The Brownian map
A continuous limit for large random planar graphs

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Outline

A planar map is just a (finite) graph drawn in the plane (or on the sphere) viewed up to continuous deformation.

It should be interpreted as a discretized model of the sphere.

Main result: A large planar map chosen uniformly at random in a suitable class (p-angulations) and viewed as a metric space (for the graph distance) is asymptotically close to a universal limiting object :

the Brownian map

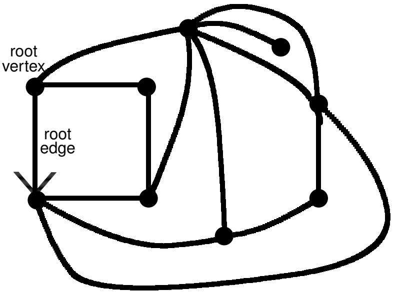
Constructed by gluing certain pairs of points in Aldous' Continuum Random Tree (CRT)

Conjectured relations with the Gaussian free field approach to Liouville theory in quantum gravity (Duplantier-Sheffield)

1. Statement of the main result

Definition

A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



A rooted quadrangulation with 7 faces

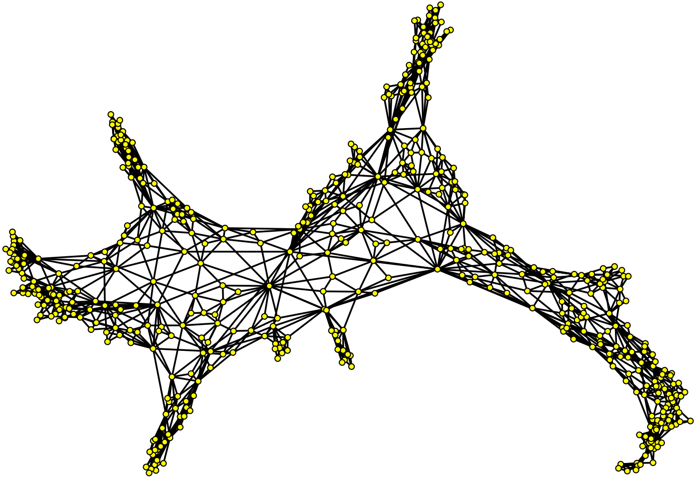
Faces = connected components of the complement of edges

p-angulation:

- each face has p adjacent edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

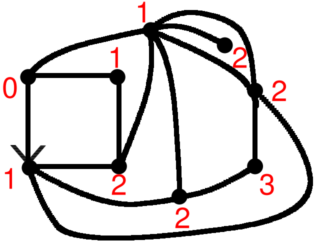


A large triangulation of the sphere (simulation by G. Schaeffer)
Can we get a continuous model out of this ?

Planar maps as metric spaces

M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\mathrm{gr}})$ is a (finite) metric space



In red : distances from the root vertex

$\mathbb{M}_n^p = \{\text{rooted } p - \text{angulations with } n \text{ faces}\}$

$\mathbb{M}_n^p$ is a finite set (*finite number of possible "shapes"*)

Choose M_n uniformly at random in $\mathbb{M}_n^p$.
View $(V(M_n), d_{\mathrm{gr}})$ as a random variable with values in

$$\mathbb{K} = \{\text{compact metric spaces, modulo isometries}\}$$

which is equipped with the Gromov-Hausdorff distance.

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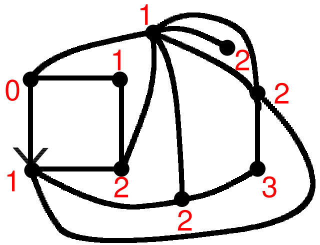

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The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

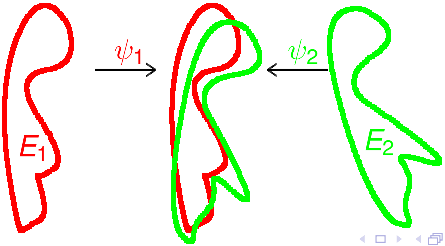
($U_\varepsilon(K_1)$ is the ε-enlargement of K_1)

Definition (Gromov-Hausdorff distance)

If (E_1, d_1) and (E_2, d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1, E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1), \psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

$$(\mathbb{K}, d_{\mathrm{GH}}) \text{ is a separable complete metric space (Polish space)}$$

$\rightarrow$ If M_n is uniformly distributed over $\{p-$angulations with n faces$\}$, it makes sense to study the convergence in distribution of

$$(V(M_n), n^{-a} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

(Problem stated for triangulations by O. Schramm [ICM06])

Choice of the rescaling parameter: $a > 0$ is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

The main theorem

$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$
M_n uniform over $\mathbb{M}_n^p$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

Theorem (The scaling limit of p-angulations)

Suppose that either $p = 3$ (triangulations) or $p \geq 4$ is even. Set

$$c_3 = 6^{1/4} \quad , \quad c_p = \left(\frac{9}{p(p-2)}\right)^{1/4} \quad \text{if } p \text{ is even.}$$

Then,

$$(V(M_n), c_p \frac{1}{n^{1/4}} d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D^*)$$

in the Gromov-Hausdorff sense. The limit $(\mathbf{m}_\infty, D^*)$ is a random compact metric space that does not depend on p (universality) and is called the Brownian map (after Marckert-Mokkadem).

Remarks. Alternative approach to the case $p = 4$: Miermont (2011)
The case $p = 3$ solves Schramm's problem (2006)

Why study planar maps and their continuous limits ?

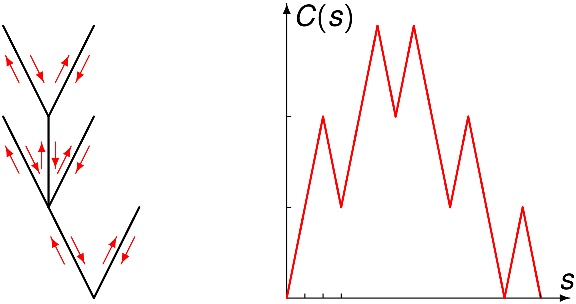
- combinatorics [Tutte '60, 4-color thm, ...]
- theoretical physics
  - enumeration of maps related to matrix integrals ['t Hooft 74, Brézin, Itzykson, Parisi, Zuber 78, etc.]
    recent work of Guionnet, Eynard, etc.
  - large random planar maps as models of random geometry (quantum gravity, cf Ambjørn, Durhuus, Jonsson 95, recent papers of Bouttier-Guitter 2005-2012, Duplantier-Sheffield 2011)
- probability theory: models for a Brownian surface
  - analogy with Brownian motion as continuous limit of discrete paths
  - universality of the limit (conjectured by physicists)
  - asymptotic properties of large planar graphs
- algebraic and geometric motivations: cf Lando-Zvonkin 04 *Graphs on surfaces and their applications*

2. The Brownian map

The Brownian map $(\mathbf{m}_\infty, D^*)$ is constructed by identifying certain pairs of points in Aldous' Brownian continuum random tree (CRT).

Constructions of the CRT (Aldous, 1991-1993):

- As the scaling limit of many classes of discrete trees
- As the random real tree whose contour is a Brownian excursion.



A discrete tree and its contour function.

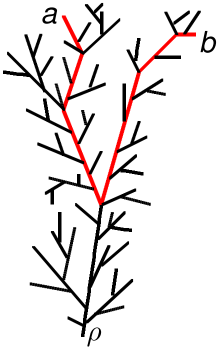
The notion of a real tree

Definition

A real tree, or $\mathbb{R}$-tree, is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique continuous and injective path (up to re-parametrization)
- this path is isometric to a line segment

$\mathcal{T}$ is a rooted real tree if there is a distinguished point ρ, called the root.



Remark. A real tree can have

- infinitely many branching points
- (uncountably) infinitely many leaves

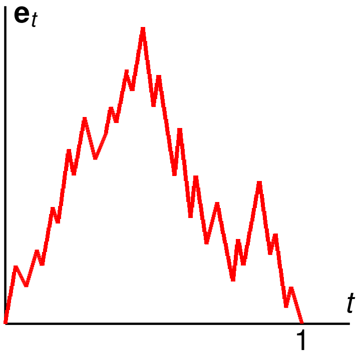
Fact. The coding of discrete trees by contour functions (Dyck paths) can be extended to real trees: also gives a "lexicographical ordering" on a real tree.

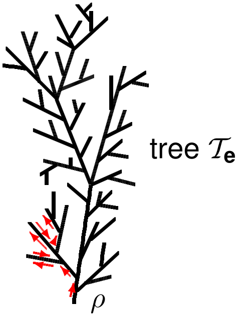
Definition of the CRT

Let $\mathbf{e} = (\mathbf{e}_t)_{0 \le t \le 1}$ be a Brownian excursion with duration 1.

Definition

The CRT $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by the Brownian excursion $\mathbf{e}$.





Assigning Brownian labels to a real tree

Let $(\mathcal{T}, d)$ be a real tree with root ρ.

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0$
- $E[(Z_a - Z_b)^2] = d(a, b), \qquad a, b \in \mathcal{T}$

Labels evolve like Brownian motion along the branches of the tree:

- The label Z_a is the value at time $d(\rho, a)$ of a standard Brownian motion
- Similar property for Z_b, but one uses
  - the same BM between 0 and $d(\rho, a \wedge b)$
  - an independent BM between $d(\rho, a \wedge b)$ and $d(\rho, b)$

Assigning Brownian labels to a real tree

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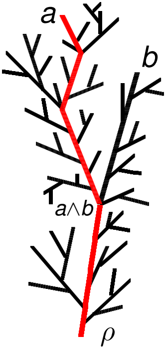

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The definition of the Brownian map

$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT, $(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ Brownian motion indexed by the CRT
Set, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

where $[a, b]$ is the "lexicographical interval" from a to b in $\mathcal{T}_{\mathbf{e}}$ (vertices visited when going from a to b in clockwise order around the tree).
Then set

$$D^*(a,b) = \inf_{a_0=a,a_1,\ldots,a_{k-1},a_k=b} \sum_{i=1}^{k} D^0(a_{i-1}, a_i),$$

$$a \approx b \text{ if and only if } D^*(a,b) = 0 \quad (\text{equivalent to } D^0(a,b) = 0).$$

Definition

The Brownian map $\mathbf{m}_\infty$ is the quotient space $\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}}/\approx$, which is equipped with the distance induced by D^*.

Interpretation

Starting from the CRT $\mathcal{T}_{\mathbf{e}}$, with Brownian labels $Z_a, a \in \mathcal{T}_{\mathbf{e}}$,
$\rightarrow$ Identify two vertices $a, b \in \mathcal{T}_{\mathbf{e}}$ if:

- they have the same label $Z_a = Z_b$,
- one can go from a to b around the tree (in clockwise or in counterclockwise order) visiting only vertices with label greater than or equal to $Z_a = Z_b$.

Remark. Not many vertices are identified:

- A "typical" equivalence class is a singleton.
- Equivalence classes may contain at most 3 points.

Still these identifications drastically change the topology.

Two theorems about the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D^*) = 4 \qquad\qquad a.s.$$

(Already known in the physics literature.)

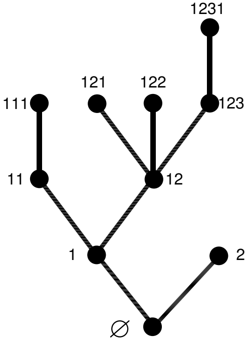
Theorem (topological type, LG-Paulin 2007)

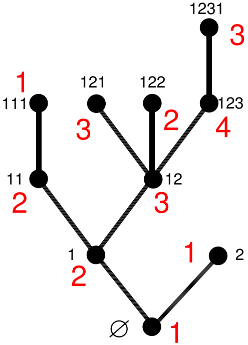
Almost surely, $(\mathbf{m}_\infty, D^*)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

k-point functions (distribution of mutual distances between k randomly chosen points of $\mathbf{m}_\infty$):

- $k = 2$ cf Chassaing-Schaeffer (2004)
- $k = 3$ Bouttier-Guitter (2008)
- $k \geq 4$??

3. The main tool: Bijections between maps and trees





A planar tree $\tau = \{\varnothing, 1, 2, 11, \ldots\}$

(rooted ordered tree)

the lexicographical order on vertices will play an important role in what follows

A well-labeled tree $(\tau, (\ell_v)_{v \in \tau})$

Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$$\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$$
$$\mathbb{M}_n^4 = \{\text{rooted quadrangulations with } n \text{ faces}\}$$

Theorem (Cori-Vauquelin, Schaeffer)

There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ such that, if $M = \Phi(\tau, (\ell_v)_{v\in\tau})$, then

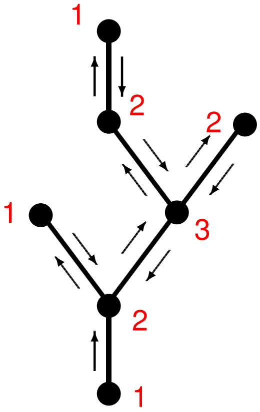
$$V(M) = \tau \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\mathrm{gr}}(\partial, v) = \ell_v \quad , \forall v \in \tau$$

Key facts.

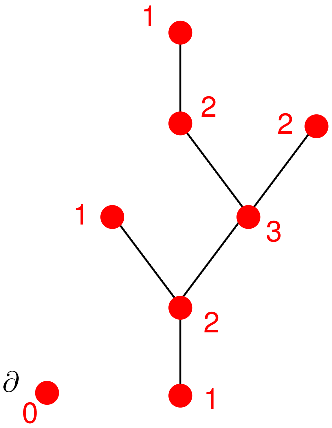
- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

Coding of more general maps: Bouttier, Di Francesco, Guitter (2004)

Schaeffer's bijection between quadrangulations and well-labeled trees



well-labeled tree



quadrangulation

Rules.

- add extra vertex ∂ labeled 0
- follow the contour of the tree, connect each corner to the last visited corner with smaller label

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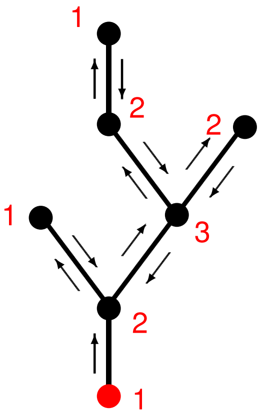

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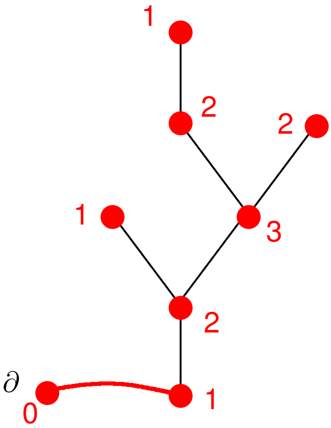

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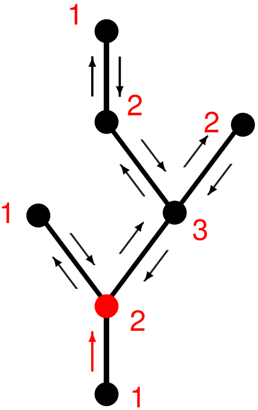

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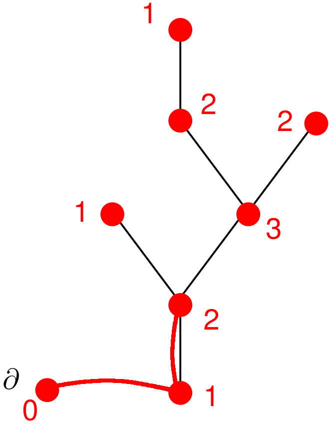

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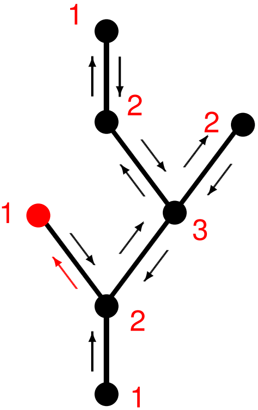

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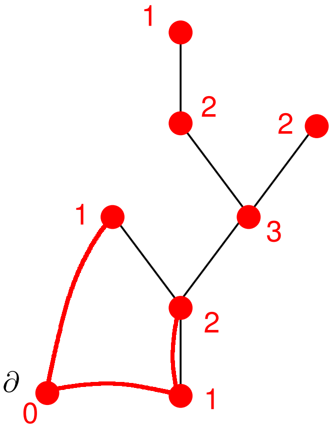

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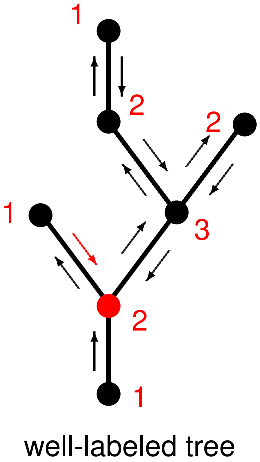

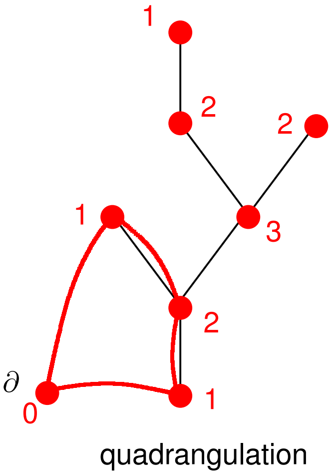

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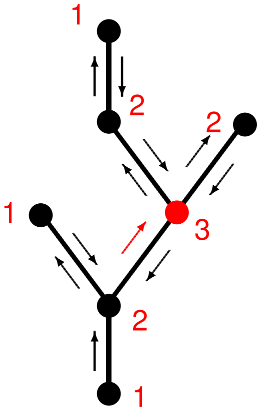

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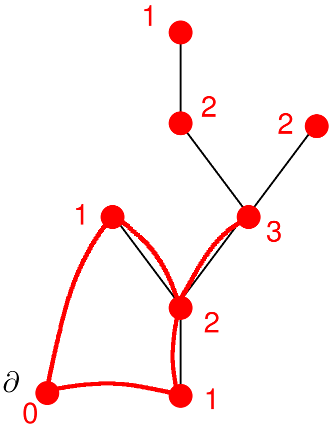

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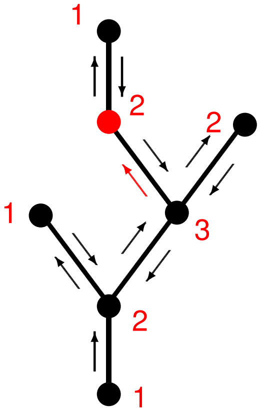

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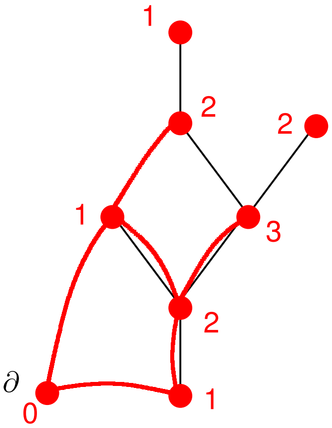

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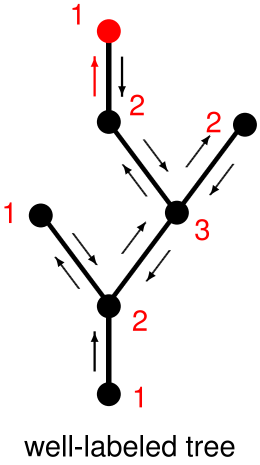

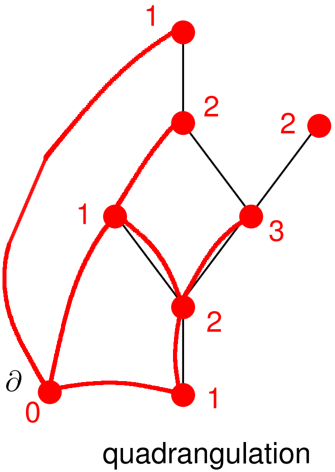

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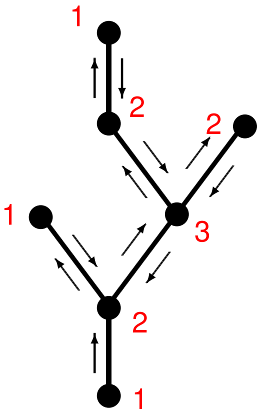

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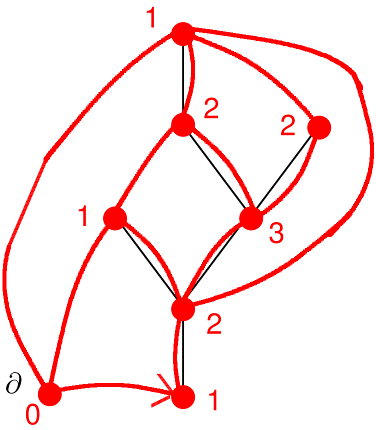

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The label in the tree becomes the distance from ∂ in the graph

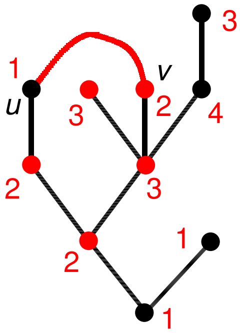
Interpretation of the equivalence relation $\approx$

In Schaeffer's bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v$, $\forall w \in]u, v]$

Explains why in the continuous limit

$$Z_a = Z_b = \min_{c \in [a,b]} Z_c$$
$$\Rightarrow a \text{ and } b \text{ are identified}$$



Key points of the proof of the main theorem:

- Prove the converse (no other pair of points are identified)
- Obtain the formula for the limiting distance D^*

A property of distances in the Brownian map

Let ρ_* be the (unique) vertex of $\mathcal{T}_{\mathbf{e}}$ such that

$$Z_{\rho_*} = \min_{c \in \mathcal{T}_{\mathbf{e}}} Z_c$$

Then, for every $a \in \mathcal{T}_{\mathbf{e}}$,

$$D^*(\rho_*, a) = Z_a - \min Z.$$

("follows" from the analogous property in the discrete setting)

No such simple expression for $D^*(a,b)$ in terms of labels, but

$$D^*(a,b) \leq D^0(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big)$$

(also easy to interpret from the discrete setting)

D^* is the maximal metric that satisfies this inequality

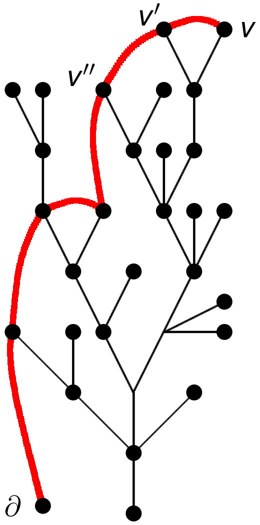
4. Geodesics in the Brownian map

Geodesics in quadrangulations

Use Schaeffer's bijection between quadrangulations and well-labeled trees.

To construct a geodesic from v to ∂:

- Look for the last visited vertex (before v) with label $\ell_v - 1$. Call it v'.
- Proceed in the same way from v' to get a vertex v''.
- And so on.
- Eventually one reaches the root ∂.



Simple geodesics in the Brownian map

Brownian map: $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}} / \approx$

$\mathcal{T}_{\mathbf{e}}$ is re-rooted at ρ_* vertex with minimal label

$\prec$ lexicographical order on $\mathcal{T}_{\mathbf{e}}$

Recall $D^*(\rho_*, a) = \overline{Z}_a := Z_a - \min Z$.

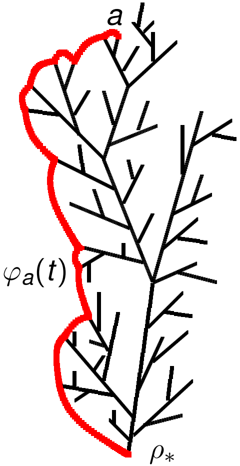
Fix $a \in \mathcal{T}_{\mathbf{e}}$ and for $t \in [0, \overline{Z}_a]$, set

$$\varphi_a(t) = \sup\{b \prec a : \overline{Z}_b = t\}$$

(same formula as in the discrete case !)

Then $(\varphi_a(t))_{0 \le t \le \overline{Z}_a}$ is a geodesic from ρ_* to a

(called a simple geodesic)



Fact

Simple geodesics visit only leaves of $\mathcal{T}_{\mathbf{e}}$ (except possibly at the endpoint)

Simple geodesics in the Brownian map

Brownian map: $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$

$\mathcal{T}_{\mathbf{e}}$ is re-rooted at ρ_* vertex with minimal label

$\prec$ lexicographical order on $\mathcal{T}_{\mathbf{e}}$

Recall $D^*(\rho_*, a) = \overline{Z}_a := Z_a - \min Z$.

Fix $a \in \mathcal{T}_{\mathbf{e}}$ and for $t \in [0, \overline{Z}_a]$, set

$$\varphi_a(t) = \sup\{b \prec a : \overline{Z}_b = t\}$$

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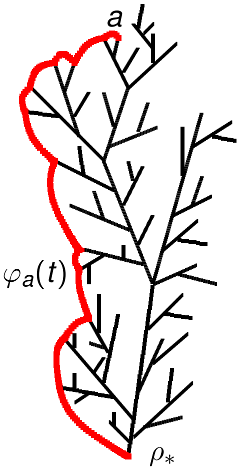


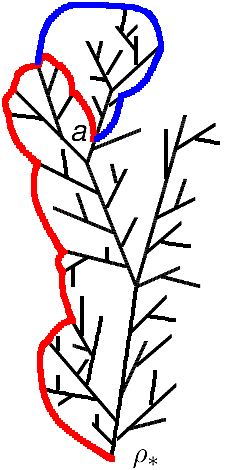
Fact

Simple geodesics visit only leaves of $\mathcal{T}_{\mathbf{e}}$ (except possibly at the endpoint)

How many simple geodesics from a given point ?

- If a is a leaf of $\mathcal{T}_{\mathbf{e}}$, there is a unique simple geodesic from ρ_* to a
- Otherwise, there are
  - 2 distinct simple geodesics if a is a simple point
  - 3 distinct simple geodesics if a is a branching point

(3 is the maximal multiplicity in $\mathcal{T}_{\mathbf{e}}$)



Proposition (key result)

All geodesics from the root are simple geodesics.

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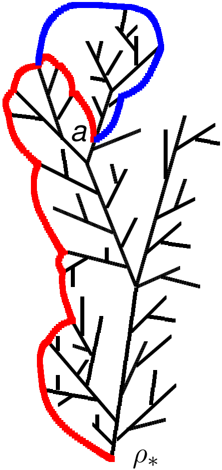

Proposition (key result)

All geodesics from the root are simple geodesics.

The main result about geodesics

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{\text{leaves of } \mathcal{T}_{\mathbf{e}}\}$ and set

$$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})) \qquad (\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\approx = \mathbf{m}_\infty \text{ canonical projection})$$

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
- $\dim(\mathrm{Skel}) = 2$ (recall $\dim(\mathbf{m}_\infty) = 4$)

Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ_*: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ_* replaced by a point chosen "at random" in $\mathbf{m}_\infty$.

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Confluence property of geodesics

Fact: Two simple geodesics coincide near ρ_*.
(easy from the definition)

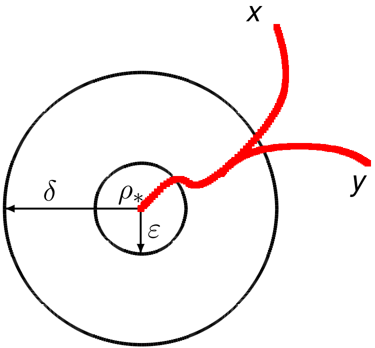
Corollary

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- *if $D^*(\rho_*, x) \geq \delta$, $D^*(\rho_*, y) \geq \delta$*
- *if γ is any geodesic from ρ_* to x*
- *if γ' is any geodesic from ρ_* to y*

then

$$\gamma(t) = \gamma'(t) \quad \text{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ_* along a geodesic.
(also true if ρ_* is replaced by a typical point of $\mathbf{m}_\infty$)

See also Bouttier-Guitter (2008)

Uniqueness of geodesics in discrete maps

M_n uniform distributed over $\mathbb{M}_n^{2p} = \{2p -$ angulations with n faces$\}$
$V(M_n)$ set of vertices of M_n, ∂ root vertex of M_n, $d_{\rm gr}$ graph distance

For $v \in V(M_n)$, $\mathrm{Geo}(\partial \to v) = \{$geodesics from ∂ to $v\}$

If γ, γ' are two discrete paths (with the same length)

$$d(\gamma, \gamma') = \max_i d_{\rm gr}(\gamma(i), \gamma'(i))$$

Corollary

Let $\delta > 0$. Then,

$$\frac{1}{n} \#\{v \in V(M_n) : \exists \gamma, \gamma' \in \mathrm{Geo}(\partial \to v),\ d(\gamma, \gamma') \geq \delta n^{1/4}\} \xrightarrow[n\to\infty]{} 0$$

Macroscopic uniqueness of geodesics, also true for "approximate geodesics"= paths with length $d_{\rm gr}(\partial, v) + o(n^{1/4})$

Other results for discrete geodesics: Bouttier-Guitter (2008)

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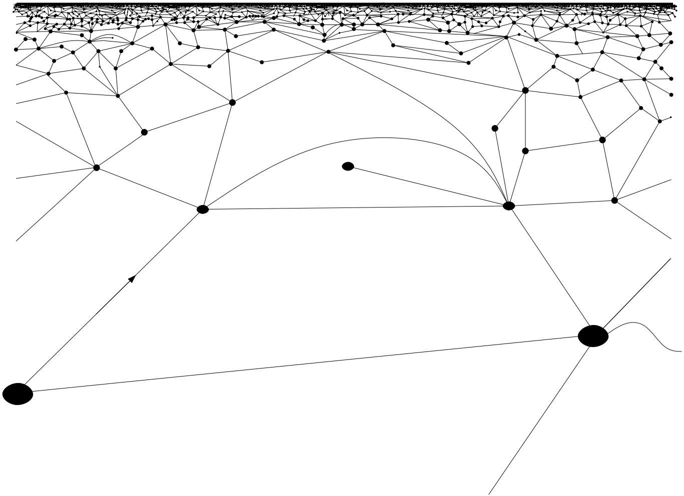
5. The Brownian plane (joint with N. Curien)

$\mathcal{P}$ non-compact version of the Brownian map,
with scale invariance property: $\lambda\mathcal{P} \overset{(d)}{=} \mathcal{P}$

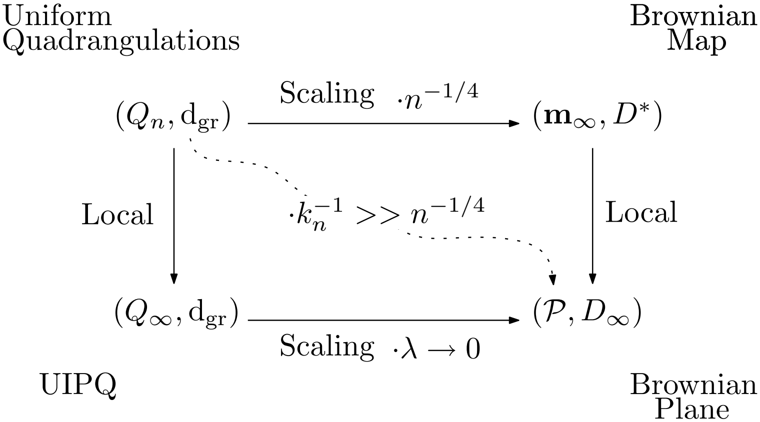
- tangent cone of the Brownian map: $\lambda \mathbf{m}_\infty \xrightarrow[\lambda\to\infty]{(d)} \mathcal{P}$
  (in the sense of Gromov-Hausdorff for pointed metric spaces)
- scaling limit of the Uniform Infinite Planar Quadrangulation (UIPQ)
- scaling limit of quadrangulations, with scaling factor $k_n^{-1} \gg n^{-1/4}$

The UIPQ



Q_n uniform over {quadrangulations with n faces},
$Q_n \longrightarrow Q_\infty$ in the sense of the local limit of graphs

Convergence to the Brownian plane



Constructing the Brownian plane

Replace the CRT by the infinite Brownian tree $\mathcal{T}_\infty$
coded by a pair of independent 3-dimensional Bessel processes.

Introduce Brownian labels Z^∞ on the infinite Brownian tree :

- Use these labels to identify certain pairs of vertices of $\mathcal{T}_\infty$
- Construct the distance D_∞ on the quotient space in a way analogous to the case of the Brownian map.

Properties of the Brownian plane

- Scale invariance : $\lambda\mathcal{P} \overset{(d)}{=} \mathcal{P}$
- $\dim \mathcal{P} = 4$, $\mathcal{P}$ homeomorphic to the plane
- Confluence of geodesic rays to infinity ($g : [0, \infty) \longrightarrow \mathcal{P}$ is a geodesic ray if $D_\infty(g(s), g(t)) = |s - t|$ for all s, t)
  Any two geodesic rays merge in finite time
- Interpretation of the labels Z^∞ as "distances from infinity":

$$Z^\infty_x - Z^\infty_y = \lim_{z\to\infty} (D^\infty(x, z) - D^\infty(y, z))$$

(similar to a result of Curien-Ménard-Miermont for UIPQ)

6. Canonical embeddings: Open problems

Recall that a planar map is defined up to (orientation-preserving) homeomorphisms of the sphere.

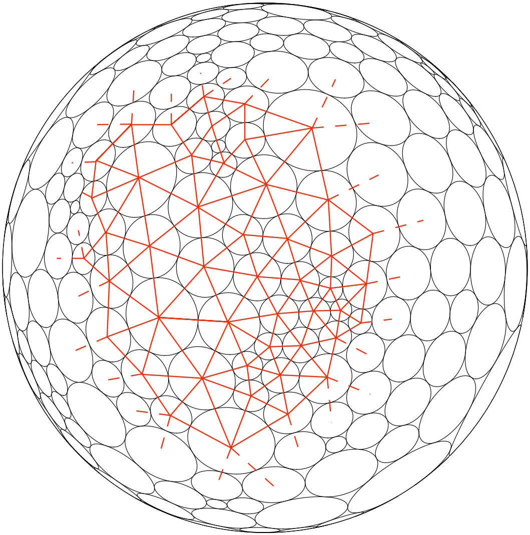
It is possible to choose a particular (canonical) embedding of the graph satisfying conformal invariance properties, and this choice is unique (at least up to the Möbius transformations, which are the conformal transformations of the sphere $\mathbb{S}^2$).

Question

Applying this canonical embedding to M_n (uniform over p-angulations with n faces), can one let n tend to infinity and get a random metric Δ on the sphere $\mathbb{S}^2$ satisfying conformal invariance properties, and such that

$$(\mathbb{S}^2, \Delta) \overset{\rm (d)}{=} (\mathbf{m}_\infty, D^*)$$

Canonical embeddings via circle packings 1



From a circle packing, construct a graph M :

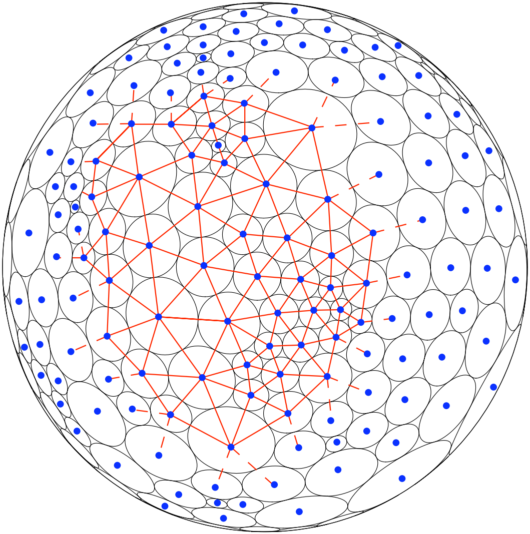
- $V(M) = \{$centers of circles$\}$
- edge between a and b if the corresponding circles are tangent.

A triangulation (without loops or multiple edges) can always be represented in this way.

Representation unique up to Möbius transformations.

Figure by N. Curien

Canonical embeddings via circle packings 2



Apply to M_n uniform over {triangulations with n faces}. Let $n \to \infty$. Expect to get

- Random metric Δ on $\mathbb{S}^2$ (with conformal invariance properties) such that $(\mathbb{S}^2, \Delta) = (\mathbf{m}_\infty, D^*)$
- Random volume measure on $\mathbb{S}^2$

Connections with the Gaussian free field and Liouville quantum gravity ? (cf Duplantier-Sheffield).

Figure by N. Curien

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