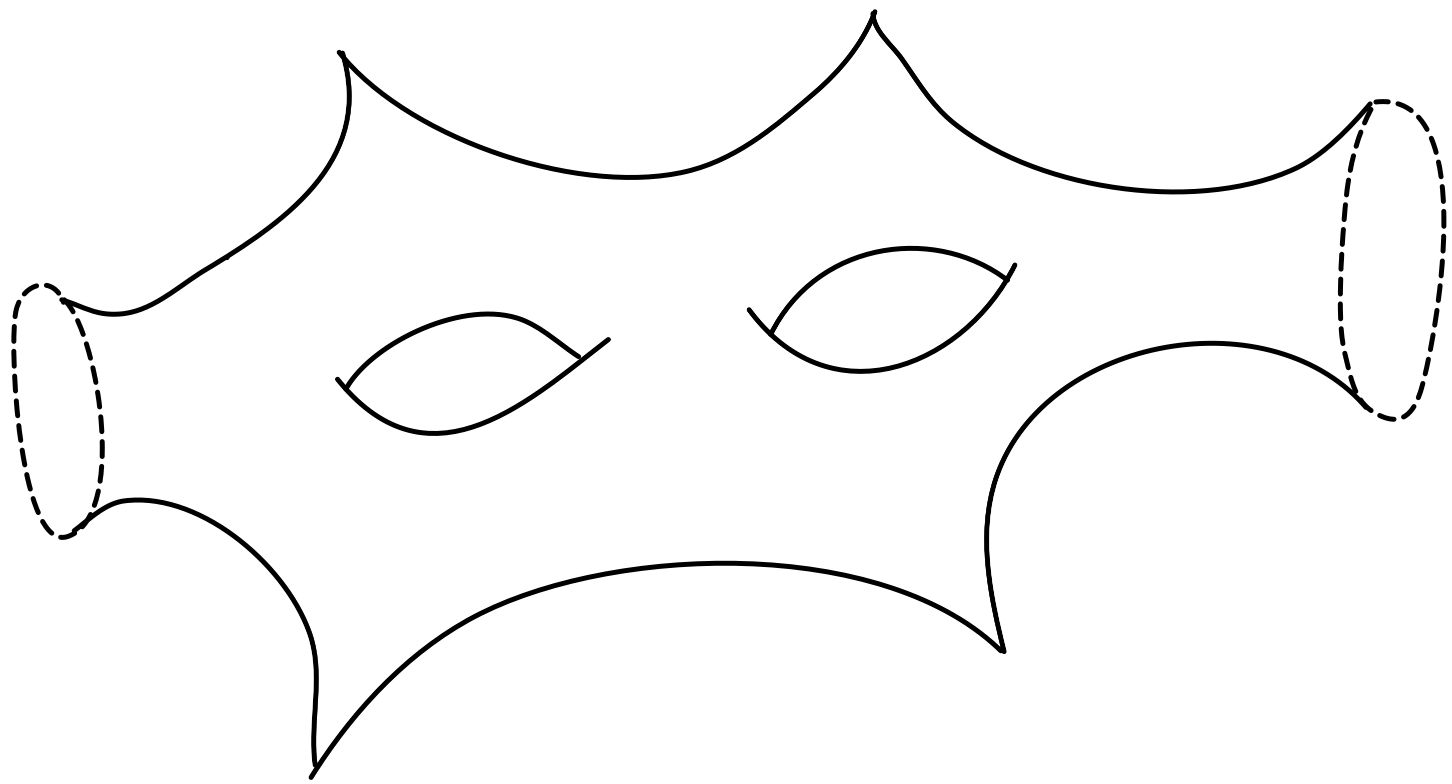
2D continuum quantum gravity

Integral over 2D Riemannian metrics + matter fields

$$\int \mathcal{D}[g_{ab}]\, e^{-\mu \int \sqrt{g}} \int \mathcal{D}[X]\, e^{-S[X,g]}$$



A fairly well understood theory

Functional integral over 2d Riemannian metrics, conformal gauge

$$g_{ab} = \hat{g}_{ab}\, e^{\phi}$$

A. Polyakov 1981

Faddeev-Popov ghost systems leads to effective action for the remaining conformal factor

$$\int \mathcal{D}[g_{ab}] = \int \mathcal{D}_{\hat{g}e^{\phi}}[\phi]\ \det(\nabla^{FP}_{\hat{g}}) = \int \mathcal{D}_{\hat{g}}[\varphi]\ e^{S_L[\varphi]}$$

Liouville theory: conformally invariant theory

$$S_L[\varphi] = \frac{1}{2\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2}(\hat{\nabla}\varphi)^2 + \frac{Q}{2}\hat{R}\,\varphi + \mu_R\, e^{\gamma\varphi}\right)$$

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} \qquad\qquad c_L = 1 + 6Q^2 = 26 - c_M$$

Real positive action for $c_M \leq 1$

Non-critical strings: conformal mode + tachyon background

No matter fields: pure gravity

Relation with the critical string

$$S_L[\phi] = \frac{1}{2\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2\gamma^2}(\hat{\nabla}\phi)^2 + \frac{Q}{2\gamma}\hat{R}\,\phi + \mu_R\, e^{\phi}\right)$$

For $c_M = 25$ $\qquad Q = 0 \qquad \gamma^2 = -4$

If matter fields taken to be 25 bosons = space coordinates X

$$S_M[X] = \frac{1}{2\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2}(\hat{\nabla}X)^2\right)$$

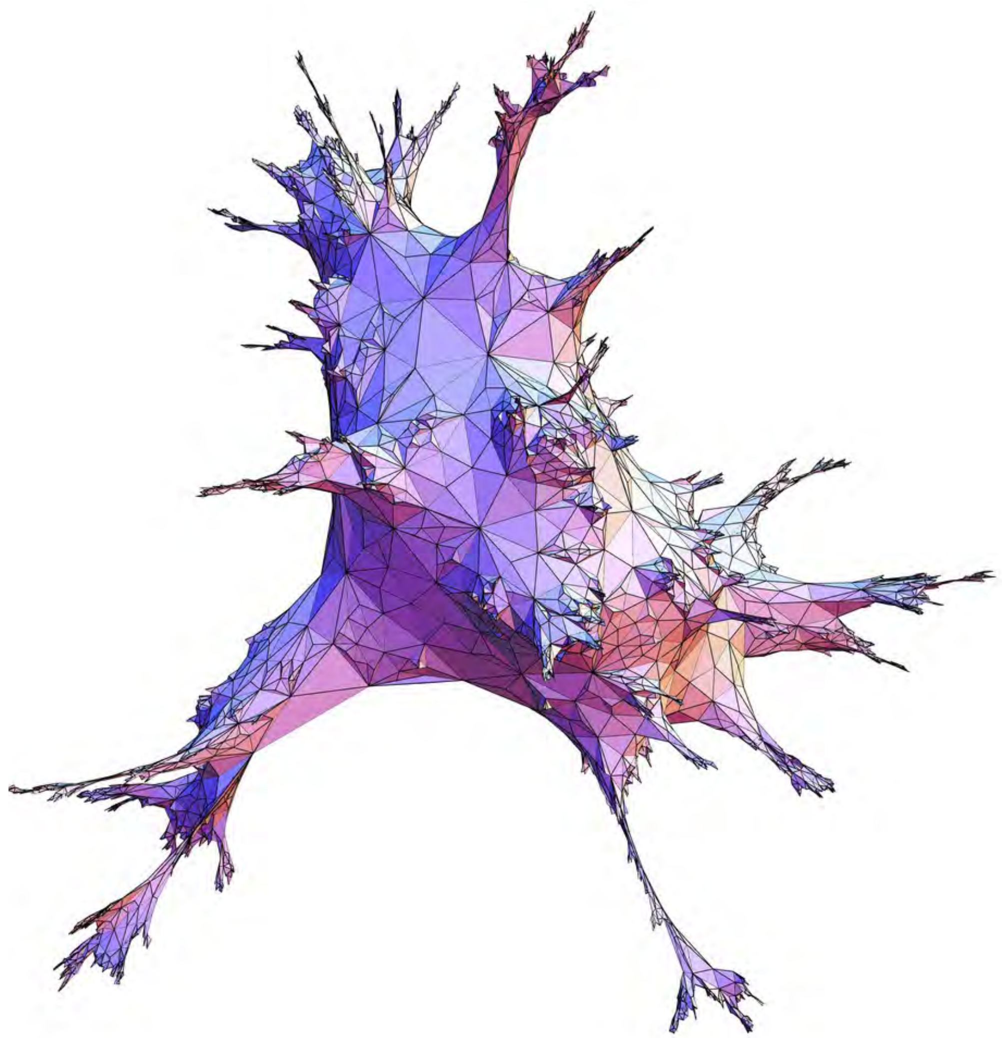
The conformal mode plays the rôle of real time T

One recovers the critical bosonic string in the flat Minkowski background !

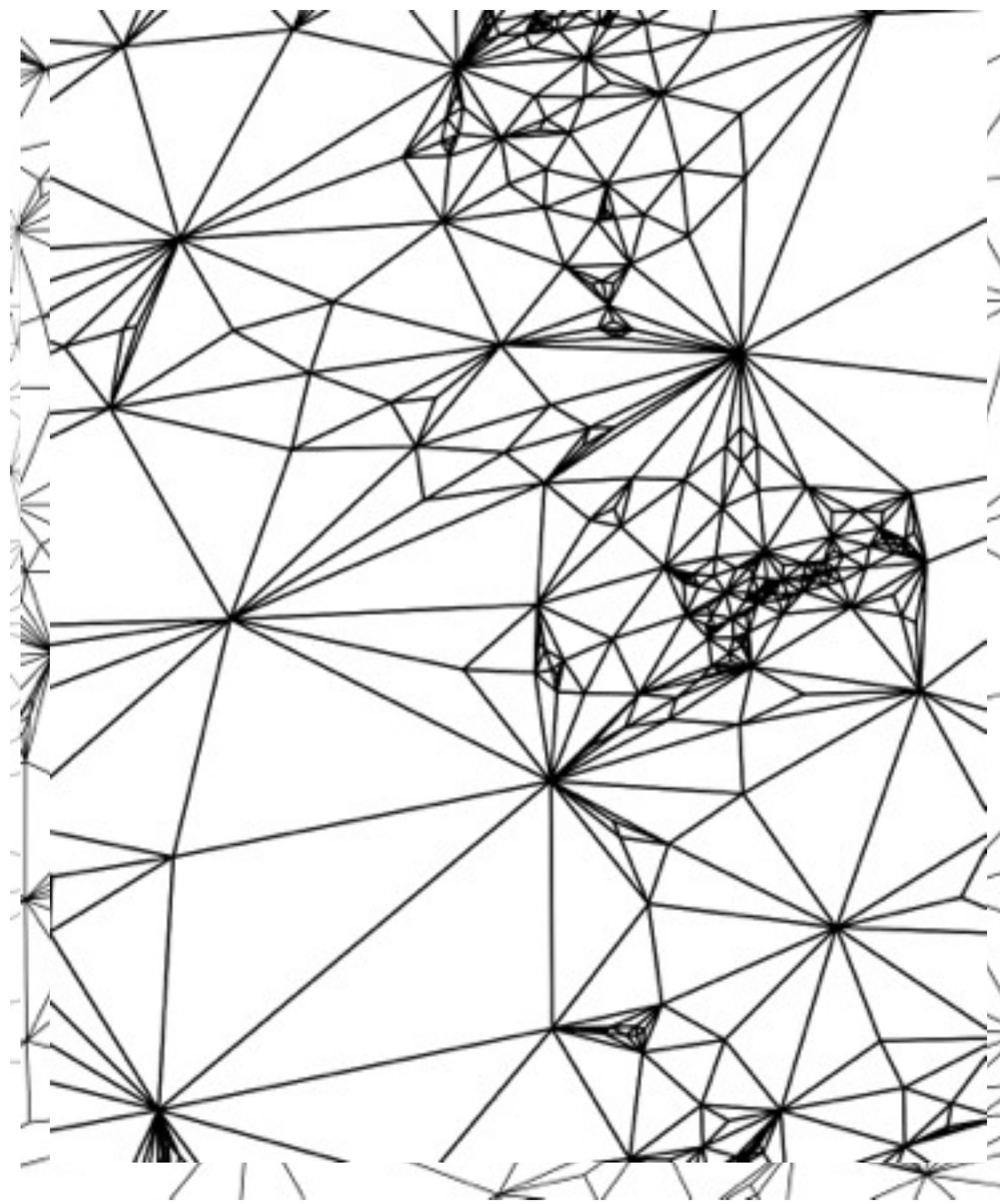
$$S[X] = \frac{1}{4\pi}\int \sqrt{\hat{g}}\left(\frac{1}{2}(G_{\mu\nu}\hat{\nabla}X^{\mu}\hat{\nabla}X^{\nu}\right)$$

2D discrete quantum gravity

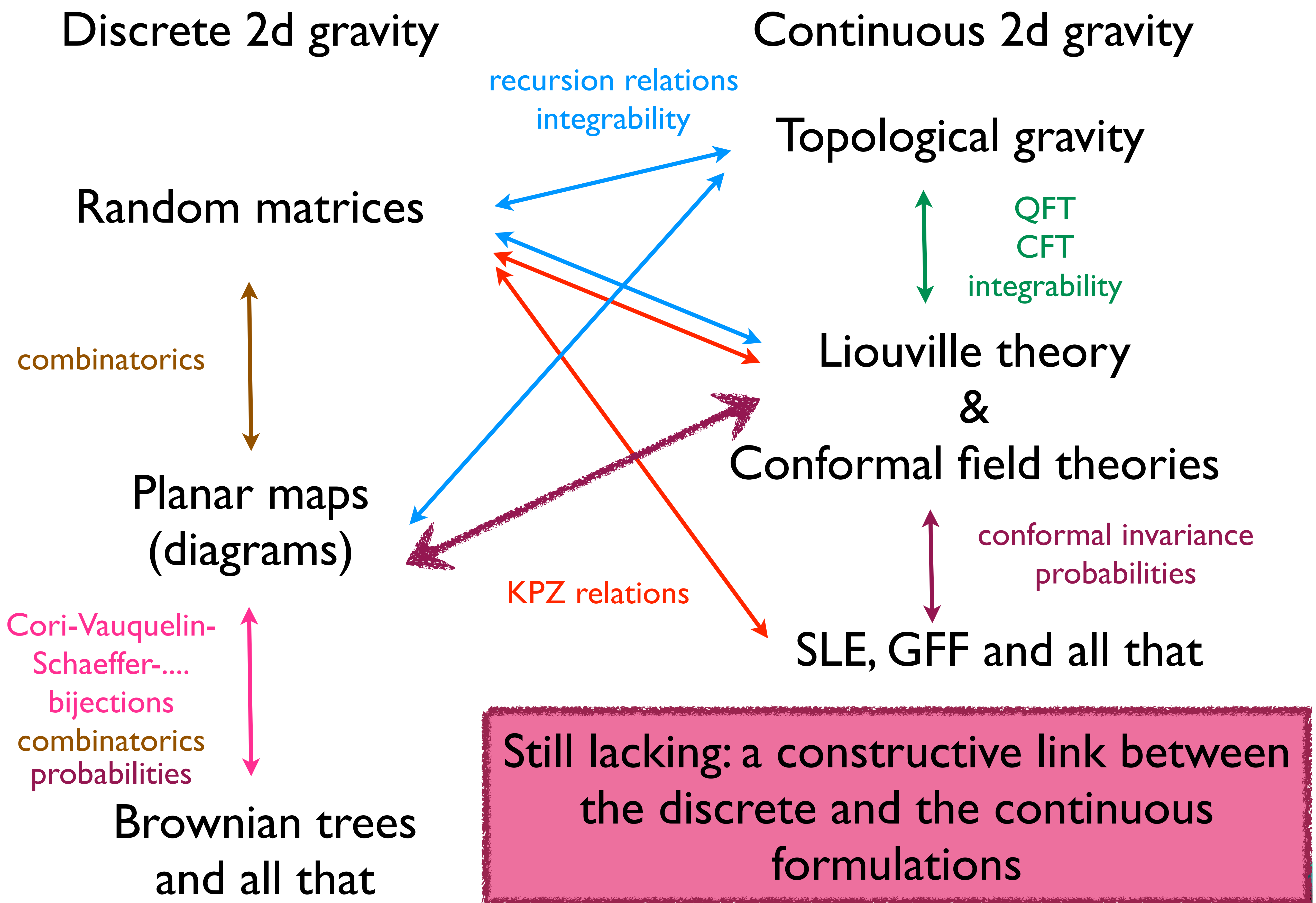
Discretize metric ⟶ Random planar lattices (maps)



Courtesy Nicolas Curien et al.



Also fairly well understood



Discrete 2d gravity
Continuous 2d gravity
recursion relations
integrability
Topological gravity
Random matrices
QFT
CFT
integrability
combinatorics
Liouville theory
&
Conformal field theories
Planar maps
(diagrams)
conformal invariance
probabilities
KPZ relations
Cori-Vauquelin-
Schaeffer-....
bijections
combinatorics
probabilities
SLE, GFF and all that
Brownian trees
and all that
Still lacking: a constructive link between the discrete and the continuous formulations

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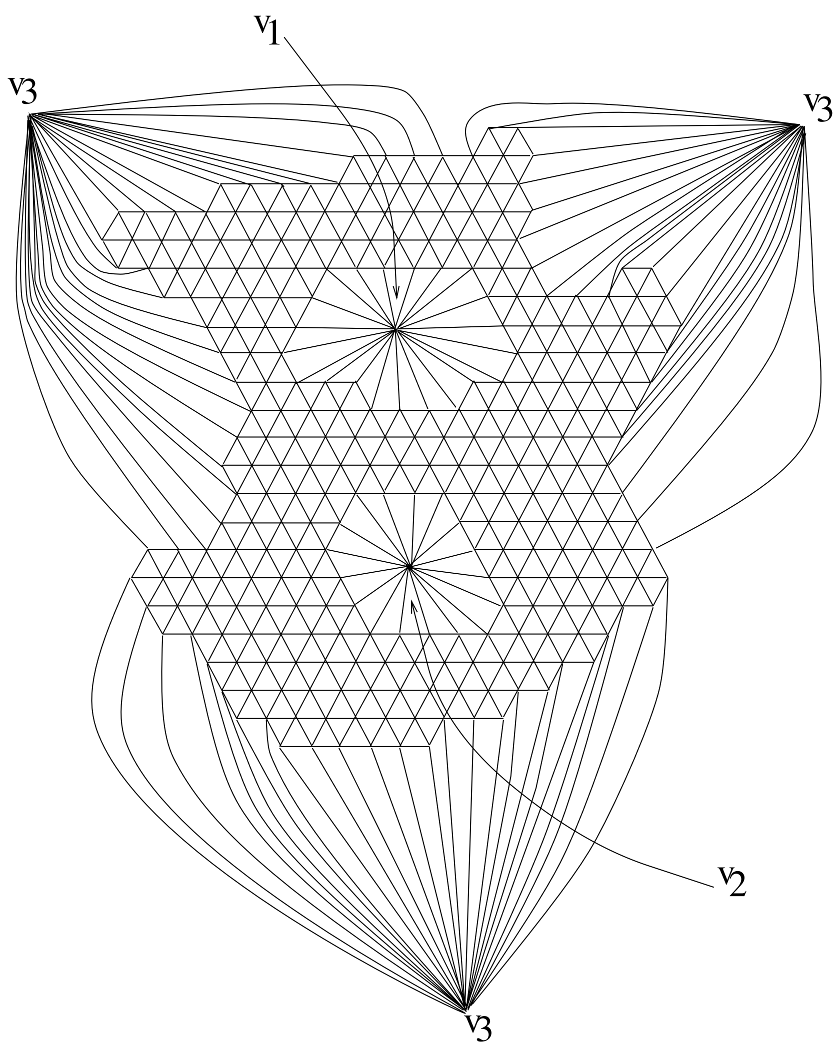
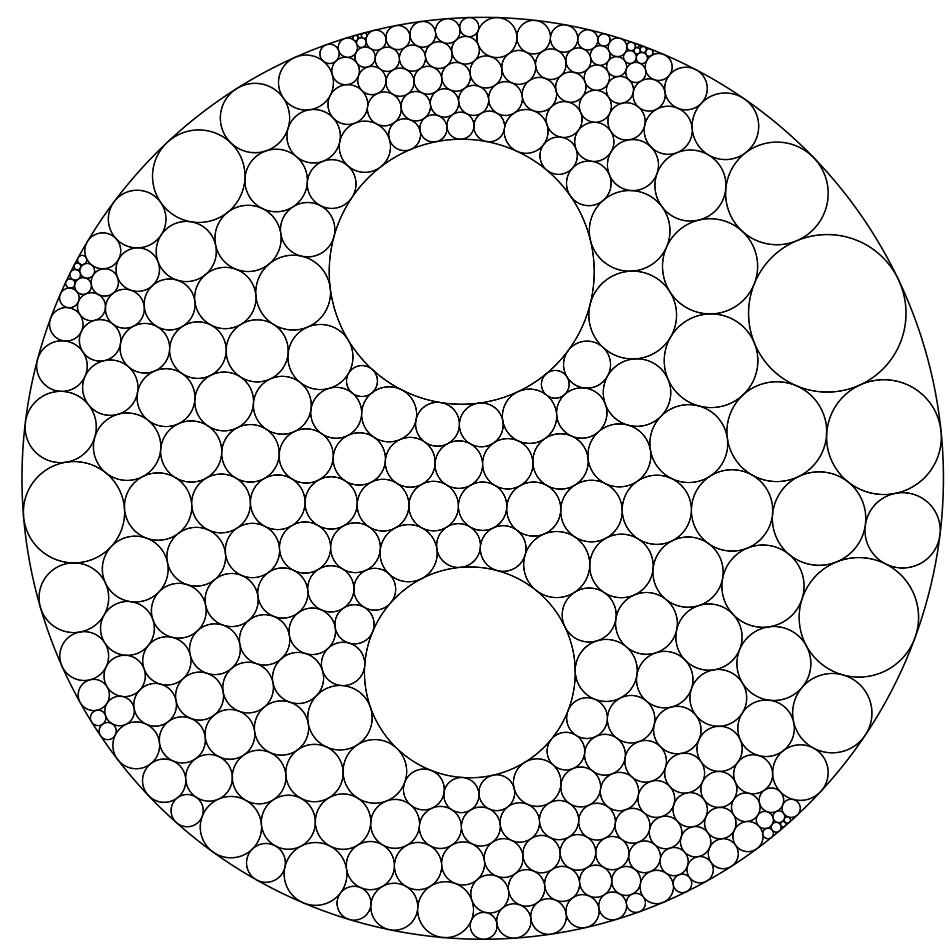
The Koebe-Andreev-Thurston theorem

There is a bijection between triangulations and circle packings, modulo SL(2,C) Möbius transformations

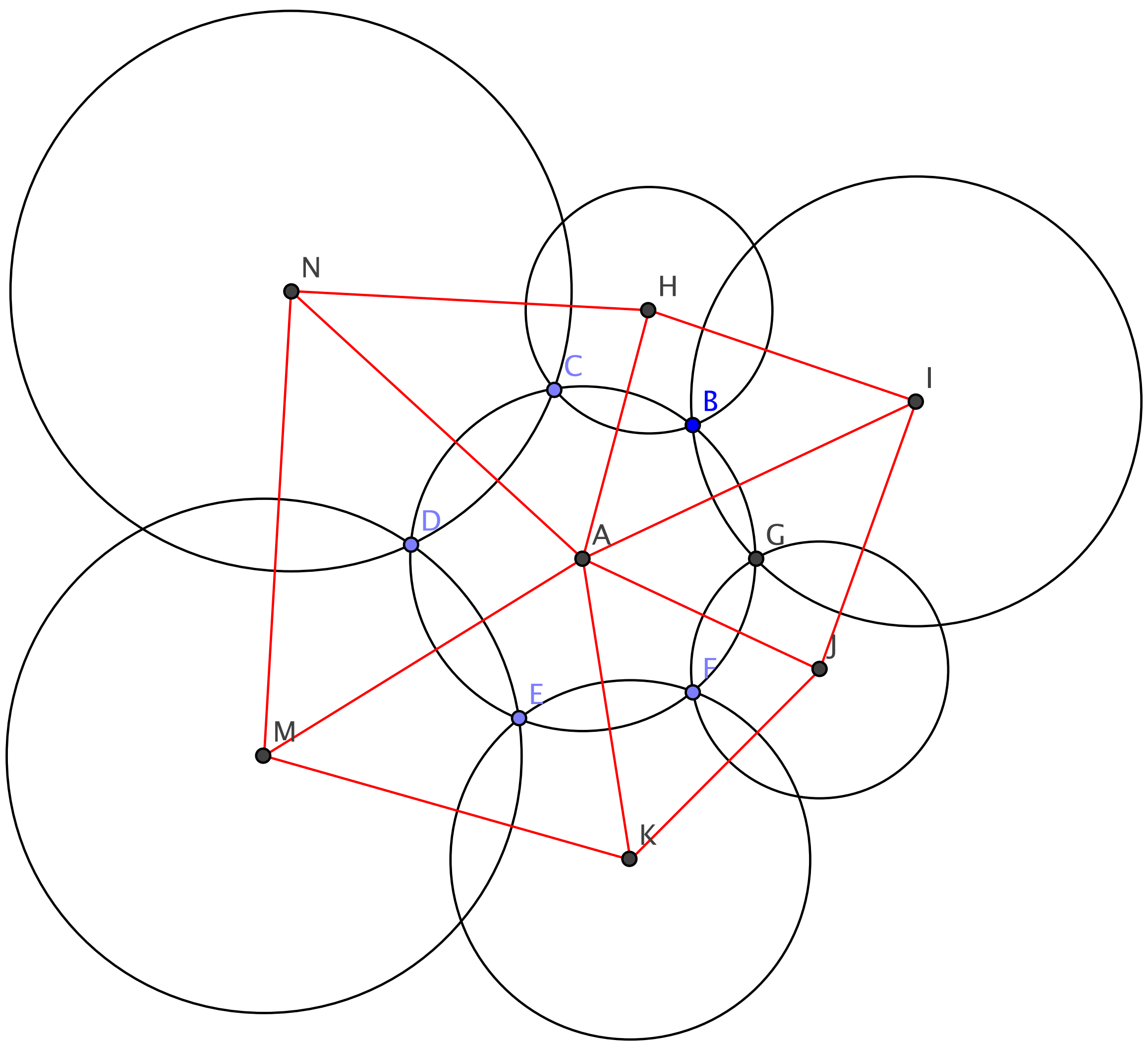


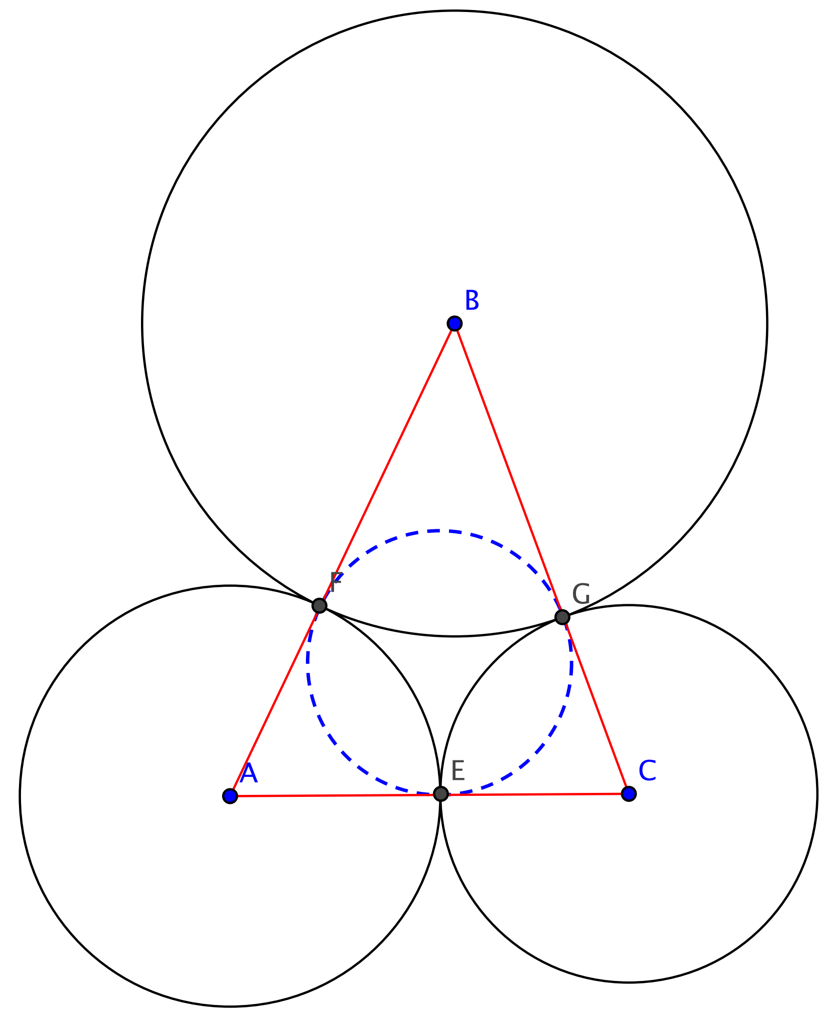


Illustrations borrowed to Schramm & Mishenko

A generalisation of circle packings: circle patterns

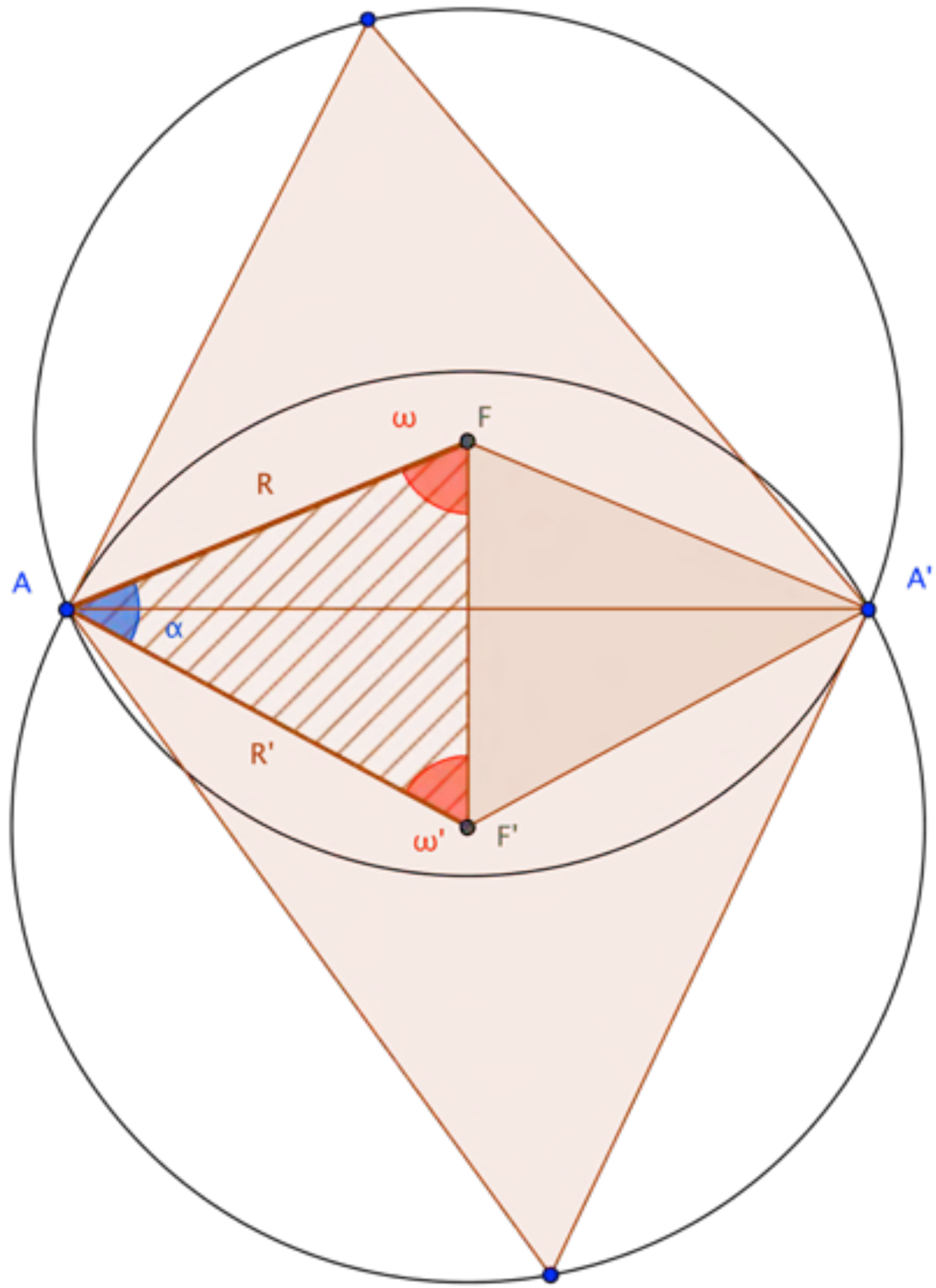
The angles of intersection of the circles are given.
Find the circle pattern and the radii.





Theorem of existence and unicity
(I. Rivin 1994, Ann. of Math.)

circle packing = circle patterns
with angles 0 or $\pi/2$ only



angle $\alpha, \alpha', \cdots$ fixed
adjust the radii R, R', ...
so that angles at faces
F, F', ... $= 2\pi$

Minimize w.r.t. R, R', ... a
convex functional

$$\Gamma[\{R\};\{\alpha\}]$$

involving the hyperbolic
volumes of the triangles

$$(A, F, F')$$

(Colin de Verdière)

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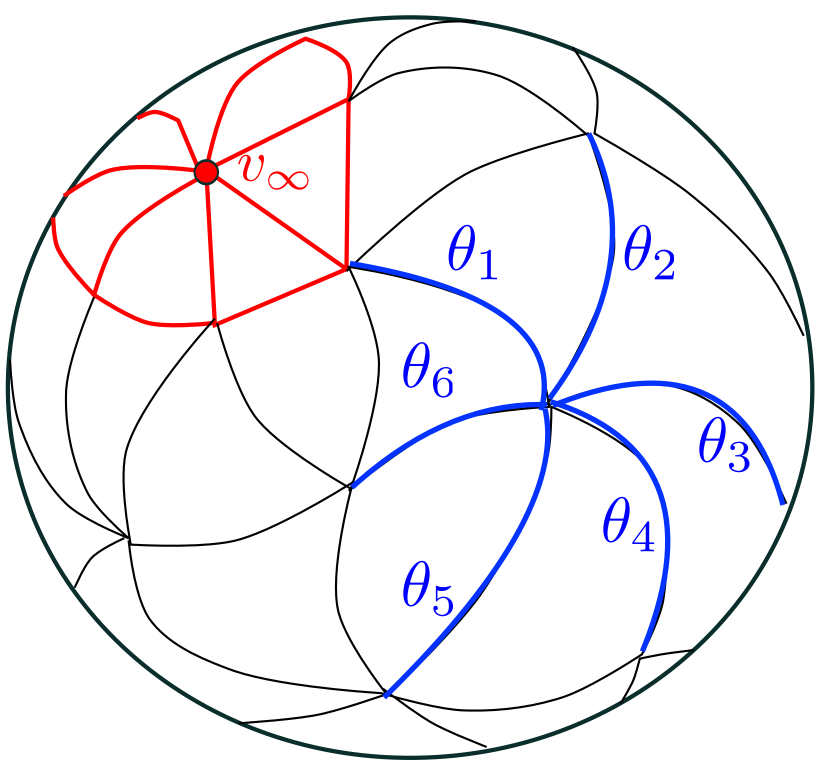
Random Delaunay triangulations

Consider an abstract triangulation of the sphere T

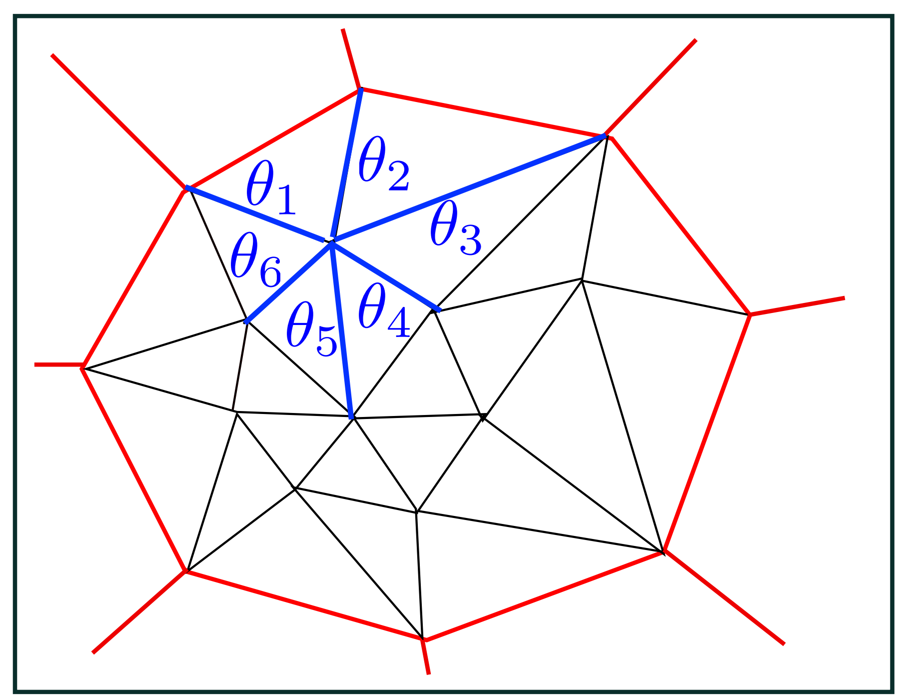
+ angles θ_e attached to the edges e of T

such that at each vertex $\sum_{e\to v}\theta_e = 2\pi$

Theorem: there is a unique (mod. SL(2,C)) Delaunay triangulation in the complex plane such that the circle angles are $\theta^*_e = \pi - \theta_e$

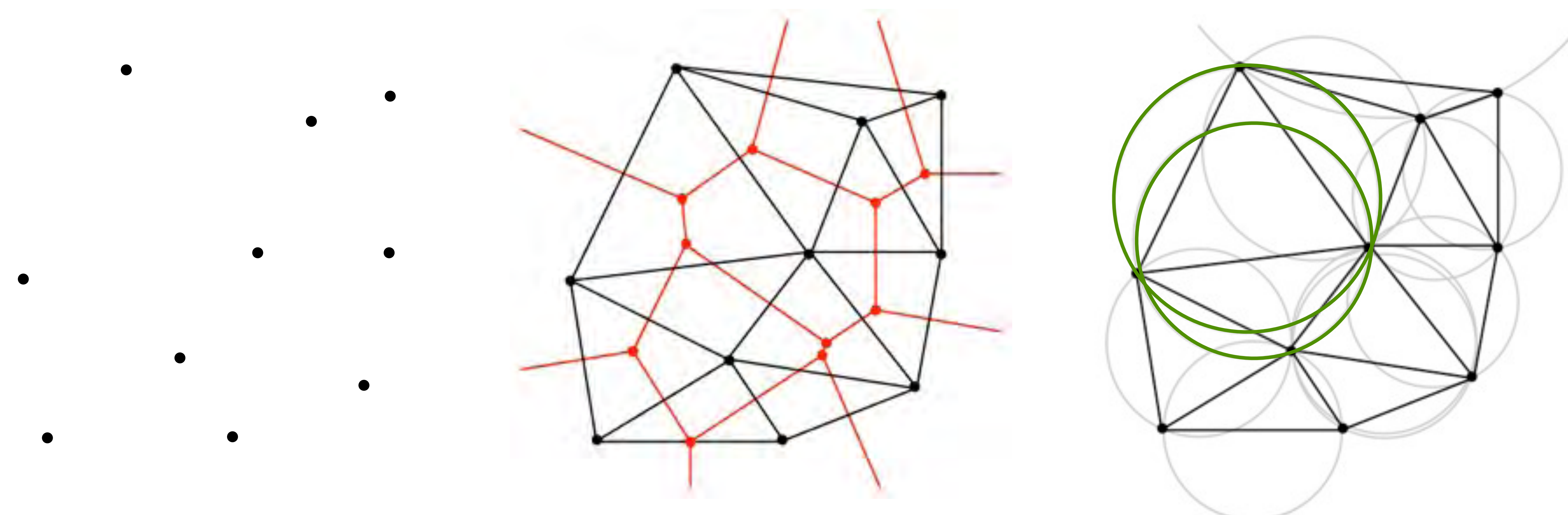


Sphere



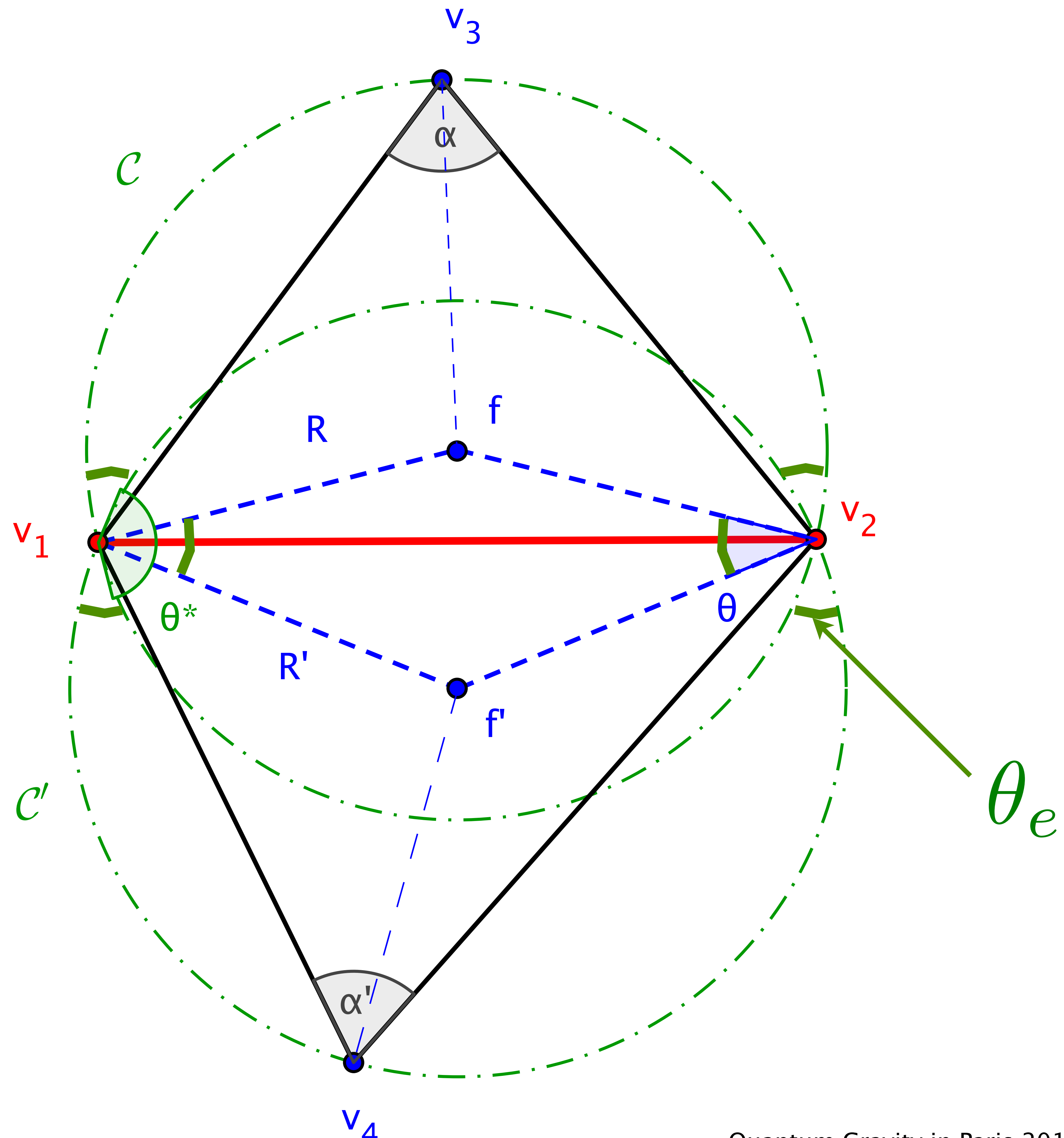
Plane

Voronoï tessellation and Delaunay triangulation in the plane



No vertex of a triangle must be inside the circumscribed circle to any another triangle

From Wikimedia Co



v3
α
C
f
R
v1
v2
θ*
θ
R'
f'
C'
θe
α'
v4

A triangulation is Delaunay if and only if f all $\theta_e \in [0, \pi)$

Warning! not all planar triangulations T admit $\boldsymbol{\theta} = \{\theta_e\}$ such that $\sum_{e \to v} \theta_e = 2\pi$

Triangulations with loops or multiple links are excluded

But one expects that admissible $(T, \boldsymbol{\theta})$ are generic and in the same universality class than generic planar triangulations and general planar maps (more later...).

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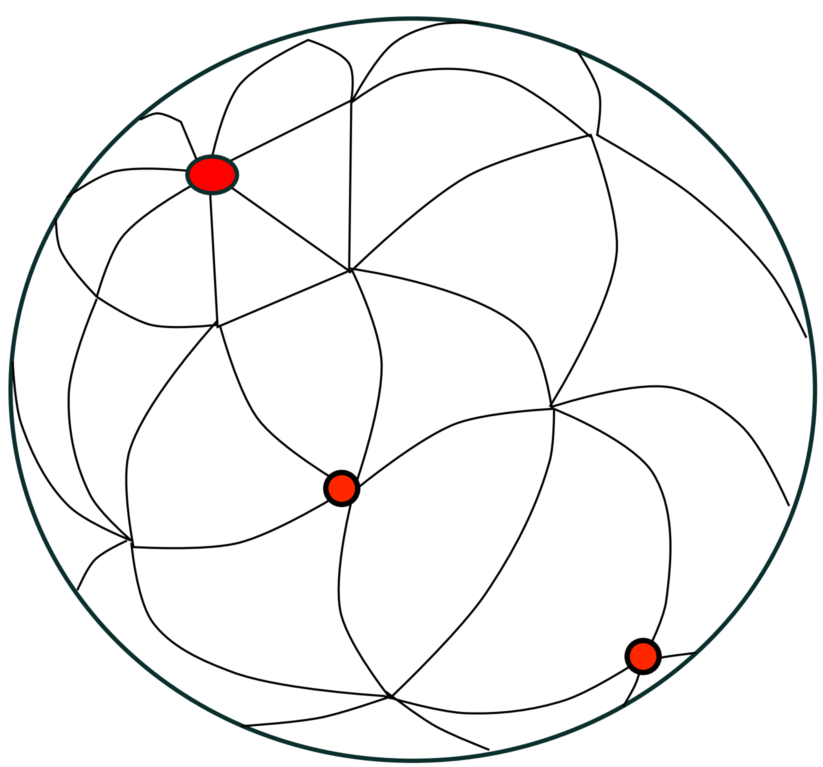
Take as initial measure on triangulations the uniform measure on triangulations and the flat measure on the angles

$$\mu(\widetilde{T}) = \mu(T, d\boldsymbol{\theta})) = \text{uniform}(T) \prod_{e\in\mathcal{E}(T)} d\theta(e) \quad \prod_{v\in\mathcal{V}(T)} \delta\left(\sum_{e\mapsto v} \theta(e) - 2\pi\right)$$

Question: which measure does this induce on Delaunay triangulations? For this consider N+3 points, 3 fixed by $\mathrm{SL}(2,\mathbb{C})$

$$\mathfrak{D}_{N+3} = \mathbb{C}^{N+3}/\mathrm{SL}(2,\mathbb{C}) \simeq \mathbb{C}^N$$

$$d\mu(z_4, \cdots, z_{N+3}) = ?$$



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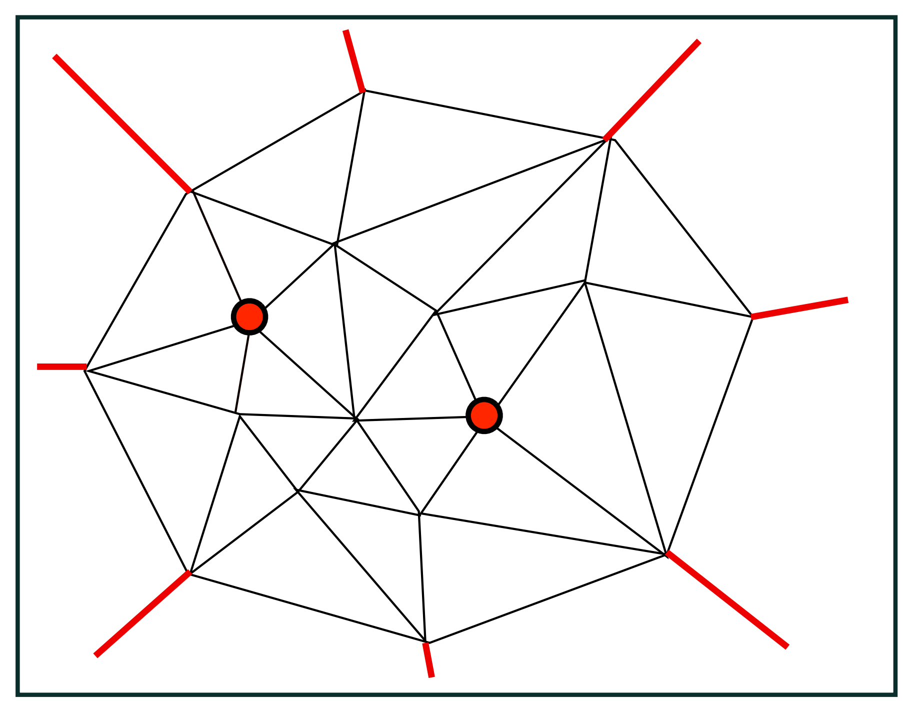
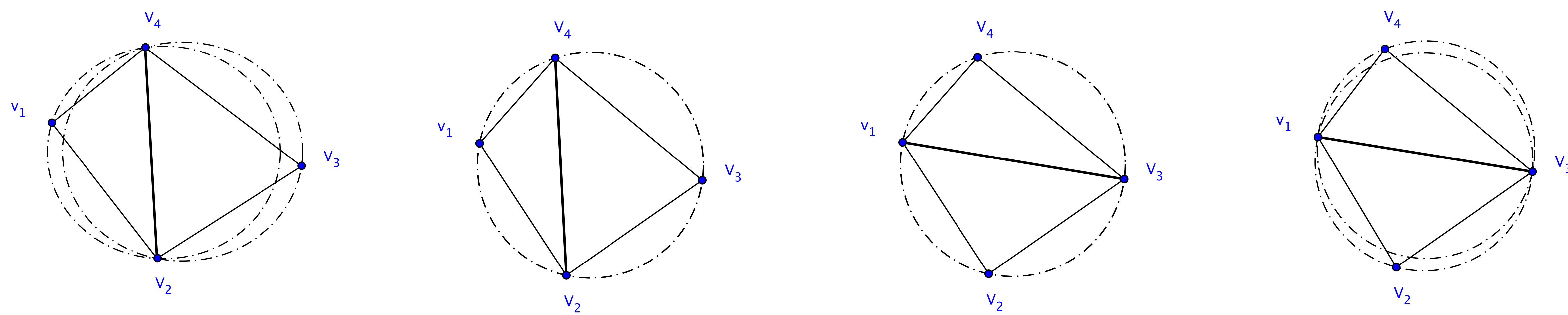


Plane

Transition between Delaunay triangulations by edge flips

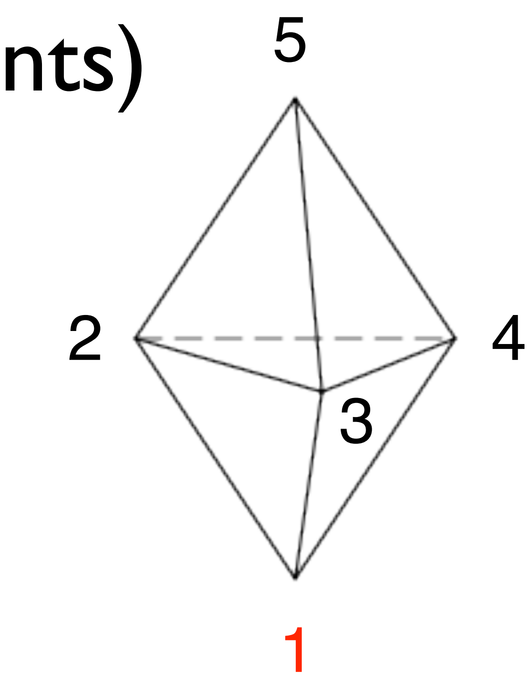


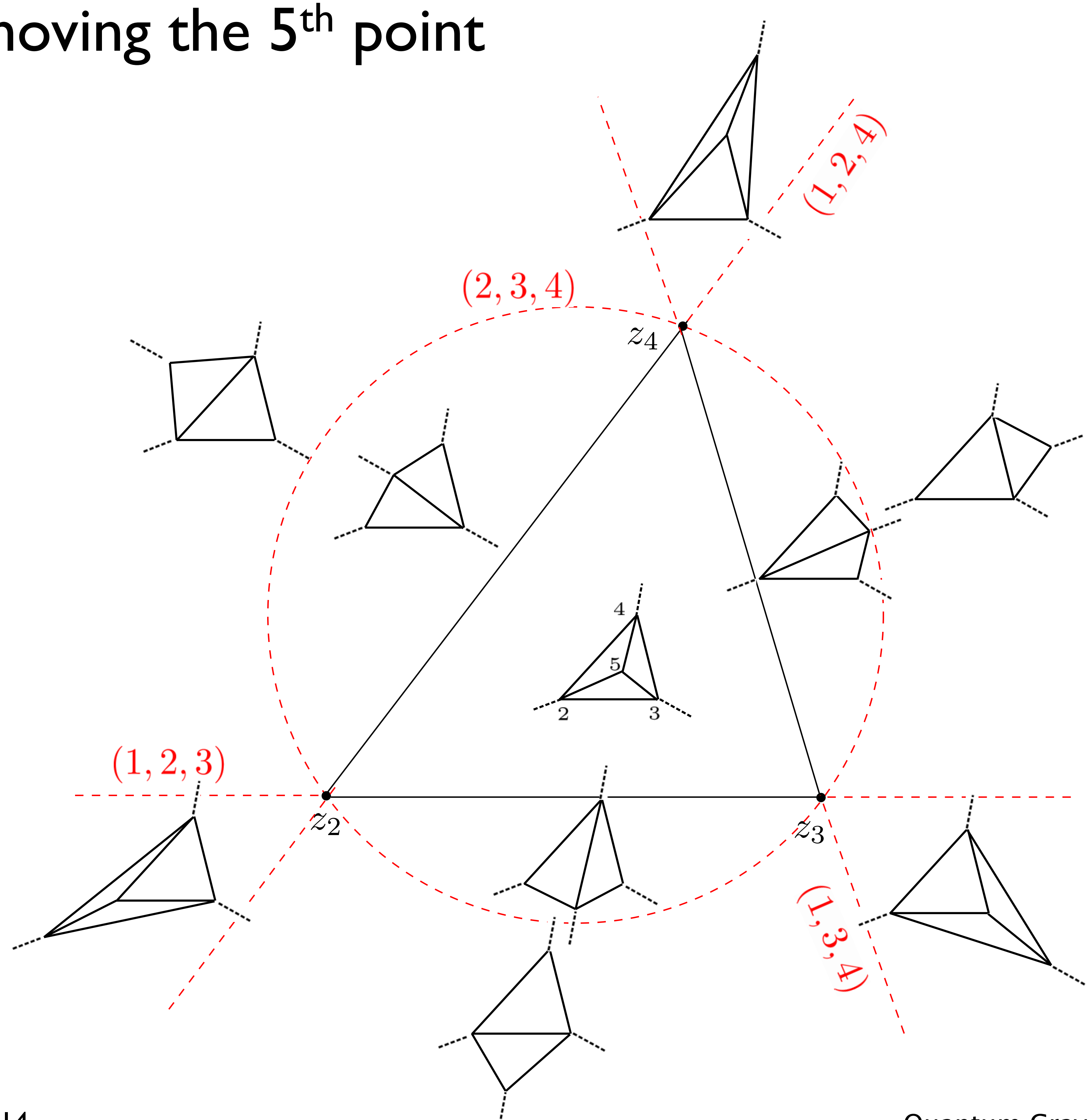
the flip of link e occurs when $\theta(e) = 0$

Moving the points allows to explore the whole space of Delaunay triangulations and of dressed abstract triangulations

Elementary example: the hexahedron (5 points)

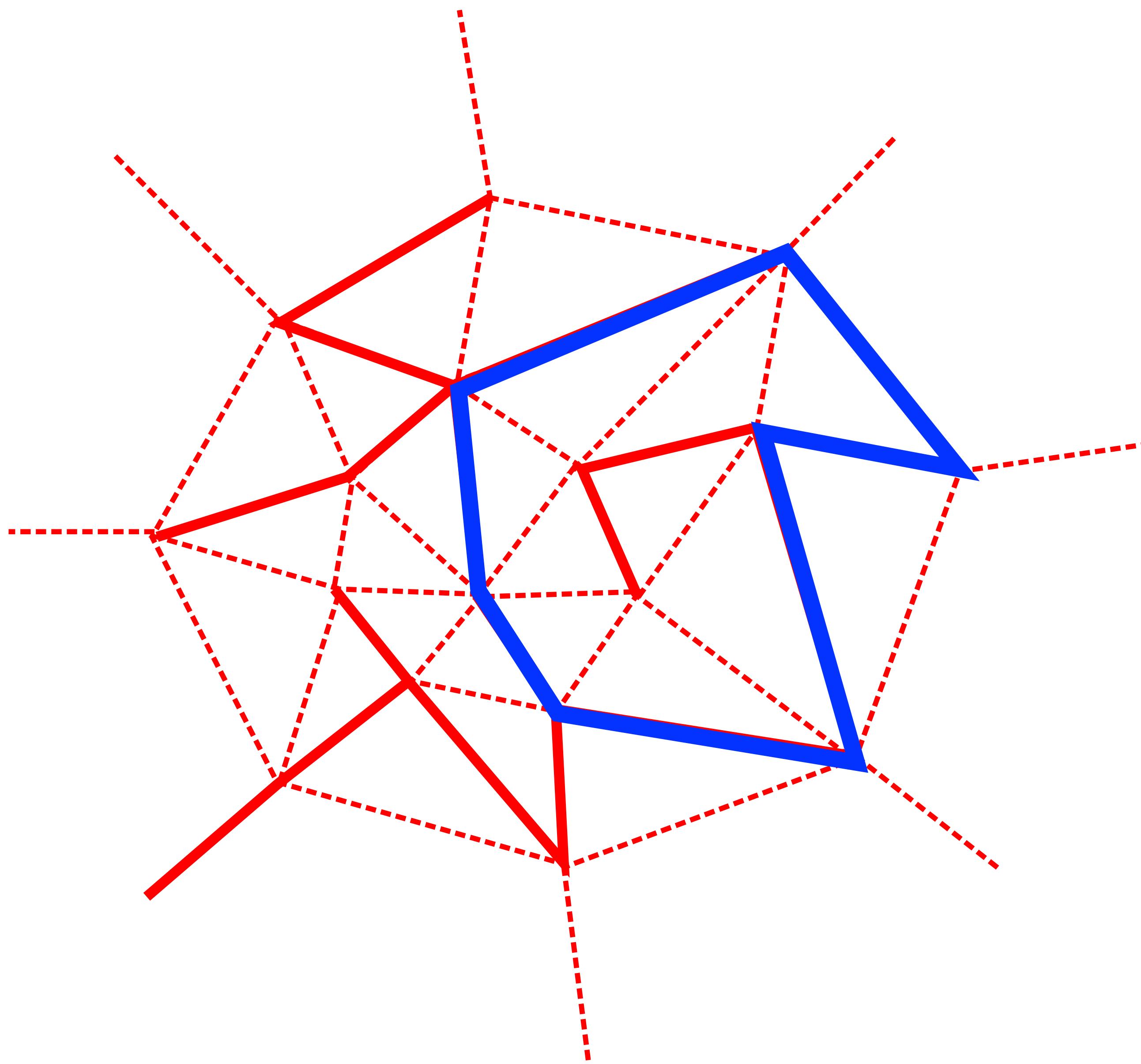
moving the 5th point





1st question: Which sets of edges form independent basis for the angles?

Answer: The sets whose complementary form a cycle-rooted-spanning-tree of the triangulation with odd length cycle



2nd question: what is the Jacobian of the change of angle variables between two basis of edges?

Answer: Jacobian = 1 !

Indeed... $\mu(T, d\boldsymbol{\theta}) \;=\; \dfrac{1}{2}\,\mathrm{uniform}(T) \times \displaystyle\prod_{e \in \mathcal{E}_0(T)} d\theta(e)$

So, the measure over the points is given locally (for a given Delaunay triangulation) by a simple Jacobian

$$\mu(T, d\boldsymbol{\theta}) \;=\; d\mu(\mathbf{z}) \;=\; \prod_{v=4}^{N+3} d^2 z_v \;\left|\det\left(J_T(z)_{\setminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

$$J_T(z) = \left(\frac{\partial \theta_e}{\partial(z_v, \bar{z}_v)}\right)_{\substack{e \in \mathcal{E}(T) \\ v \in \mathcal{V}(T)}}$$

The matrix elements of the Jacobian are made of simple poles

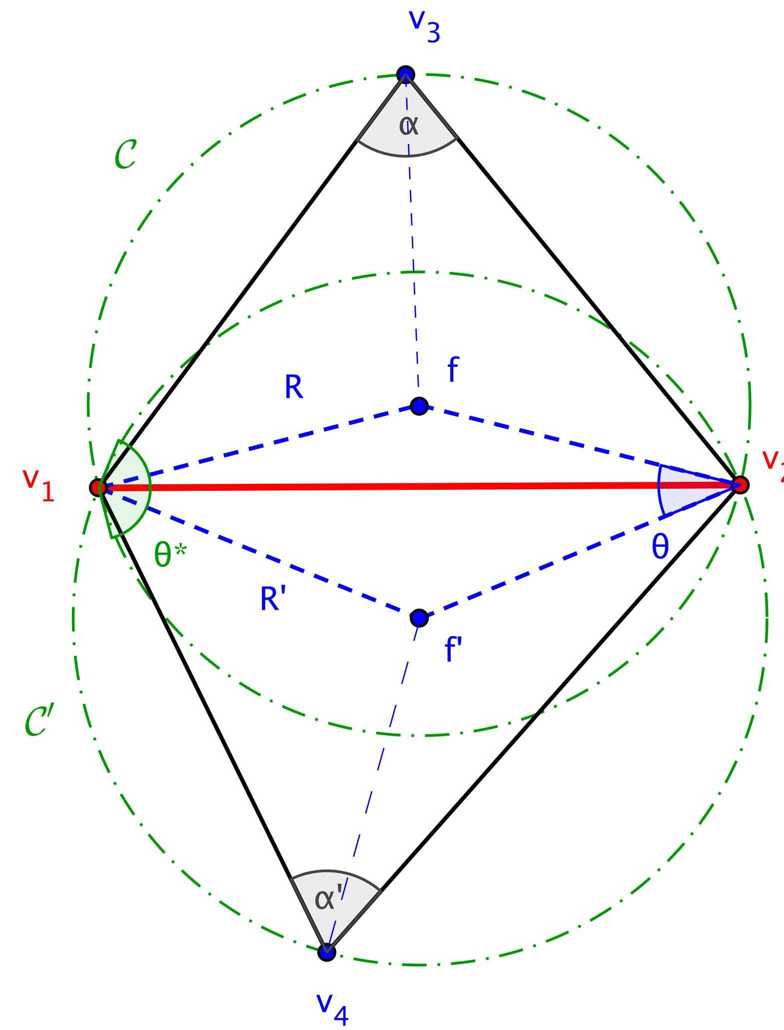
$$J_{v,e} = \frac{\partial \theta_e}{\partial z_v} \quad , \qquad J_{\bar{v},e} = \frac{\partial \theta_e}{\partial \bar{z}_v}$$

$$J_{v_1,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4}-z_{v_1}} - \frac{1}{z_{v_3}-z_{v_1}}\right)$$

$$J_{v_3,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3}-z_{v_1}} - \frac{1}{z_{v_3}-z_{v_2}}\right)$$

$$J_{v_2,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_3}-z_{v_2}} - \frac{1}{z_{v_4}-z_{v_2}}\right)$$

$$J_{v_4,e} = \frac{\mathrm{i}}{2}\left(\frac{1}{z_{v_4}-z_{v_2}} - \frac{1}{z_{v_4}-z_{v_1}}\right)$$



The determinant of the Jacobian matrix is locally a rational function of the z_v's and $\bar{z}_v$'s

$$\mathcal{D}_T(z)_{\smallsetminus\{1,2,3\}} = \left|\det\left(J_T(z)_{\smallsetminus\{1,2,3\}\times\bar{\mathcal{E}}_0}\right)\right|$$

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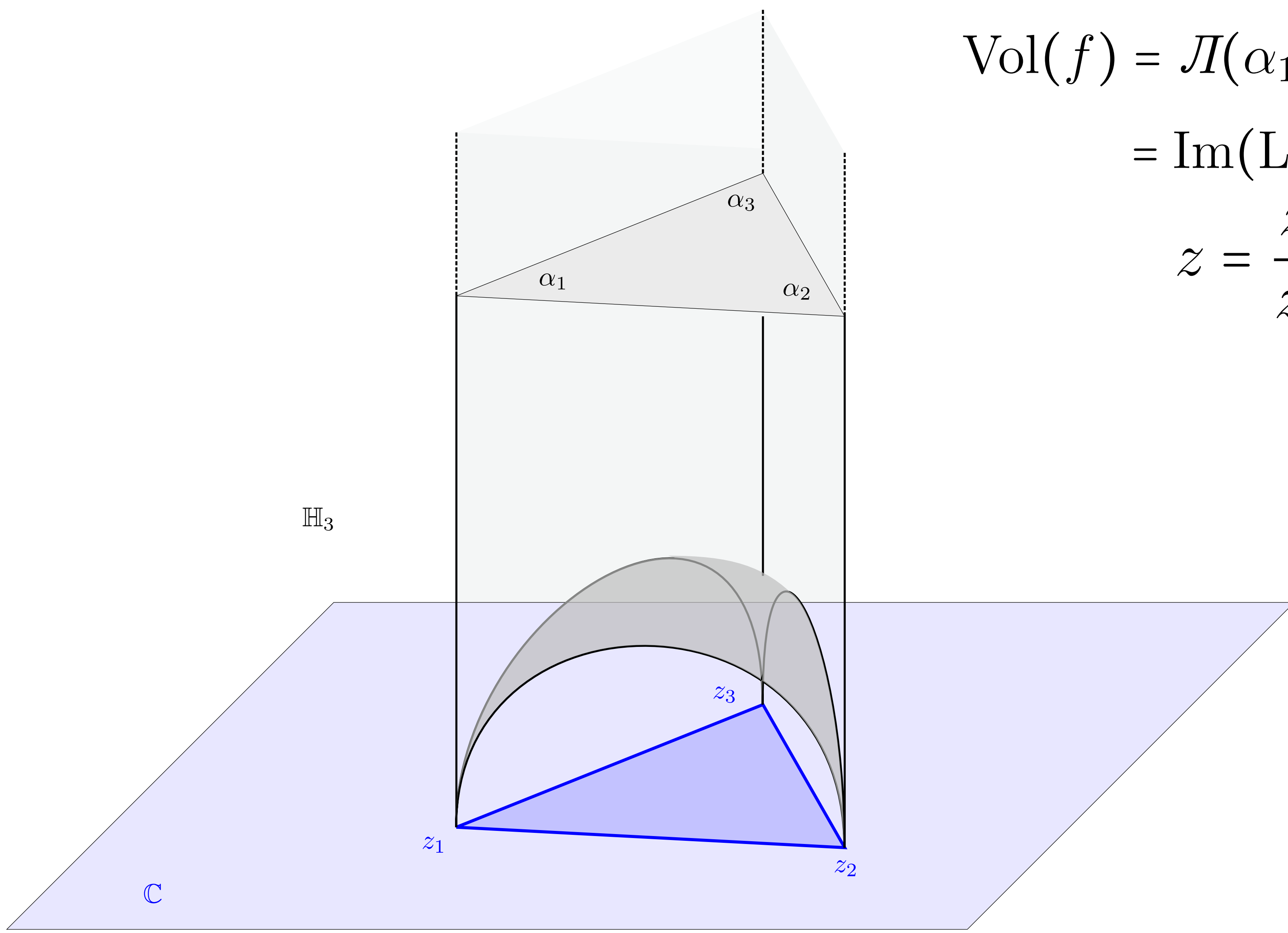
Hyperbolic volume of triangle = volume of ideal tetrahedron above the triangle in hyperbolic Poincaré half-space



$$\mathrm{Vol}(f) = \text{Л}(\alpha_1) + \text{Л}(\alpha_2) + \text{Л}(\alpha_3)$$

$$= \mathrm{Im}(\mathrm{Li}_2(z)) + \ln(|z|)\mathrm{Arg}\,(1-z)$$

$$z = \frac{z_3 - z_1}{z_2 - z1}$$

Action of a triangulation = sum over volumes

$$\mathcal{A}_T = - \sum_{triangles\ \ f \in \mathcal{F}(T)} \mathrm{Vol}(f)$$

Define the matrix $D_{u\bar{v}}(z) = \dfrac{\partial}{\partial z_u}\dfrac{\partial}{\partial \bar{z}_v}\mathcal{A}_T(z)$

Theorem:

1. $D_{u\bar{v}}$ is a Kähler form on $\mathfrak{D}_{N+3}$ i.e. $D > 0$

2. $D_{u\bar{v}}$ is countinuous (no discontinuity when a flip occurs)

3. The measure determinant is the Kähler volume form

$$\mathcal{D}_T(z)_{\setminus\{1,2,3\})} = \det\left[(D_{u,\bar{v}})_{\substack{u,v\neq\\ \{1,2,3\}}}\right]$$

The *(2N x 2N)* Jacobian has been reduced to a *N x N* Kähler determinant! But this is not a determinantal process!

$d\mu(z) = d^2 z \;\; \det(D)$ is a conformal point process

Independence of the 3 fixed points and $\mathrm{SL}(2,\mathbb{C})$ invariance

$$H = \frac{\det\left(D_{\backslash a,b,c}(z)\right)}{\left|\Delta_3(z_a, z_b, z_c)\right|^2}$$

$$\Delta_3(z_a, z_b, z_c) = (z_a - z_b)(z_a - z_c)(z_b - z_c)$$

is independent of the choice of points

$$z \to w = \frac{az+b}{cz+d} \quad \text{with} \quad ad - bc = 1$$

$$H(z) = \left| \prod_{i=1}^{N+3} w'(z_i) \right|^2 H(w) = \prod_{i=1}^{N+3} \frac{1}{|cz_i + d|^2} H(w)$$

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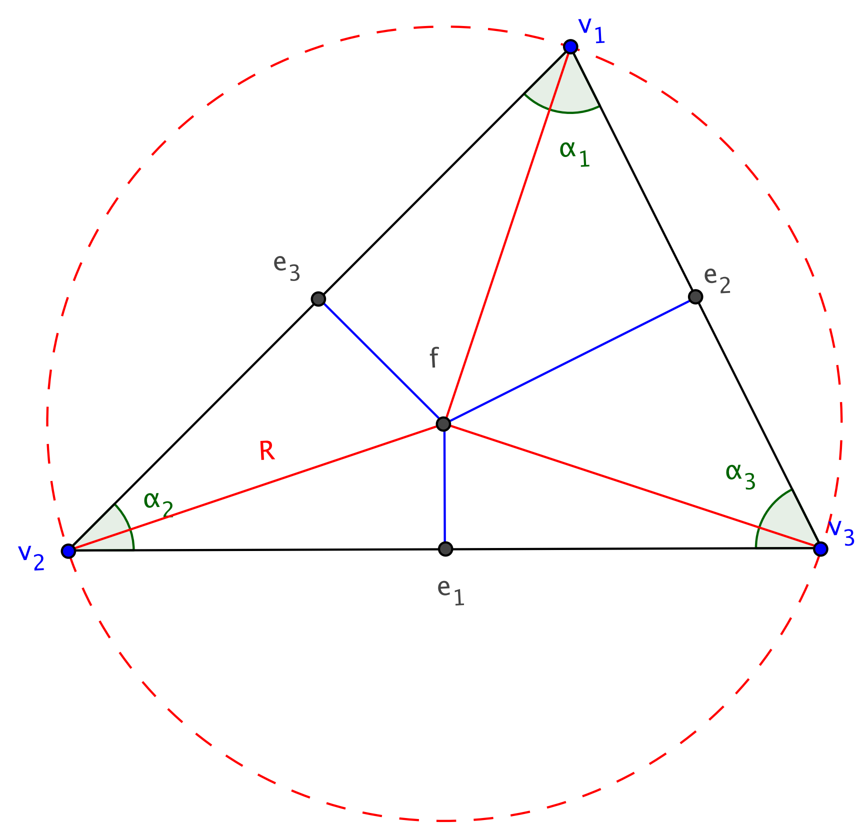
Local representation of D as a sum over triangles

$$D_{u,\bar v} = \sum_f D_{u,\bar v}(f) \quad , \qquad D_{u,\bar v}(f) = -\frac{\partial^2}{\partial z_u \partial \bar z_v}\mathrm{Vol}(f)$$



$$D(f) = \frac{1}{8\,R(f)^2}\begin{pmatrix} \cot(\alpha_2)+\cot(\alpha_3) & -\cot(\alpha_3)-\mathrm{i} & -\cot(\alpha_2)+\mathrm{i} \\ -\cot(\alpha_3)+\mathrm{i} & \cot(\alpha_3)+\cot(\alpha_1) & -\cot(\alpha_1)-\mathrm{i} \\ -\cot(\alpha_2)-\mathrm{i} & -\cot(\alpha_1)+\mathrm{i} & \cot(\alpha_1)+\cot(\alpha_2) \end{pmatrix}$$

D as a discretized Fadeev-Popov determinant!

Local derivatives from vertices $\longrightarrow$ faces

$$\nabla\Phi(f) = \frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(\bar z_3-\bar z_2)+\Phi(v_2)(\bar z_1-\bar z_3)+\Phi(v_3)(\bar z_2-\bar z_1)}{\mathrm{Area}(f)}$$

$$\overline{\nabla}\Phi(f) = -\frac{1}{2\mathrm{i}}\frac{\Phi(v_1)(z_3-z_2)+\Phi(v_2)(z_1-z_3)+\Phi(v_3)(z_2-z_1)}{\mathrm{Area}(f)}$$

D can be written as

$$\Phi\cdot D(f)\cdot\overline{\Psi} = \sum_{i,j\ \text{vertices of}\ f}\Phi(v_i)D_{i\bar\jmath}(f)\overline{\Psi}(v_j) = \frac{\mathrm{Area}(f)}{R(f)^2}\ \overline{\nabla}\Phi(f)\ \nabla\overline{\Psi}(f)$$

«continuous form» with complex functions treated as real vector fields

$$\mathrm{Area}(f) = d^2w_f \qquad\qquad \frac{1}{R(f)^2} = \mathrm{e}^{\,\phi(w_f)}$$

$$\Phi\cdot D\cdot\overline{\Psi} = \int d^2w\ \ \mathrm{e}^{\,\phi(w)}\ \partial_{\bar z}\Phi^z(w)\ \partial_z\Psi^{\bar z}(w)$$

This is a discrete version of the Faddeev-Popov determinant in Polyakov's formulation of two dimensional gravity and of non-crititical string theory !

Functional integral over 2d Riemannian metrics, conformal gauge

$$g_{ab}(z) = \delta_{ab}\ \mathrm{e}^{\phi(z)} \qquad\qquad \int \mathcal{D}[g_{ab}] = \int \mathcal{D}[\phi]\ \det(\nabla_{\mathrm{FP}})$$

Faddeev-Popov ghost systems

$$\det(\nabla_{\mathrm{FP}}) = \int \mathcal{D}[\mathbf{c},\mathbf{b}]\ \exp\left(\int d^2z\ \mathrm{e}^{\phi}\ \left(b_{zz}(\nabla c)^{zz} + b_{\bar{z}\bar{z}}(\nabla c)^{\bar{z}\bar{z}}\right)\right)$$

Integrating over the b's only one gets

$$\det(\nabla_{\mathrm{FP}}) = \int \mathcal{D}[\mathbf{c}]\ \exp\left(\int d^2z\ \mathrm{e}^{\phi}\, \partial_z c^{\bar{z}}\ \partial_{\bar{z}} c^{z}\right)$$

D is nothing but the discretised FP determinant

$$D = \nabla_{\mathrm{FP}}$$

and $\phi(f) = -2\log(R(f))$
plays the role of a Liouville field on the Voronoï lattice

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- Discrete model with exact conformal invariance!
- Is it possible to define a stress-energy tensor?

$$(T(z)\,,\,\bar{T}(\bar{z}))$$

Yes, but not very useful (yet) since $c_{\mathrm{total}} = 0$

- Positivity and convexity properties for the measure?

Yes!

Delaunay triangulations maximize the prepotential (easy)

But also the Kahler volume form (non trivial)

- Can one find an explicit expression for the measure?

Unclear, except in the isoradial case (curvature = 0)

- Can we derive the Liouville action as the effective action for the (coarse-grained or exact?) local conformal factor ?

$$\varphi_{\mathrm{Liouville}} = -2\log(R_{\mathrm{triangle}}) = \log(\rho_{\mathrm{points}})$$

- Is this model related to topological gravity?

  Yes, the measure can be written in term of Chern classes of the moduli space of punctured surfaces

  But not exactly the usual ones

- Other (quasi)-conformal embeddings of planar maps?

  Exact uniformization? (see e.g. N. Curien recent work)

  Other representations from algebraic geometry?

  Work in progress

- Integrability?

  Unclear yet.

- Coupling to background classical metric?

  Yes. Work in progress.

- Surfaces with boundaries & higher genus? Matter fields?

  Not difficult to formulate, to be done.