Exact Quantum Black hole entropy:
a macroscopic window into quantum gravity

Sameer Murthy
King's College London

Quantum gravity in Paris
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We do not have a direct microscopic probe of quantum gravity.

(Not yet! March 17, 14.45h London)

Black hole entropy is a precious clue to understand quantum gravity

$$k_B \log d_{\text{micro}} = S_{\text{BH}}^{\text{class}} + \cdots$$

(Boltzmann)

Universal law in GR

$$S_{\text{BH}}^{\text{class}} = \frac{1}{4}\frac{A_{\text{H}}}{\ell_{\text{P}}^2} = \frac{A_{\text{H}}\, c^3}{4\,\hbar\, G_N}$$

(Bekenstein-Hawking '74)

Deviations from GR!

Recent progress on this front

What is a good microscopic theory of quantum gravity?

String theory?

- Perturbatively UV finite in flat space
- Weak-strong dualities
- AdS/CFT holography

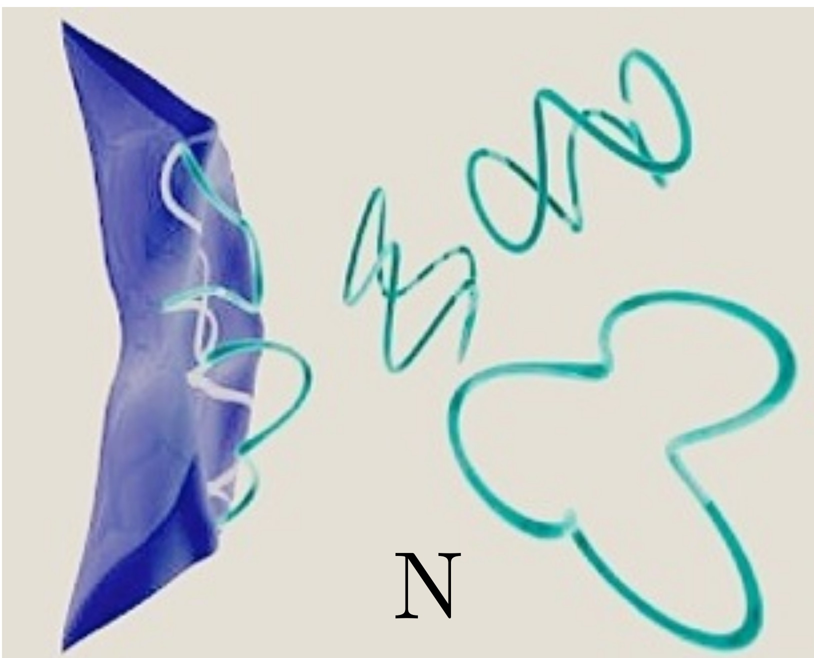
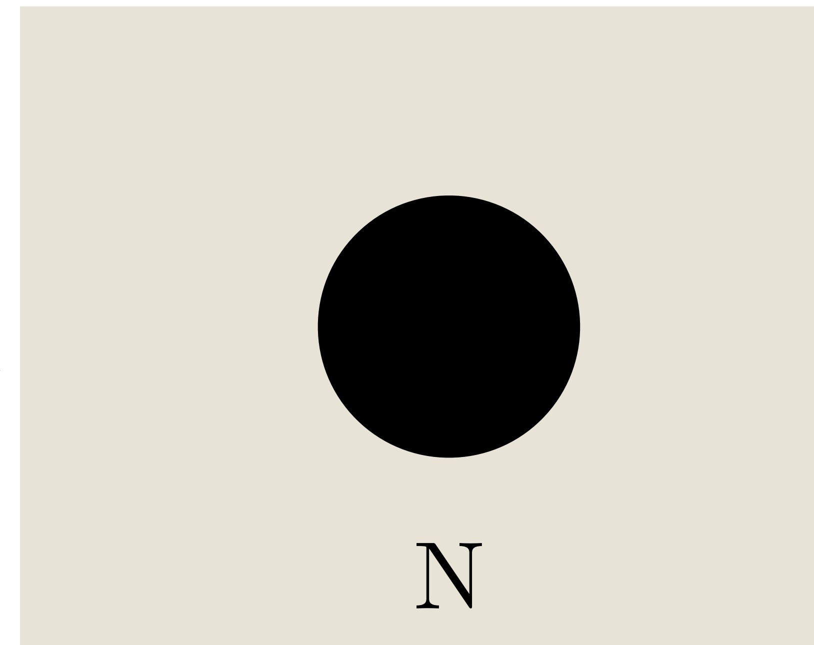
However, we do not know what phase (vacuum/compactification) corresponds to the real world!

(c.f. talk of Ambjorn)

Focus on universal requirements which should hold in all phases of the theory.

Interpret a BH as a statistical ensemble of states.

Black holes in string theory are ensembles of microscopic excitations

Microscopic		Macroscopic
	$g_s\mathrm{N} \ll 1$ ← g_s → $g_s\mathrm{N} \gg 1$	
Strominger-Vafa '96		Bekenstein-Hawking '74

$$d_{\mathrm{micro}}(\mathrm{N}) = e^{\pi\sqrt{\mathrm{N}}} + \cdots \quad (\mathrm{N} \to \infty)$$

$$S_{\mathrm{BH}}^{\mathrm{class}} = \frac{A_H}{4\ell_{\mathrm{Pl}}^2} = \pi\sqrt{N}$$

$$\log d_{\mathrm{micro}} = S_{\mathrm{BH}}^{\mathrm{class}} + \cdots \;\rightarrow\; S_{\mathrm{BH}}^{\mathrm{quant}}$$

(finite N)

What is new? Finite size quantum effects!

$$S_{\rm BH}^{\rm quant} = \frac{1}{4}A + a_0 \log(A) + a_1 \frac{1}{A} + a_2 \frac{1}{A^2} + \cdots$$
$$+ b_1(A) e^{-A} + \cdots$$

Questions

1. What is the physics of these corrections?
2. How to compute them in a concrete model?
3. Can we compare them to a similar expansion in the microscopic theory?

Exact AdS/CFT

Supersymmetric Localization

Mock modular forms

Finite size corrections arise from quantum fluctuations in the black hole

Wald Entropy formalism

- Obeys the first law of thermodynamics
- Extends Bekenstein-Hawking area law in GR
- Applicable to any *local* effective action of gravity
- Successfully applied to BH models in supergravity

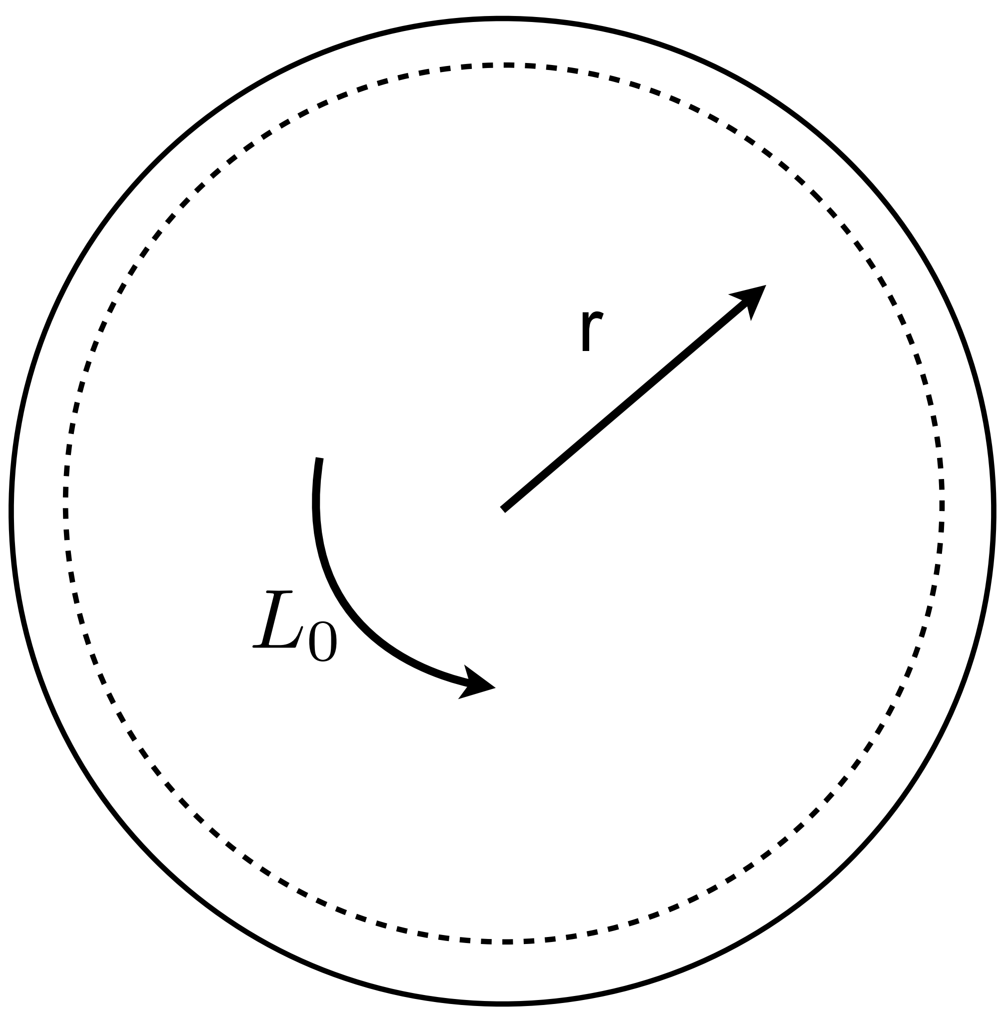
(Cardoso, de Wit, Mohaupt ’99)

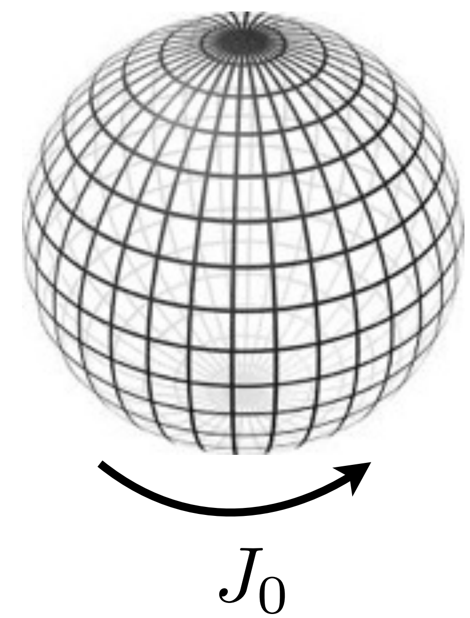
We still need a good formalism to study Quantum BH entropy including non-analytic and non-local terms.

Supersymmetric black holes and $\mathbf{AdS_2}$

4d extremal Reissner-Nordstrom solution
⇨ near-horizon geometry $AdS_2 \times S^2$.

All known supersymmetric BHs develop near-horizon AdS_2 factor.





Euclidean $\mathbf{AdS_2 \times S^2}$

Quantum BH entropy is a functional integral over $\mathrm{AdS_2}$ configurations (Sen '08)

$$\exp(S^{\rm qu}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\left[- i\, q_I \oint A^I \right] \right\rangle^{\rm reg}_{\rm AdS_2}$$

- Boundary conditions fixed by classical BH configuration

  (c.f. attractor mechanism Ferrara, Kallosh, Strominger)

- $AdS_2 \Rightarrow$ microcanonical ensemble with fixed charges

- Saddle point evaluation ⇨ classical Wald entropy

- Logarithmic one-loop corrections can be computed.

  (Sen + Banerjee, Gupta, Mandal, 2010-2012)

$\mathrm{AdS/CFT}$ correspondence has been extremely successful...

$\mathbf{CFT_{p+1}} \equiv$ Quantum gravity on $\mathbf{AdS_{p+2}}$

→

- Quark-gluon plasma
- Fluid dynamics
- Quantum phase transitions
- Superconductivity?

Good progress in understanding the classical planar limit $(\mathrm{N} \to \infty)$.

…But *Quantum* gravity = 1/N effects

Dual theory for BPS BH is a collection of supersymmetric ground states

Dual $\mathbf{CFT_1}$ obtained as IR limit of brane configuration that makes up the black hole.

In d=0+1, no space for long-wavelength fluctuations.

$$Z_{\mathrm{CFT}_1}(q) = \mathrm{Tr}_{\mathcal{H}(q)} 1 = d_{\mathrm{micro}}(q) .$$

AdS/CFT correspondence

$$\Rightarrow Z_{\mathrm{AdS}_2}(q) = d_{\mathrm{micro}}(q)$$

Prototype: N=8 string theory in 4d (macro)

(Cremmer, Julia '78)

Macroscopic description: d=4 supergravity coupled to 28 U(1) gauge fields + superpartner scalars + fermions.

1/8 BPS dyonic BH solutions. (Cvetic, Youm '96)

BH Charges $(q_I, p^I),\ I = 1, \ldots, 28\,,$

U-duality symmetry $E_{7,7}(\mathbb{Z})$

Quartic invariant $N(q,p) = q^2 p^2 - (q.p)^2$

Classical BH Entropy $S_{BH} = \pi \sqrt{N} + \cdots$

Prototype: N=8 string theory in 4d (micro)

Microscopic degeneracies $d_{\rm micro}(\mathrm{N})$ computed using representation as D1-D5-P-K system in Type II string theory. (Maldacena, Moore, Strominger '99)

They depend only on U-duality invariant N.

Modular form!

With $q = e^{2\pi i\tau}$,

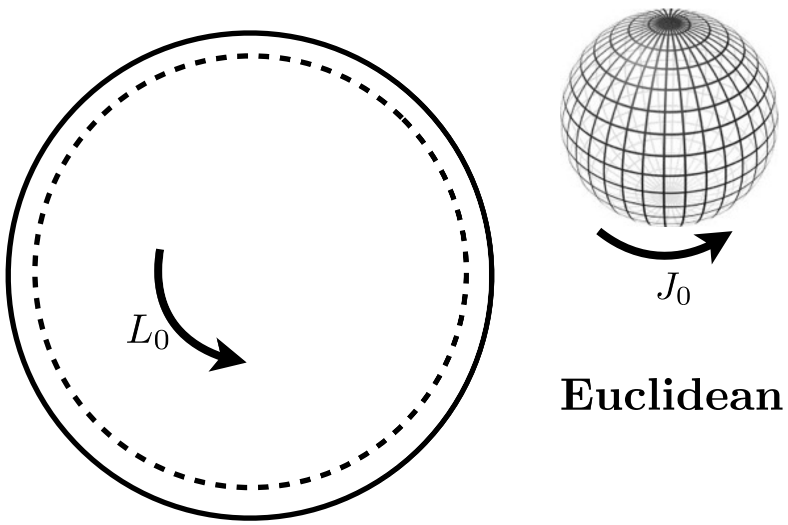
$$\sum_{\mathrm{N}} d_{\rm micro}(\mathrm{N})\, e^{2\pi i \mathrm{N}\tau} = \theta(\tau)/\eta(\tau)^6$$

$$= q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + 56q^8 + \cdots$$

BPS quantum black hole entropy

$$\exp(S^{\rm qu}_{BH}(q_I)) \equiv Z_{AdS_2}(q_I) = \left\langle \exp\Big[-i\,q_I \oint A^I\Big] \right\rangle^{\rm reg}_{\rm AdS_2} .$$

Supercharge Q with $Q^2 = L_0 - J_0$.



Euclidean $\mathbf{AdS_2 \times S^2}$

$\mathcal{M}$: Field space of supergravity.

$d\mu$: Measure on this field space.

$\mathcal{O}$: Wilson line.

$\mathcal{S}$: Action of graviton and other massless fields.

Localization

Witten '88, Duistermaat-Heckmann '82, Atiyah-Bott '84, Pestun '07

Consider a supermanifold $\mathcal{M}$ with an odd vector field Q and an off-shell algebra $Q^2 = H$ with H a compact $U(1)$.

We would like to evaluate an integral of a $Q-$invariant operator $\mathcal{O}$

$$I := \int_{\mathcal{M}} d\mu \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

The functional integral localizes onto the submanifold $\mathcal{M}_Q$ of solutions of the off-shell BPS equations $Q\,\Psi = 0$

$$I = \int_{\mathcal{M}_Q} d\mu_Q \, \mathcal{O} \, e^{-\mathcal{S}} \, .$$

How to compute the functional integral

(A.Dabholkar, J.Gomes, S.M. '10, '11)

1. Formalism: N=2 off-shell supergravity. (de Wit, van Holten, Van Proeyen '80)

2. Find all solutions of localization equations $Q_1 \Psi = 0$, subject to $AdS_2 \times S^2$ boundary conditions. (R.Gupta, S.M. '12)

3. Evaluate action on these solutions (including all higher derivative terms). Compute the measure.

4. Only chiral-superspace integrals in the action contribute. These are exactly known in string theory. (V.Reys, S.M. '13)

Evaluation of the functional integral

- QG path integral reduces to an 8-dimensional integral.
- 7 of the integrals are Gaussian

$$e^{S_{BH}^{\mathrm{qu}}}(N) = \int \frac{d\sigma}{\sigma^{9/2}} \exp\left(\sigma + \pi^2 N/4\sigma\right) = \widetilde{I}_{7/2}(\pi\sqrt{N})$$

A quantitative test

(Classical entropy)

N	$d_{\rm micro}(\mathrm{N})$		$\exp\left(S^{\rm cl}(\mathrm{N})\right)$
3	8		230.76
4	12		535.49
7	39		4071.93
8	56		7228.35
11	152		33506.14
12	208		53252.29
15	513		192400.81
...	...		...
10^5	exp(295.7)		exp(314.2)

$$\log(d_{\rm micro}) \xrightarrow{\Delta\to\infty} S^{\rm cl}_{BH} \, .$$

A quantitative test

(A.Dabholkar, J.Gomes, S.M. '11)

N	$d_{\text{micro}}(\text{N})$	$\exp\big(S^{\text{qu}}(\text{N})\big)$	$\exp\big(S^{\text{cl}}(\text{N})\big)$
3	8	7.97	230.76
4	12	12.2	535.49
7	39	38.99	4071.93
8	56	55.72	7228.35
11	152	152.04	33506.14
12	208	208.45	53252.29
15	513	512.96	192400.81
...	...	...	...
10^5	exp(295.7)	exp(295.7)	exp(314.2)

$$d_{\text{micro}}(\Delta) = e^{S^{\text{qu}}_{BH}(\Delta)}\big(1 + O(e^{-\pi\sqrt{\Delta}/2})\big)$$

Why does this work so well?

The Fourier series of the microscopic degeneracies

$$Z(\tau) \equiv \sum_N d_{\text{micro}}(N)\ e^{2\pi i N\tau} = \theta(\tau)/\eta(\tau)^6$$

is a modular form.

Strong-weak coupling symmetry: $Z(-1/\tau) = \tau^{5/2} Z(\tau)$

$$\tau \rightarrow \tau + 1$$
$$\tau \rightarrow -1/\tau\,.$$

⇒ $SL_2(\mathbb{Z})$ Modular symmetry group

Highly constraining

Exact formula for degeneracies

Hardy-Ramanujan-Rademacher expansion

$$d_{\text{micro}}(N) = \sum_{c=1}^{\infty} c^{-9/2} K_c(N)\, \widetilde{I}_{7/2}\left(\frac{\pi\sqrt{N}}{c}\right)$$

$$= \widetilde{I}_{7/2}(\pi\sqrt{N}) + O(e^{-\pi\sqrt{N}/2})$$

$$= e^{\pi\sqrt{N}}\left(1 - \frac{15}{4}\log N + O\left(\frac{1}{N}\right)\right).$$

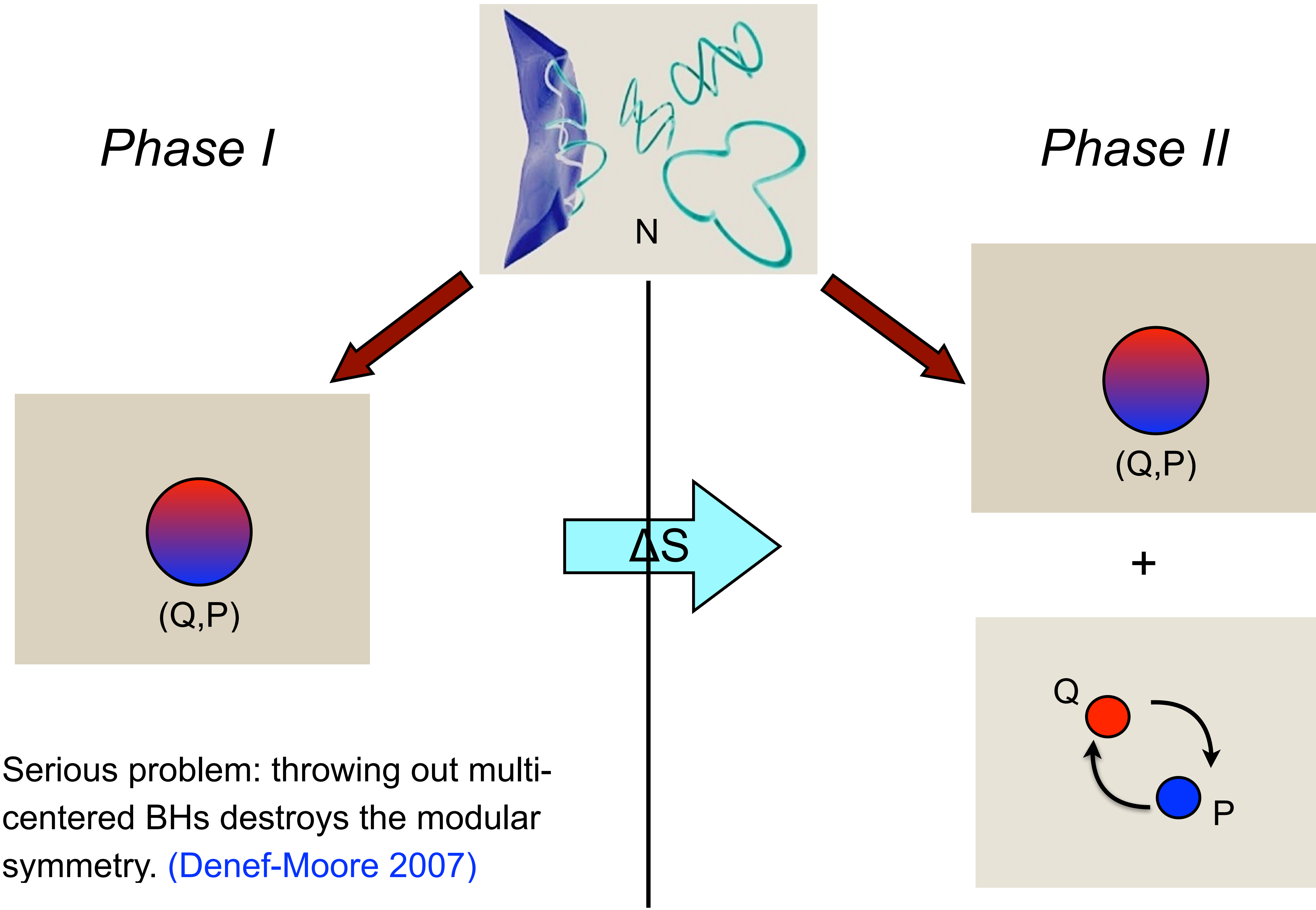
Orbifolds of AdS_2

(A.Dabholkar, J.Gomes, S.M. to appear)

Bekenstein-Hawking

One-loop corrections

Wall-crossing and BH phase transitions



Serious problem: throwing out multi-centered BHs destroys the modular symmetry. (Denef-Moore 2007)

Mock modular forms provide the answer

(A.Dabholkar, S.M., D.Zagier '12)

These functions were described by Ramanujan, who gave a list of examples, but did not give a definition!

Their definition and structural properties were finally understood by S. Zwegers in 2000.

Surprisingly, this is exactly what we need to solve the BH wall-crossing problem.

For the N=4 theory, we could solve it fully (based on formula due to Dijkgraaf, Verlinde, Verlinde '96), and explicitly compute the partition function of a single BH as a function of its charges.

What is the partition function of a single-centered black hole?

We have a canonical decomposition of the partition function:

$$Z_{\text{micro}}(\tau) = Z_{\text{BH}}(\tau) + Z_{\text{multi}}(\tau)$$

- $Z_{\text{multi}}(\tau)$ contains all the wall-crossing information.
- $Z_{\text{BH}}(\tau)$ is the partition function of the single centered BH. It is a mock modular form.

One can now use modular symmetry to make Rademacher expansions as before. (e.g. Manschot, Bringmann '13).

Many new explorations have opened up as a result.
e.g. Large discrete symmetry groups (moonshine) of BHs in string theory (J. Harvey, S.M. '13)

Conclusions and outlook

- Finite size effects in BH thermodynamics can be computed.
- Localization methods give us convergent perturbation expansions for the quantum gravity partition function.
- Emergence of quantum structure from continuum gravity, inclusion of sub-leading saddle points are important.
- Mathematical structures: New mock modular symmetries seem to play a key role in the BH wall-crossing problem.
- Effective low-energy theory provides strong constraints on quantum theory of gravity.

Lunch time!