Exact Quantum Black hole entropy: a macroscopic window into quantum gravity

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Quantum gravity in Paris
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We do not have a direct microscopic probe of quantum gravity.

(Not yet! March 17, 14.45h London)
Black hole entropy is a precious clue to understand quantum gravity

\[ k_B \log d_{\text{micro}} = S_{\text{BH}}^{\text{class}} + \cdots \]  
(Boltzmann)

Universal law in GR

\[ S_{\text{BH}}^{\text{class}} = \frac{1}{4} \frac{A_H}{\ell_P^2} = \frac{A_H c^3}{4 \hbar G_N} \]  
(Bekenstein-Hawking '74)

Recent progress on this front

Deviations from GR!
What is a good microscopic theory of quantum gravity?

String theory?
- Perturbatively UV finite in flat space ✓
- Weak-strong dualities ✓
- AdS/CFT holography ✓

However, we do not know what phase (vacuum/compactification) corresponds to the real world!

(c.f. talk of Ambjorn)

Focus on universal requirements which should hold in all phases of the theory.

Interpret a BH as a statistical ensemble of states.
Black holes in string theory are ensembles of microscopic excitations

Microscopic

Strominger-Vafa '96

\[ d_{\text{micro}}(N) = e^{\pi \sqrt{N}} + \cdots \quad (N \to \infty) \]

Macroscopic

Bekenstein-Hawking '74

\[ S_{\text{BH}} = \frac{A_H}{4\ell_P^2} = \pi \sqrt{N} \]

\[ \log d_{\text{micro}} = S_{\text{BH}} + \cdots \]

\[ S_{\text{BH}} \quad \text{(finite } N) \]
What is new? Finite size quantum effects!

\[ S_{\text{BH}}^{\text{quant}} = \frac{1}{4} A + a_0 \log(A) + a_1 \frac{1}{A} + a_2 \frac{1}{A^2} + \cdots + b_1(A)e^{-A} + \cdots \]

Questions

1. What is the physics of these corrections?
2. How to compute them in a concrete model?
3. Can we compare them to a similar expansion in the microscopic theory?

Exact AdS/CFT

Supersymmetric Localization

Mock modular forms
Finite size corrections arise from quantum fluctuations in the black hole

Wald Entropy formalism

- Obeys the first law of thermodynamics
- Extends Bekenstein-Hawking area law in GR
- Applicable to any local effective action of gravity
- Successfully applied to BH models in supergravity

(Cardoso, de Wit, Mohaupt '99)

We still need a good formalism to study Quantum BH entropy including non-analytic and non-local terms.
Supersymmetric black holes and $\text{AdS}_2$

4d extremal Reissner-Nordstrom solution $\rightarrow$ near-horizon geometry $\text{AdS}_2 \times S^2$.

All known supersymmetric BHs develop near-horizon $\text{AdS}_2$ factor.

Euclidean $\text{AdS}_2 \times S^2$
Quantum BH entropy is a functional integral over $\text{AdS}_2$ configurations \(^{(Sen '08)}\)

\[
\exp(S_{BH}^{qu}(q_I)) \equiv Z_{\text{AdS}_2}(q_I) = \left\langle \exp \left[ -i q_I \int A^I \right] \right\rangle_{\text{AdS}_2}^{\text{reg}}
\]

- Boundary conditions fixed by classical BH configuration
  (c.f. attractor mechanism Ferrara, Kallosh, Strominger)
- $\text{AdS}_2 \Rightarrow \text{microcanonical ensemble}$ with fixed charges
- Saddle point evaluation $\iff$ classical Wald entropy
- Logarithmic one-loop corrections can be computed.
  \(^{(Sen + Banerjee, Gupta, Mandal, 2010-2012)}\)
AdS/CFT correspondence has been extremely successful...

\[ \text{CFT}_{p+1} \equiv \text{Quantum gravity on AdS}_{p+2} \]

- Quark-gluon plasma
- Fluid dynamics
- Quantum phase transitions
- Superconductivity?

Good progress in understanding the classical planar limit \((N \to \infty)\).

...But \textbf{Quantum} gravity = \(1/N\) effects
Dual theory for BPS BH is a collection of supersymmetric ground states

Dual $\text{CFT}_1$ obtained as IR limit of brane configuration that makes up the black hole.

In $d=0+1$, no space for long-wavelength fluctuations.

$$Z_{\text{CFT}_1}(q) = \text{Tr}_{\mathcal{H}(q)} 1 = d_{\text{micro}}(q).$$

$\text{AdS/CFT correspondence}$

$$\Rightarrow Z_{\text{AdS}_2}(q) = d_{\text{micro}}(q)$$
Prototype: \( N=8 \) string theory in 4d (macro)

(Cremmer, Julia '78)

Macroscopic description: \( d=4 \) supergravity coupled to 28 U(1) gauge fields + superpartner scalars + fermions.

1/8 BPS dyonic BH solutions. (Cvetic, Youm '96)

BH Charges \((q_I, p^I), I = 1, \ldots, 28,\)

U-duality symmetry \( E_{7,7}(\mathbb{Z}) \)

Quartic invariant \( N(q, p) = q^2 p^2 - (q \cdot p)^2 \)

Classical BH Entropy \( S_{BH} = \pi \sqrt{N} + \cdots \)
Prototype: N=8 string theory in 4d (micro)

Microscopic degeneracies $d_{\text{micro}}(N)$ computed using representation as D1-D5-P-K system in Type II string theory. \hspace{1cm} (Maldacena, Moore, Strominger ’99)

They depend only on U-duality invariant N.

With $q = e^{2\pi i \tau}$, \hspace{1cm} Modular form!

$$\sum_N d_{\text{micro}}(N) \ e^{2\pi i N \tau} = \frac{\theta(\tau)}{\eta(\tau)^6}$$

$$= q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + 56q^8 + \cdots$$
BPS quantum black hole entropy

\[
\exp(S_{BH}^{\text{qu}}(q_\mathcal{I}))) \equiv Z_{\text{AdS}_2}(q_\mathcal{I}) = \left\langle \exp \left[ -i q_\mathcal{I} \int A^I \right] \right\rangle_{\text{reg}}^{\text{AdS}_2}.
\]

Supercharge \( Q \) with \( Q^2 = L_0 - J_0 \).

\( M \): Field space of supergravity.

\( d\mu \): Measure on this field space.

\( \mathcal{O} \): Wilson line.

\( S \): Action of graviton and other massless fields.
Localization

Witten ’88, Duistermaat-Heckmann ’82, Atiyah-Bott ’84, Pestun ’07

Consider a supermanifold $\mathcal{M}$ with an odd vector field $Q$ and an off-shell algebra $Q^2 = H$ with $H$ a compact $U(1)$.

We would like to evaluate an integral of a $Q$–invariant operator $\mathcal{O}$

$$I := \int_{\mathcal{M}} d\mu \mathcal{O} e^{-S}.$$ 

The functional integral localizes onto the submanifold $\mathcal{M}_Q$ of solutions of the off-shell BPS equations $Q \Psi = 0$

$$I = \int_{\mathcal{M}_Q} d\mu_Q \mathcal{O} e^{-S}.$$
How to compute the functional integral

(A.Dabholkar, J.Gomes, S.M. ’10, ’11)

1. Formalism: N=2 off-shell supergravity.  
   (de Wit, van Holten, Van Proeyen ’80)

2. Find all solutions of localization equations
   \[ Q_1 \Psi = 0 \],
   subject to \( AdS_2 \times S^2 \) boundary conditions.
   (R.Gupta, S.M. ’12)

3. Evaluate action on these solutions (including all higher
derivative terms). Compute the measure.

4. Only chiral-superspace integrals in the action contribute.
   These are exactly known in string theory.
   (V.Reys, S.M. ’13)
Evaluation of the functional integral

- QG path integral reduces to an 8-dimensional integral.
- 7 of the integrals are Gaussian

\[ e^{S_{BH}^{\text{qu}}(N)} = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \frac{\pi^2 N}{4\sigma}\right) = \tilde{I}_{7/2}(\pi \sqrt{N}) \]
### A quantitative test

**Classical entropy**

<table>
<thead>
<tr>
<th>(N)</th>
<th>(d_{\text{micro}}(N))</th>
<th>(\exp(S_{\text{cl}}(N)))</th>
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<tr>
<td>3</td>
<td>8</td>
<td>230.76</td>
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<tr>
<td>4</td>
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<td>208</td>
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<td>15</td>
<td>513</td>
<td>192400.81</td>
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<td>(\ldots)</td>
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<tr>
<td>(10^5)</td>
<td>(\exp(295.7))</td>
<td>(\exp(314.2))</td>
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</tbody>
</table>

\[
\log(d_{\text{micro}}) \xrightarrow{\Delta \to \infty} S_{BH}^{\text{cl}}.
\]
A quantitative test  

(A.Dabholkar, J.Gomes, S.M. ’11)

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<tr>
<th>N</th>
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$$d_{micro}(\Delta) = e^{S_{BH}^{qu}(\Delta)} (1 + O(e^{-\pi \sqrt{\Delta}/2}))$$
Why does this work so well?

The Fourier series of the microscopic degeneracies

\[ Z(\tau) \equiv \sum_{N} d_{\text{micro}}(N) e^{2\pi i N \tau} = \frac{\theta(\tau)}{\eta(\tau)^6} \]

is a modular form.

Strong-weak coupling symmetry:  \[ Z(-1/\tau) = \tau^{5/2} Z(\tau) \]
Exact formula for degeneracies

Hardy-Ramanujan-Rademacher expansion

\[
d_{\text{micro}}(N) = \sum_{c=1}^{\infty} c^{-9/2} K_c(N) \tilde{I}_{7/2}(\frac{\pi \sqrt{N}}{c})
\]

\[
= \tilde{I}_{7/2}(\pi \sqrt{N}) + O(e^{-\pi \sqrt{N}/2})
\]

\[
= e^{\pi \sqrt{N}} \left(1 - \frac{15}{4} \log N + O\left(\frac{1}{N}\right)\right).
\]
Wall-crossing and BH phase transitions

Phase I

(Q,P)

Phase II

(Q,P)

ΔS

Serious problem: throwing out multi-centered BHs destroys the modular symmetry. (Denef-Moore 2007)
Mock modular forms provide the answer

(A. Dabholkar, S. M., D. Zagier ’12)

These functions were described by Ramanujan, who gave a list of examples, but did not give a definition!

Their definition and structural properties were finally understood by S. Zwegers in 2000.

Surprisingly, this is exactly what we need to solve the BH wall-crossing problem.

For the N=4 theory, we could solve it fully (based on formula due to Dijkgraaf, Verlinde, Verlinde ’96), and explicitly compute the partition function of a single BH as a function of its charges.
What is the partition function of a single-centered black hole?

We have a canonical decomposition of the partition function:

\[
Z_{\text{micro}}(\tau) = Z_{\text{BH}}(\tau) + Z_{\text{multi}}(\tau)
\]

- \(Z_{\text{multi}}(\tau)\) contains all the wall-crossing information.
- \(Z_{\text{BH}}(\tau)\) is the partition function of the single centered BH. It is a mock modular form.

One can now use modular symmetry to make Rademacher expansions as before. (e.g. Manschot, Bringmann ’13).

Many new explorations have opened up as a result. e.g. Large discrete symmetry groups (moonshine) of BHs in string theory (J. Harvey, S.M. ’13).
Conclusions and outlook

• Finite size effects in BH thermodynamics can be computed.

• Localization methods give us convergent perturbation expansions for the quantum gravity partition function.

• Emergence of quantum structure from continuum gravity, inclusion of sub-leading saddle points are important.

• Mathematical structures: New mock modular symmetries seem to play a key role in the BH wall-crossing problem.

• Effective low-energy theory provides strong constraints on quantum theory of gravity.

Lunch time!