Black Hole Bound States

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Quantum Gravity in Paris
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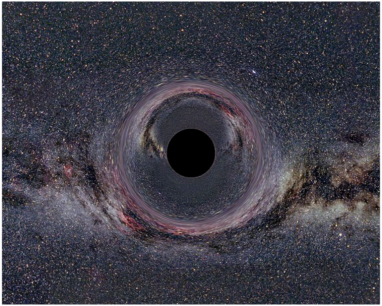
Based on joint work with B. Pioline and A. Sen.

Black holes

Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{4\pi G}{c^2} T_{\mu\nu}$$

Black hole simulation:



Kraus, Zahn

Horizon area:

$A_{\rm hor} = 16\pi G^2 M^2 / c^4$

Black hole entropy:

$\Rightarrow S = A_{\rm hor}/4$

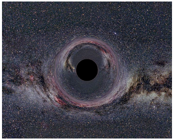
Bekenstein (1973), Hawking (1975)

Can we understand this entropy quantum mechanically, i.e. $\log \Omega(M) \sim S$?

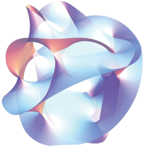
String theory

Theory of quantum gravity in 10 dimensions:

$\mathbb{R}^4$ $\times$ Calabi-Yau



$\times$



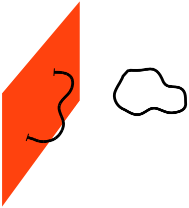
3-dim. projection of 6-dim. CY.
en.wikipedia.org

Physics on $\mathbb{R}^4$ $\Leftrightarrow$ Physics on Calabi-Yau!

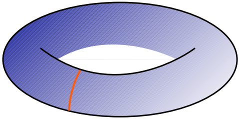
String theory provides examples of quantum black holes

Strominger, Vafa (1996); Dijkgraaf, Verlinde, Verlinde (1996); Maldacena, Strominger, Witten (1997),...

Ingredients:



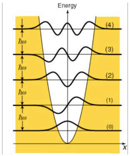
D-brane



D-brane around cycle with mass $M(\gamma, t)$

$$\begin{aligned} \{\mathcal{Q}^I, \bar{\mathcal{Q}}^J\} &= \sigma^\mu P_\mu \delta^{IJ} \\ \{\mathcal{Q}^I, \mathcal{Q}^J\} &= \varepsilon^{IJ} Z(\gamma, t) \end{aligned}$$

Supersymmetry



Analyze spectrum using CFT

$\Rightarrow$

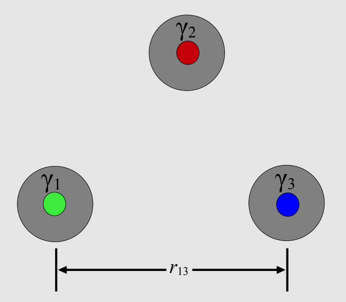
Confirmation!

$$\log \Omega(\gamma) \sim S/k_{\rm B}$$

Many remaining questions:

- Does $A_{\mathrm{hor}}/4$ agree with $\log \Omega(\gamma)$ beyond leading order?
- What is the contribution from multi-center black holes to $\Omega(\gamma)$?
- Can agreement be shown for less or no supersymmetry?
- ...

Black hole bound states



n black holes with charges γ_i located at $\vec{r}_i$ in $\mathbb{R}^3$

Static solutions exist due to interplay between gravitational attraction and electro-magnetic repulsion

Motivation:

Microscopic counting of BPS black holes does not accurately distinguish single center and multi-center solutions with charge γ

Strominger, Vafa (1996); Maldacena, Strominger, Witten (1997); Bena, Wang, Warner (2006); Strominger, Gaiotto, Yin (2006); Denef, Moore (2007); De Boer, Denef, El-Showk, Messamah, Van den Bleeken (2008);...

$\Rightarrow$ Understanding of bound states is crucial for precision tests of black hole entropy

Black hole bound states resolve a puzzle for the "light" spectrum:

$$S_{\mathrm{BH}} \sim \pi \sqrt{\tfrac{2}{3} P^3 Q_0}$$

Puzzle: $Q_0 \geq -\frac{1}{24} P^3$ in CFT

Solution: The states with $-\frac{P^3}{24} \leq Q_0 \leq 0$ are realized as BH bound states: Gaiotto, Dabholkar, Nampuri (2007); Denef, Moore (2008); Sen (2011);...

Bound state with total charge $(0, P, 0, Q_0) = (0, P, 0, -P^3/24)$:

(-1, P/2, P^2/8, -P^3/48)   (1, P/2, -P^2/8, -P^3/48)

Denef equations

$\mathcal{N} = 2$ BPS equations of motion require the distances $r_{ij} = |\vec{r}_i - \vec{r}_j| \in \mathbb{R}_+$ to satisfy:

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

- $\gamma_{ij} = \langle \gamma_i, \gamma_j \rangle \in \mathbb{Z}$: Dirac-Schwinger-Zwanziger innerproduct
- $c_i(\{\gamma_j\}; t) \in \mathbb{R}$: stability parameters depending on vector multiplet scalars t

Denef (2000)

Phase space $M(\{\gamma_i\}, \{c_i\})$:

- parametrizes $\vec{r}_i \in \mathbb{R}^3, i = 1, \ldots, N$
- has dimension $2N - 2$

De Boer, El-Showk, Messamah, Van den Bleeken (2008)

Denef equations: Two aspects

Wall-crossing:

Solutions might decay or recombine upon varying $c_i \in \mathbb{R}$:

Denef (2000); Denef, Moore (2007),...

For example $N = 2$: $\lim_{c_1 \to 0} r_{12} = \lim_{c_1 \to 0} \frac{\gamma_{12}}{c_1} = \pm\infty$

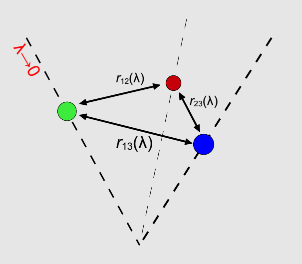
Scaling solutions:

Centers could get arbitrarily close, depending on $\{\gamma_i\}$

Bena, Wang, Warner (2006); Denef, Moore (2007),...

For example $N = 3$: If $\gamma_{12} + \gamma_{23} \geq \gamma_{31}$, and cyclic perm. $\Rightarrow$

$$\lim_{\lambda \to 0} r_{ij}(\lambda) = \lambda\gamma_{ij} + \mathcal{O}(\lambda^2) \in M(\{\gamma_i\}, \{c_i\})$$



BPS index

$\mathcal{H}(\gamma, t)$: Hilbert space of $\mathcal{N} = 2$ supergravity with fixed γ and vector multiplet scalars t

BPS index: $$\Omega(\gamma; t) = -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}(\gamma,t)}\,(2J_3)^2\,(-1)^{2J_3},$$
with J_3 generator of $SU(2)_{spin}$ in $\mathbb{R}^3$

Single-centered index: $\Omega_{\mathrm{S}}(\gamma)$= # of states associated to the single center BH with charge γ, independent of t

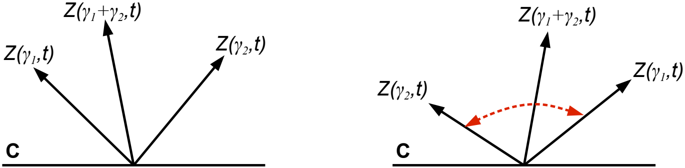
Generically a protected quantity $\Rightarrow$ independent of

- coupling constant
- hypermultiplet scalars

Wall-crossing

Dependence of $\Omega(\gamma; t)$ on vector multiplet scalars t

A central charge $Z(\gamma, t) \in \mathbb{C}$ is associated to every BPS state



- If $Z(\gamma_1, t) || Z(\gamma_2, t)$: $M(\gamma_1) + M(\gamma_2) = M(\gamma_1 + \gamma_2)$
  $\Rightarrow$ bound states are marginally stable
- $\Omega(\gamma_1 + \gamma_2; t)$ is only locally constant as function of t;
  it might jump across walls where $Z(\gamma_1, t) || Z(\gamma_2, t)$.

Wall-crossing

The jump $\Delta\Omega(\gamma; t \to t') = \Omega(\gamma; t') - \Omega(\gamma; t')$ is given by the wall-crossing formula

Kontsevich, Soibelmann (2008); Joyce, Song (2008); ...

Put strong constraints on the BPS indices $\Omega(\gamma; t)$ and the physics of bound states.

BPS index

Decomposition formula:

$$\bar{\Omega}(\gamma;t) = \sum_{\substack{\sum_i N_i\gamma_i=\gamma,\\ \gamma_i\neq\gamma_j, i\neq j}} g(\{N_i\gamma_i\};\{c_i(t)\}) \prod_j \frac{\bar{\Omega}_{\mathrm{S}}(\gamma_j)^{N_j}}{N_j!}$$

- $\bar{\Omega}_{\mathrm{S}}(\gamma_j) = \sum_{n|\gamma_j} \frac{\Omega_{\mathrm{S}}(\gamma_j/n)}{n^2}$: rational invariant associated to center j Familiar from Schwinger pair creation and D-instanton measure. Gopakumar, Vafa (1998); Kontsevich, Soibelman (2008); Joyce, Song (2008); Kim, Park, Wang, Yi (2011),...
- $\frac{\bar{\Omega}_{\mathrm{S}}(\gamma)^N}{N!}$: Maxwell-Boltzmann distribution
- $g(\{N_i\gamma_i\};\{c_i\}) \in \mathbb{Z}$: # of “binding” states

JM, Pioline, Sen (2010)

Main question: How to interpret and determine $g(\{N_i\gamma_i\};\{c_i\})$?

Quiver quantum mechanics: Field content

Low energy excitations of the black hole bound state

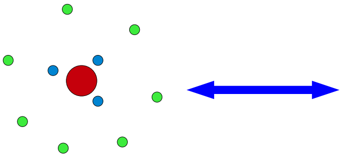
$$\Downarrow$$

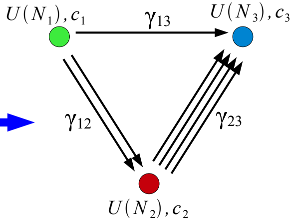
$\mathcal{N} = 4$ quiver quantum mechanics

Denef (2002); Denef, Moore (2007); ... Familiar from BPS monopoles: Cederwall, Ferretti, Nilsson, Salomonson (1995); Sethi, Stern, Zaslow (1995); Gauntlett, Harvey (1996); Gauntlett, Kim, Park, Yi (2000),...

Field content is:

- determined by multiplicities $\{N_i\}$ and innerproducts $\{\gamma_{ij}\}$:
  - vector multiplets $(\vec{r}_i, A_i, \lambda_i)$ with gauge group $U(N_i)$
  - $|\gamma_{ij}|$ bifundamental chiral multiplets $(\phi^a_{ij}, F^a_{ij}, \psi^a_{ij}), a = 1, \ldots, |\gamma_{ij}|$
- parametrized by a quiver $(\vec{N}, \vec{c})$:





Quiver quantum mechanics: Higgs branch

$g_s \ll 1$: Higgs branch

- D-term eqs: $\displaystyle\sum_{j,i\to j}\sum_{a=1}^{\gamma_{ij}} \phi_{ij}^a(\phi_{ij}^a)^\dagger - \sum_{j,j\to i}\sum_{a=1}^{\gamma_{ji}} (\phi_{ij}^a)^\dagger \phi_{ij}^a = c_i \mathbf{1}_{N_i}$

  F-term eqs: $\displaystyle\frac{\partial W(\{\phi_{ij}^a\})}{\partial \phi_{kl}^b} = 0$

  $\Rightarrow$ equations for Kähler quotient $\mathcal{M}(\vec{N};\vec{c})$
- Witten index: $\mathrm{Tr}_{\mathcal{H}_{\mathrm{Higgs}}(\vec{N},\vec{c})}(-1)^F = \chi(\mathcal{M}(\vec{N};\vec{c}))$

No oriented loops: $\chi(\mathcal{M}(\vec{d},\vec{\theta}))$ (and $p(\mathcal{M}(\vec{d},\vec{\theta}),y)$) are given by the Reineke formula Reineke (2002)

Higgs branch: Abelianization formula

Maxwell-Boltzmann distribution

$\Downarrow$

$g(\{N_i\gamma_i\};\{c_i\})$: # of ground states of Abelian quiver theory

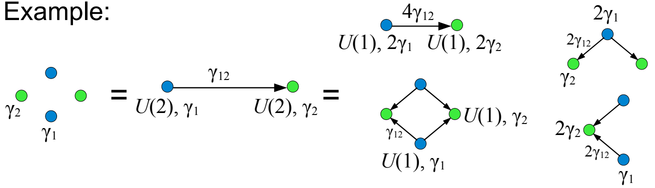
$\Rightarrow$ non-Abelian from Abelian:

1. degeneracies: $\Omega_{\rm S}(\ell\gamma_i) = \delta_{\ell,1}$
2. decomposition formula implies:

$$\chi(\mathcal{M}_Q(\vec{N};\vec{c})) \sim \sum_{Q'} \chi(\mathcal{M}_{Q'}(\vec{1}_{N'};\vec{c}\,'))$$

JM, Pioline, Sen (2010)

Example:



Mathematical studies by Mozgovoy, Okada, Reineke, Stoppa, Weist,. . .

Coulomb branch

$g_s \gg 1$: Coulomb branch

- vector multiplets: $$\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$$

  Denef (2002); Kim, Park, Wang, Yi (2011),...

N.B.: BPS solutions require more conditions to be physical, in particular regularity of metric

Coulomb branch

Goal: determine $g(\{\gamma_i\};\{c_i\})$ in space-time

1. $g(\{\gamma_i\},\{c_i\})$ is the (twisted) Dirac index of the space $M(\{\gamma_i\},\{c_i\})$
2. $\exists$ symplectic form on $\mathcal{M}_n$:

$$\omega = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\epsilon^{abc}\frac{\mathrm{d}r_{ij}^a\wedge\mathrm{d}r_{ij}^b\,r_{ij}^c}{|r_{ij}|^3}$$

3. $g(\{\gamma_i\},\{c_i\})$ can be determined using geometric quantization in special cases

   De Boer, El-Showk, Messamah, Van den Bleeken (2008); ...

Coulomb branch

General computation is feasible by refining the index:

$$\Omega(\gamma, y; t) = \frac{\mathrm{Tr}_{\mathcal{H}_{\mathsf{BPS}}(\gamma,t)}\,(-y)^{2J_3}}{-y^{-2}+2-y^2}$$

$\Rightarrow$ $g(\{\gamma_i\}, y; \{c_i\})$: equivariant Dirac index of $M(\{\gamma_i\},\{c_i\})$

Index theorem: $g(\{\gamma_i\}, y; \{c_i\}) = \int_M \mathrm{Ch}(\mathcal{L},\nu)\,\hat{A}(M,\nu)|_{2N-2}$
with $\nu = \log(y)$, $\mathrm{Ch}(\mathcal{L},\nu)$ = equivariant Chern character of $\mathcal{L}$,
$\hat{A}(M,\nu)$ = equivariant $\hat{A}-$genus of M

Berline, Vergne (1985)

Coulomb branch: Localization

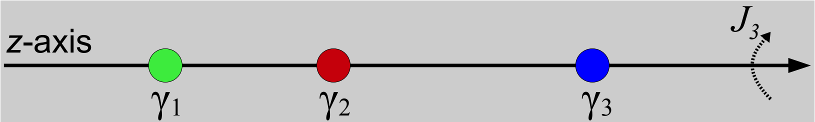
Evaluate integral by localization with respect to J_3

Duistermaat, Heckman (1982); Berline, Vergne (1985);...

$\Downarrow$

Sum over isolated fixed points $\in M(\{\gamma_i\}, \{c_i\})$ of J_3

The solutions which contribute are of the form:



JM, Pioline, Sen (2011)

Coulomb branch formula

Fixed point formula:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \sum_{p\in\{\text{f.p. of } J_3\}} s(p)\, y^{2J_3(p)}$$

- angular momentum:

$$J_3(p) = \frac{1}{2}\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)$$

- sign:

$$s(p) = \mathrm{sign}\left(\det\left(\frac{\partial^2 W}{\partial z_i \partial z_j}\right)\right)$$

with $W(\{z_i\}) = -\sum_{i<j}\gamma_{ij}\,\mathrm{sign}(z_j - z_i)\,\log|z_i - z_j| - \sum_{i=1}^N c_i z_i$

Coulomb branch formula: Example

Example: $\gamma_i, i = 1, \ldots, 3$, $c_3 < c_2 < 0 < c_1$:

- Fixed points have orderings:
  $\{1,2,3;+\}, \{2,1,3;-\}, \{3,1,2;-\}, \{3,2,1;+\}$,
- Enumerate:

$$g(\{\gamma_i\}, y; \{c_i\}) = (-1)^{\gamma_{12}+\gamma_{23}+\gamma_{13}} (y - y^{-1})^{-2}$$
$$\left(y^{\gamma_{12}+\gamma_{13}+\gamma_{23}} - y^{\gamma_{12}-\gamma_{23}-\gamma_{13}} - y^{\gamma_{13}+\gamma_{23}-\gamma_{12}} + y^{-\gamma_{12}-\gamma_{13}-\gamma_{23}}\right)$$

agreement with wall-crossing formula's proven by A. Sen (2011)

NB: Hard to find fixed points numerically.

Algorithm: Deforming γ_{ij}

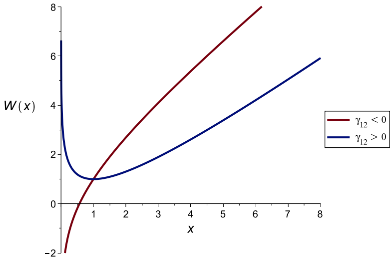
Problem: (numerical) determination of fixed points is tedious and time-consuming

Resolution: Recursive determination of $g(\{\gamma_i\}, y; \{c_i\})$ using:

1. γ_{ij} can be deformed from $\mathbb{Z}$ to $\mathbb{R}$ in Denef equations
2. take a convenient choice $\gamma_{0,ij}$ for γ_{ij}
3. determine $s_0(p)$
4. study the (dis)appearance of extrema of $W(\{z_i\})$ during the reverse deformation: $s(p) = s_0(p) + \sum_A s_A(p)$

   For example:

   $N = 2$, $x = z_2 - z_1$, $c_1 < 0$:



$$W(x) = -\gamma_{12} \ln(x) - c_1 x$$

JM, Pioline, Sen (2013)

Algorithm: Results

For quivers without closed loops:

- explicit expression for $s(p)$ is obtained:

$$s(p) = \prod_{k=1}^{N} \Theta(\tilde{\gamma}_{k,k+1}\, \tilde{c}_k)\, (-1)^{\sum_{k=1}^{N-1} \Theta(-\tilde{\gamma}_{k,k+1})}$$

where $\Theta(x) =$ step function

- agrees with Higgs branch result

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{BPS,Higgs}}(\vec{1}_N, \vec{c})}(-y)^{2J_3} = P(\mathcal{M}(\vec{1}_N, \vec{c}), -y)$$

Reineke (2002); JM, Pioline, Sen (2013)

Algorithm: Minimal modification hypothesis

With loops:

- scaling solutions are possible
- explicit algorithm, recursive in the number of centers
- sum over regular fixed points $\neq$ SU(2) character

Problem: What is the contribution of the scaling fixed point?

Proposal: Minimal modification hypothesis:

$$g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{i<j}\gamma_{ij}+N-1}}{(y-y^{-1})^{N-1}} \left({\sum_p}' s(p)\, y^{2J_3(p)} + p_{\rm scal}(y) \right)$$

Determine $p_{\rm scal}(y)$ iteratively by:

- $g(\{\gamma_i\}, y; \{c_i\})$ is an SU(2) character
- classically $J_3(p_{\rm scal}) = 0 \quad \Rightarrow \quad \lim_{y\to\infty} \frac{p_{\rm scal}(y)}{(y-y^{-1})^{N-1}} = 0$

Pure Higgs states

Higgs-Coulomb map $B : \mathcal{H}_{\text{Higgs}}(\gamma, t) \to \mathcal{H}_{\text{Coulomb}}(\gamma, t)$
surjective map with kernel $\ker(B)$: pure Higgs states

Berkooz, Verlinde (1999); Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012);...

$$\text{If } \ker(B) \neq \emptyset \quad \Rightarrow \quad P(\mathcal{M}(\vec{1}_N, \vec{c}), -y) - g(\{\gamma_i\}, y; \{c_i\}) \neq 0$$

- $\ker(B) \neq 0$ only occurs with loops
- extra states are associated to scaling fixed points
- not distinguishable from single center black hole states
  $\Rightarrow$ contribute to $\Omega_{\mathrm{S}}(\sum_i \gamma_i; y)$
- supersymmetric single center black hole has $J = 0$
  $\Rightarrow$ $\Omega_{\mathrm{S}}(\gamma, y) = \Omega_{\mathrm{S}}(\gamma) \in \mathbb{N}$

  Sen (2009)

Pure Higgs states

Coulomb branch formula computes the invariant $\Omega(\gamma, y; t)$ with as input the $\Omega_{\rm S}(\gamma, y)$

Different physical systems lead to different $\Omega_{\rm S}(\gamma, y)$:

- Quiver quantum mechanics: $\Omega_{\rm S}(\ell\gamma_i) = \delta_{1,\ell}$, $i \in V$
  Abelian quivers: Lefshetz hyperplane theorem $\Rightarrow$ $\Omega_{\rm S}(\gamma, y) \in \mathbb{N}$
  Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012); JM, Pioline, Sen (2013), ...
- Quantum field theories (with line operators) Cordova, Neitzke (2013)
- Supergravity

Quiver mutations

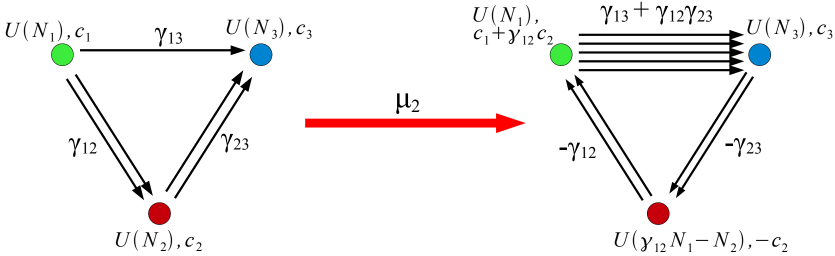
Quiver mutation $\Leftrightarrow$ Seiberg duality

Change of gauge groups $U(N_i)$ and number of hypermultiplets γ_{ij}

Math: Bernstein, Gelfand, Ponomarev (1978); Derksen, Weyman, Zelevinsky (2008); Kontsevich, Soibelman (2008); Keller, Yang (2011), ... Physics: Seiberg (1994); Feng, Hanany, He, Uranga (2001); Berenstein, Douglas (2002); Mukhopadhya, Ray (2004),...

Quiver quantum mechanics: $\Omega_{\mathrm{S}}(\ell\gamma_i) = \delta_{1,\ell}$, $i \in V$

Example: mutation on node 3 of 3-node quiver with $c_2 < 0$, $\sum_i c_i = 0$.



$$\mathcal{M}(\vec{N}; \vec{c}) \cong \mathcal{M}'(\vec{N}'; \vec{c}')$$

Generalized quiver mutations

Coulomb branch formula has more general input
$\Omega_{\mathrm{S}}(\gamma, y) = \sum_n \Omega_n(\gamma)\, y^n \in \mathbb{Z}[y, y^{-1}]$

Generalized mutation symmetry motivated by:

- quiver mutation
- Fermi flip (particle-hole duality)

Andriyash, Jafferis, Denef, Moore (2010)

Conditions:

1. Fermionic particle: $\Omega_n(\gamma_2) \geq 0$
2. $M = \sum_{\ell \geq 1} \sum_n \ell^2 \Omega_{\mathrm{S}}(\ell \gamma_2) < 0$
3. $\Omega_{\mathrm{S}}(\alpha, y; c) = \begin{cases} \Omega'_{\mathrm{S}}(\alpha + M \langle \alpha, \gamma_2 \rangle \gamma_2, y; c) & \text{for } \alpha \nparallel \gamma_k \\ \Omega'_{\mathrm{S}}(-\alpha) & \text{for } \alpha \parallel \gamma_k \end{cases}$

Generalized quiver mutations

Charges transform as:

$$(\gamma_1, \gamma_2, \gamma_3) \longrightarrow (\gamma_1 + M\gamma_{12}\gamma_2, -\gamma_2, \gamma_3)$$

induce transformations:
$(N_1, N_2, N_3) \longrightarrow (N_1, M\gamma_{12}N_1 - N_2, N_3)$
$(c_1, c_2, c_3) \longrightarrow \dots$ Proposal:

$$\Omega(\gamma, y; c) = \Omega'(\gamma', y; c')$$

Verified in many cases, but the generalized mutation symmetry remains to be proven.

Puts strong constraints on the $\Omega_{\mathrm{S}}(\gamma, y)$ JM, Pioline, Sen (2013)

Summary & Conclusion

Using quiver quantum mechanics and the Coulomb branch formula black hole bound states can be explicitly enumerated in terms of the single centered indices $\Omega_{\mathrm{S}}(\gamma)$.

These techniques are now ready to be combined with earlier results on quantum black hole counting to make precision tests on quantum black hole entropy.

`CoulombHiggs.m`:

- MATHEMATICA package for Coulomb and Higgs computations
- available at `arXiv:1302.5498`