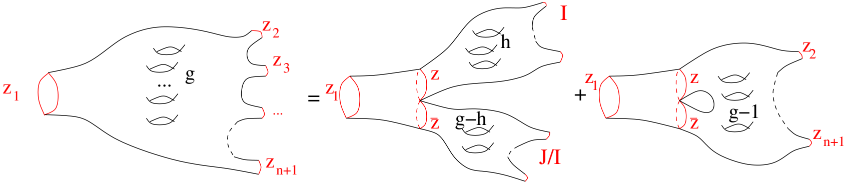
# Counting surfaces of any topology, with Topological Recursion

Bertrand Eynard, IPHT CEA Saclay,




Quatum Gravity, Orsay, March 2013

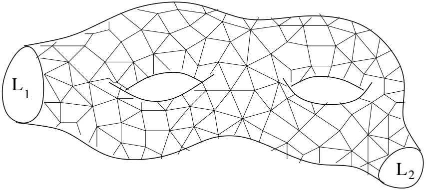
# Contents

## Outline

# 1. Introduction: Counting surfaces

# Counting Discrete surfaces

• In Combinatorics or Statistical Physics:
count discrete surfaces ("maps") made of polygons, possibly carrying a "color":




Generating series for counting:

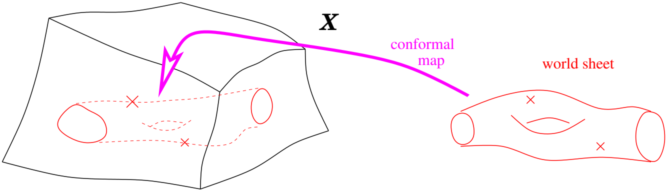
$$W_{g,n} = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}} \frac{t_3^{\#3-\mathrm{gons}}\, t_4^{\#4-\mathrm{gons}}\, t_5^{\#5-\mathrm{gons}} \dots}{x_1^{L_1+1}\, x_2^{L_2+1} \dots x_n^{L_n+1}}$$

$g =$ genus, $n = \#$boundaries, $v = \#$vertices.

Link with Tutte's equations [Tutte 60], with Random matrices [BIPZ 78], with trees [Schaeffer, BDG,...]

# Counting Riemann surfaces

• In (topological) String theory or in Enumerative Geometry: "count" Riemann surfaces ("world-sheets") conformally embedded into a "target space" $\mathfrak{X}$:

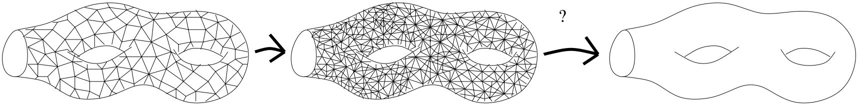



Generating series for counting:

$$W_{g,n} = \sum_{\beta} \mathrm{e}^{-t.\beta} \int_{\Sigma \in \overline{\mathcal{M}}_{g,n}(\mathfrak{X},\beta)} \frac{\Lambda(\Sigma)}{\#\mathrm{Aut}(\Sigma)} \prod_{i=1}^{n} \mathrm{e}^{-x_i \,.\#\mathrm{winding}(\partial_i \Sigma)}$$

$g$ = genus, $n = \#$boundaries, $\beta \in H_2(\mathfrak{X}, \mathbb{Z}) = \#$degree.
$\Lambda$ = cohomology class, e.g. Chern class, Hodge class,...

## Continuum limit ?




Question: limit of large discrete surfaces $\to$ Riemann surfaces ?

2 ways of viewing continuous surfaces:

- Quantun gravity CFT: Liouville CFT, KPZ, ...
- Topological gravity: Witten Kontsevich, Intersection numbers, Gromov-Witten invariants, Chern classes, Chern-Simons, ...

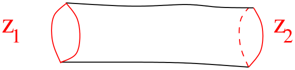
• in 1991, Kontsevich proved Witten's claim: generating series of topological gravity = KdV Tau-function (= scaling limit of random matrices = scaling limits of numbers of larges maps).

# 2. Topological Recursion

## Topological Recursion

• Assume you know the "Disc" generating function $W_{0,1}(x)$:
Disc : genus =0, $\#$ boundary=1.

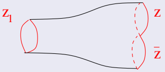
• Assume you know the "Cylinder" generating function $W_{0,2}(x_1, x_2)$: Cylinder : genus=0, $\#$ boundary=2.




Out of them, define:

**Definition (the "Recursion kernel")**

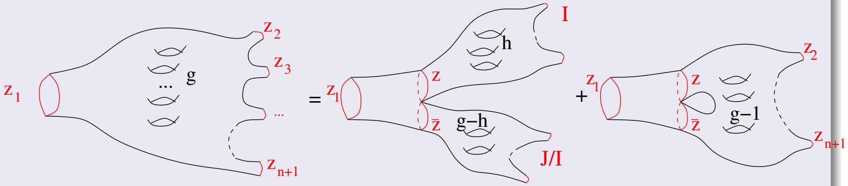
$$K_a(z_1, z) = \frac{1}{2} \frac{\int_{s_a(z)}^{z} W_{0,2}(z_1, .)}{W_{0,1}(z) - W_{0,1}(s_a(z))}$$



# Topological Recursion

## Definition (Topological Recursion)

The $W_{g,n}$ are said to satisfy the "topological recursion" iff:

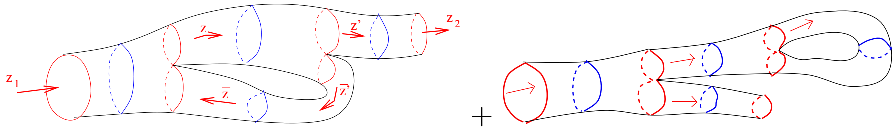



$$W_{g,n+1}(x_0, \overbrace{x_1, \dots, x_n}^{J}) = \sum_a \operatorname*{Res}_{x \to a} K_a(x_0, x) \Big[ W_{g-1,n+2}(x, s_a(x), J)$$

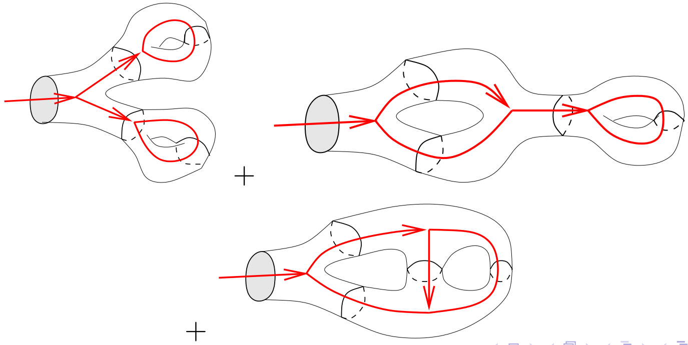
$$+ \sum_{h+h'=g,\, I \uplus I' = J}^{\prime} W_{h,1+|I|}(x, I)\, W_{h',1+|I'|}(s_a(x), I') \Big] \, dx$$

# Example: Topological Recursion

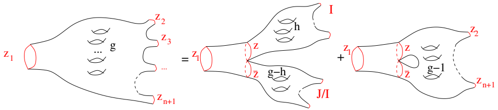
Example: $W_{1,2}$




Example: $W_{2,1}$

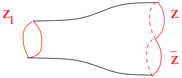


# Interpretation: Topological Recursion

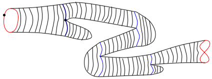


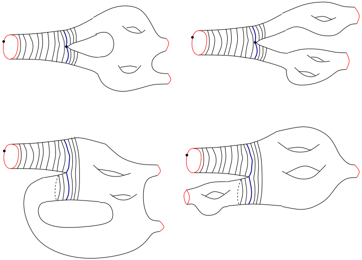

Recursion kernel = "short":

$$K(z_1, z) = \frac{1}{2} \frac{\int_{\bar{z}}^{z} W_{0,2}(z_1, .)}{W_{0,1}(z) - W_{0,1}(\bar{z})} =$$



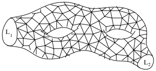

=





# 3. Examples of surfaces which satisfy the TR

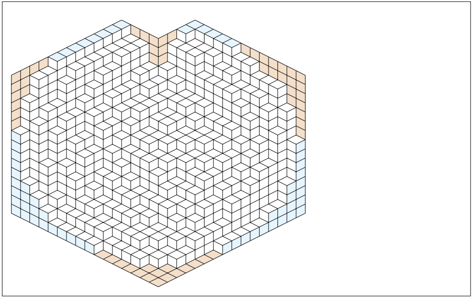
## Many surface counting problems do satisfy the TR

- Counting Discrete surfaces 

- Counting Riemann surfaces with Weil-Petersson metrics
- Counting Riemann surfaces with products of Chern classes of cotangent bundles (Intersection numbers, Kontsevich)
- Counting Riemann surfaces with Hodge class (Hurwitz numbers) = counting ramified coverings of the sphere
- Gromov-Witten Invariants = counting holomorphic maps of Riemann surfaces into a Calabi-Yau target space
- Many more ... (random matrices, 2d, 3d Young tableaux, crystal growth models, knot theory,...)

## Many surface counting problems do satisfy the TR

- Many more ... (random matrices, 2d, 3d Young tableaux, crystal growth models, knot theory,...)



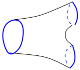
# Weil-Petersson volumes

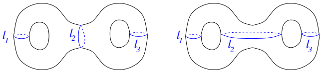
$\mathcal{M}_{g,n}$= moduli space of Riemann surfaces of genus $g$, with $n$ boundaries.

$\rightarrow$ decompose into $2g-2+n$ "pairs of pants"

$\rightarrow$ $3g-3+2n$ geodesic lengths $l_i$, twist angles $\theta_i$= Fenchel-Nielsen coordinates on $\mathcal{M}_{g,n}$.

$$W_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \prod_i dl_i \wedge d\theta_i \quad \prod_{j=\text{boundaries}} \mathrm{e}^{-x_j l_j}.$$



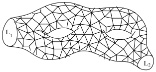



## Theorem

*$W_{g,n}$'s Weil-Petersson volumes satisfy the TR with*
$W_{0,1}(x) = \frac{1}{\pi}\sin 2\pi\sqrt{z}$, $W_{0,2}(x_1,x_2) = \frac{1}{4\sqrt{x_1x_2}\,(\sqrt{x_1}-\sqrt{x_2})^2}$

Proof: TR recursion for $W_{g,n}$ $\Leftrightarrow$ Laplace transform of Mirzakhani's recursions.

# Counting Discrete surfaces




$$W_{g,n} = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}} \frac{t_3^{\#3-\mathrm{gons}}\, t_4^{\#4-\mathrm{gons}}\, t_5^{\#5-\mathrm{gons}} \dots}{x_1^{L_1+1}\, x_2^{L_2+1} \dots x_n^{L_n+1}}$$

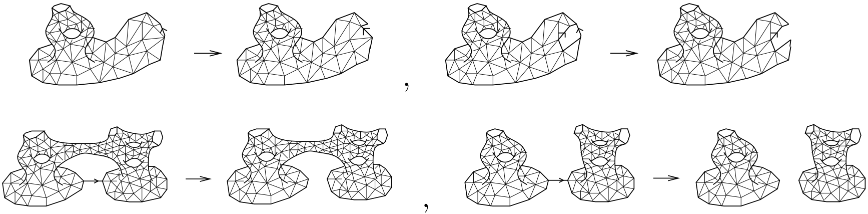
**Theorem (E2004)**

*Generating functions of discrete surfaces satisfy the topological recursion*

$$W_{g,n+1}(x_0, \overbrace{x_1, \dots, x_n}^{J}) = \operatorname*{Res}_{x \to a} K(x_0, x) \Big[ W_{g-1,n+2}(x, x, J) + \sum_{h+h'=g,\, I \uplus I' = J}' W_{h,1+|I|}(x, I)\, W_{h',1+|I'|}(x, I') \Big]$$

## Counting Discrete surfaces

Proof: by Tutte's recursion = edge removal



then, use Cauchy residue formula and move integration contours.

Remark: Tutte's equation = bijective, but moving integration contours $\neq$ bijective.

# Limit of Large Discrete surfaces

$$W_{g,n} = \sum_v t^v \sum_{\Sigma \in \mathbb{M}_{g,n}(v)} \frac{1}{\#\mathrm{Aut}} \frac{t_3^{\#3-\mathrm{gons}}\, t_4^{\#4-\mathrm{gons}}\, t_5^{\#5-\mathrm{gons}} \cdots}{x_1^{L_1+1}\, x_2^{L_2+1} \cdots x_n^{L_n+1}}$$

Take a scaling limit $t \to t_c$, $t_k \to t_{kc}$, $x_i \to a_c$, then:

**Theorem (Bergère, E 2009)**

$$W_{g,n}(x_1, \ldots, x_n; t_3, t_4, \ldots; t)$$
$$\sim (t - t_c)^{(2-2g-n)\mu + n\nu}\, \tilde{W}_{g,n}\left(\frac{x_i - a_c}{(t - t_c)^{\nu}}; \frac{t_k - t_{kc}}{(t - t_c)^{\nu_k}}\right)(1 + o(t - t_c))$$

*where exponents $\nu, \mu, \nu_k$ given by KPZ,*
*and $\tilde{W}_{g,n}$ satisfy the TR, and $\tilde{W}_{g,n}$'s are determinants of KdV Baker-Akhiezer kernel.*

In other words: asymptotic number of large maps $\leftrightarrow$ topological gravity (Witten's conjecture).

# 4. General Properties of TR

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees...
  $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then $\text{scaling} \lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

## General properties of the $F_g$ and $W_n^{(g)}$

For arbitrary $W_{0,1}$ (and $W_{0,2}$ =fund. 2nd kind form of $W_{0,1}$), the $W_{g,n}$ have many interesting properties:

- Integrability $\to$ (formal) Hirota equation, MKP hierarchy
- Symplectic invariance $F_g = W_{g,0}$ is left invariant under $y = W_{0,1}(x) \to \tilde{y} = \tilde{W}_{0,1}(\tilde{x})$ if they ate related by transformations which conserves $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$.
- Modular properties $\to$ Holomorphic anomaly equation
- Singular limits if the curve $\mathcal{E}$ becomes singular, then scaling $\lim W_{g,n}$ satisfy TR.
- Dilaton equation. $W_{g,n}$ =homogeneous of deg $2-2g-n$.
- Diagramatic representation: Givental's formalism, trees... $g$-trees$\leftrightarrow$map" bijection ?
- and many other properties ... etc

# 5. Conclusion

## Conclusion:

- The TR is a universal recursion relation satisfied by many enumeration problems (also appears in random matrices, enumeration of partitions, 3d partitions,...)
- Once we know $W_{0,1}$ and $W_{0,2}$, the TR is very efficient at actually computing the higher genus generating functions.
- Question: find geometric (bijective) proofs of TR for various models ? (for the moment all known proofs use non-bijective steps).

### Further prospects:

- $\exists$ other generalizations of the topological recursion with boundary operators. What does it compute on the A-model side ? link with AdS/CFT ?
- $\exists$ recent new conjecture: topological recursion with $\mathcal{S} =$A-polynomial, computes asymptotics of Jones polynomial of knots ? [Dijkgraaf-Fuji-Manabe 2010, Borot-E 2012]. Find a proof ???
- categorification, refinement, AGT

## The end

# Thank you for your attention

**Acknowledgments:**
This work was supported by:

by the ANR project GranMa ANR-08-BLAN-0311-01, by the ANR project GIMP Géométrie et intégrabilité en physique mathématique ANR-05-BLAN-0029-01, by the European Science Foundation through the Misgam program, by the Quebec government with the FQRNT.