Tensor Models/Group Field Theories: An overview

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March 27, 2014
Quantum Gravity in Paris 2014
LPT-Orsay & IHES

Outline

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Generalities: Quantizing Gravity

• Several issues. The Eistein theory of gravity is perturbatively divergent and nonrenormalizable [DeWitt PR '67, Goroff & Sagnotti, NPB '86].

• Remarkable Sakharov's idea: "An induced theory of gravity" [Sakharov, '67, SPD 68, Visser MPLA '02].

• Alternative scenarii (coupling gravity to other fields, Asymptotic Safety, . . .) and more "daring" scenarii (extra-dimensions, susy, background field independent methods, . . .).

• Focus on *Discrete* methods. Mid 80's: In particular, *Matrix Models* [Di Francesco et al.., PR '95] prove to be a solvable framework and concrete realization of an "emergent gravity" scenario.

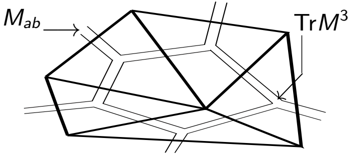
• Matrix Models and Random 2D geometry: To "replace" the sum over topologies and geometry of a 2D surface by a sum over random triangulations of surfaces.

From Matrices . . .

Matrix Models (MM): Probability measures for matrices M of *large size* N and describe 2D gravity. Archetype:

$$Z_{\mathrm{matrix}} = \int dM \, e^{-\frac{1}{2}\mathrm{Tr} M^2 + \frac{\lambda}{\sqrt{N}}\mathrm{Tr} M^3} = e^{Z_{2DQG}}$$

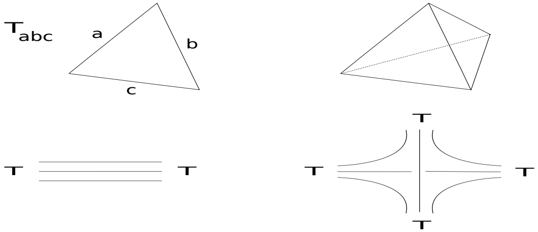
• A triangulated surface $\equiv$ A Feynman ribbon graph:



• 't Hooft's Large N limit: Planar graphs' sector $\equiv$ surfaces of genus 0 (can be counted like trees).
• Stat. Mech.: $\exists$ phase transition ($N \to \infty; \lambda \to \lambda_c$) $\rightsquigarrow$ a continuum limit (infinitely refined Riemann surfaces) as a 2D theory of gravity (Liouville + CF matter).

. . . to Tensors

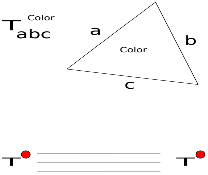
• Tensor Models (TM) of rank D: Tool for randomizing geometry in dimension D
Basic building blocks $(D-1)$-simplexes & Interaction forms a D-simplex;
For e.g. in 3D:

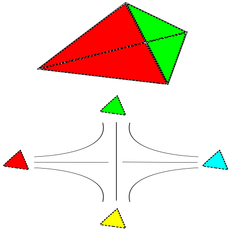


• Some results [Ambjorn et al. '91, Sasakura '91, Gross '92, Boulatov-Ooguri '92]
- Ambjorn et al., 91', numeric phase transition;
- Boulatov-Ooguri-type TM for 3D-4D simplicial gravity: (Topological/Lattice Field Theory like theory) 92'/93'; Related to Ponzano-Regge partition function 3D;
- Loop Quantum Gravity Connection ([Reisenberger-Rovelli, '00, Freidel, '05, Oriti, '06]) and "birth" of Group Field Theory.
• Lack of $1/N$ expansion $\Rightarrow$ all nice exact results of MM cannot be extended to TM.
• TM need improvement(s): C + DT's, Boulatov-Ooguri models and GFT's;

Colored Tensor Models

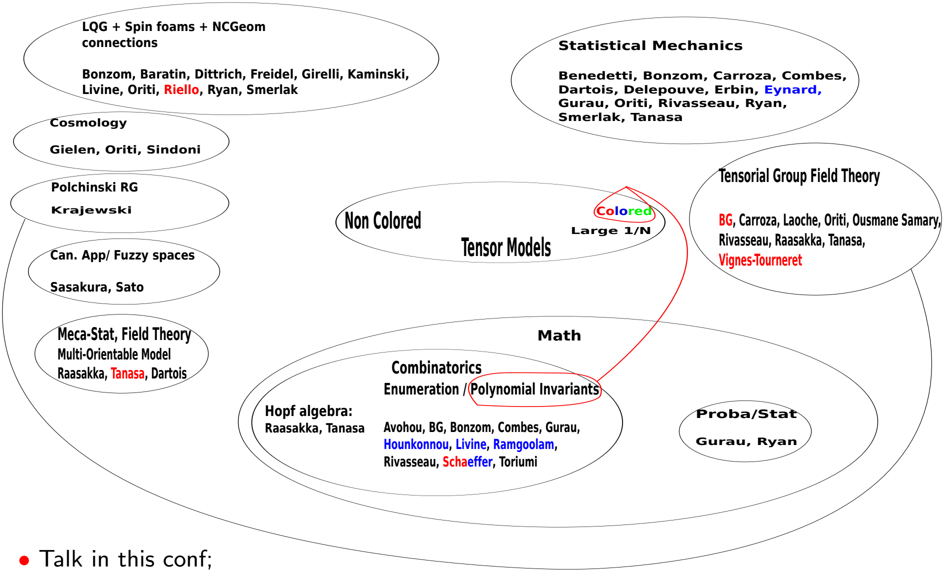
• '10 Gurau's $1/N$ expansion for colored TM [Gurau, AHP, '11]
3D:





- triangulate better objects (pseudo-manifolds) [Gurau, CMP '11]
- Leading graphs triangulate only spheres in any D [Gurau, AHP '11]
- have computable critical exponent [Bonzom, Gurau, Riello, Rivasseau, NPB, '11];
- with possible matter fields [Bonzom et al., PRD '12 ; Benedetti et al, NPB '12];
- could lead to extension of the Virasoro-DeWitt algebra [Gurau, NPB '11];
- underlie universal theory for general (ie unsymmetrized) tensors [Gurau, '11];
- Existence of a double-scaling limit [Dartois et al. '13, Gurau & Schaeffer, '13]
- Expand a natural $U(N)^{\otimes D}$ invariance [Gurau, '12, Bonzom et al. '12]
- Define renormalizable field theories;

TM People



- Talk in this conf;
- Recent Honorable Guest Contributors;
- Ordinary Contributors (running the program with usual/stressful deadlines);

Foundation questions:

• Tensor: Only a discretization tool or a "real" quanta of some "thing" ?

Big bang is identified with geometrogenesis, ie emergence of classical space-time through one or several phase transitions. Pre-space (analog of space-time before condensation) is treated as a physical transplanckian early phase of the universe (not just as a mathematical trick) [Oriti, '06, Konopka '08].

• What if $N = \infty$ from the beginning ? Several MM will be already divergent, worse are the behavior of TM.

• Divergences $\Rightarrow$ Need of Renormalization. Basic axioms for a QFT with Tensors ? [Rivasseau: The Tensor Track '11, '12, '13] Renormalization group is a guiding/selecting thread in models' space.

• Does it work for matrices ? The Grosse-Wulkenhaar model: Renormalizable and Asymptotically Safe model (without any extra-symmetry required) and induces important NEW developments in Field Theory. Closed equations might be extended for higher rank tensor [Ousmane Samary, '13] but the resolution is truly challenging.

• Motto: "Randomizing Geometry = Quantum Gravity"

• Goal: Achieve a universal scenario for an emergent spacetime through one or several phase transitions.

• Tools/Methods/Motivations: At the crossroad of Matrix models (large N limit), Group Field Theory, QFT/Constructive renormalization, Proba/Stats.

Outline

Tensor Models [Gurau, '10, '11]

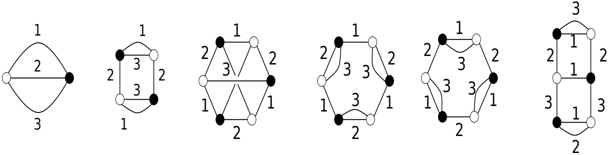
• Study of probability measures of random tensor spaces $+$
Geometric/Topological/Physical inputs.
• A covariant complex tensor $T_{p_1,\dots,p_d}$ with transformation rule

$$T^{U}_{p_1,\dots,p_d} = \sum_{q_k} U^{(1)}_{p_1 q_1} \dots U^{(d)}_{p_d q_d} T_{q_1,\dots,q_d} \,, \qquad U^{(a)} \in U(N_a) \tag{1}$$

• G/T/Physics input: T is viewed as a $(d-1)$-simplex.
• Tensor Invariance for defining the interactions

$$S^{\rm int}_{\mathbf{b}}(T,\bar T) = \mathrm{Tr}_{\mathbf{b}}(T\dots T) = S^{\rm int}_{\mathbf{b}}(T^U,\bar T^U) \tag{2}$$

b a colored graph encoding the contraction pattern; $S^{\rm int}_{\mathbf{b}}$ "is" a gluing of simplexes and represents a d-simplex. Ex:



(courtesy of Gurau)

Action/Partition function

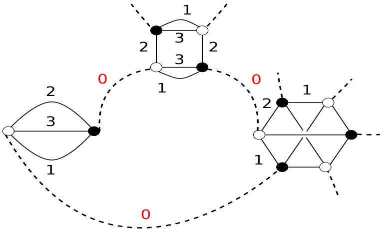
- Invariant action:

$$S[T,\bar{T},\{\lambda_{\mathbf{b}}\}_{\mathbf{b}}] = \sum_{\mathbf{b}} \lambda_{\mathbf{b}} N^{-\omega(\mathbf{b},d)} S^{\text{int}}_{\mathbf{b}}(T,\bar{T}), \qquad \omega(\mathbf{b},d)\geq 0 \tag{3}$$

- Partition function

$$Z[\{\lambda_{\mathbf{b}}\}_{\mathbf{b}}] = \int \prod_{p_i} [dT_{p_1,\dots,p_d}\, d\bar{T}_{p_1,\dots,p_d}] e^{-N^{D-1} S[T,\bar{T},\{\lambda_{\mathbf{b}}\}_{\mathbf{b}}]} \tag{4}$$

- Expanding the partition function: Feynman graphs

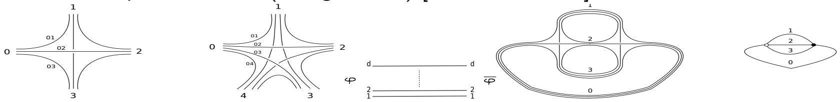


- But what is the analogue of matrix models' ribbon graphs ?

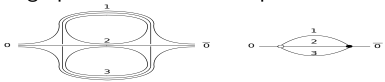
Graph Anatomy:
On the case of a Colored Tensor Graph

Colored Tensor Graphs I

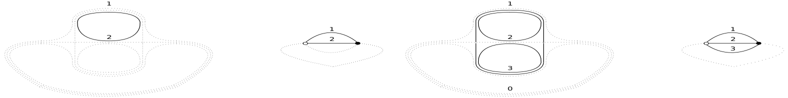
- Colored bi-partite models (triangulations) [Gurau, CMP '11]:



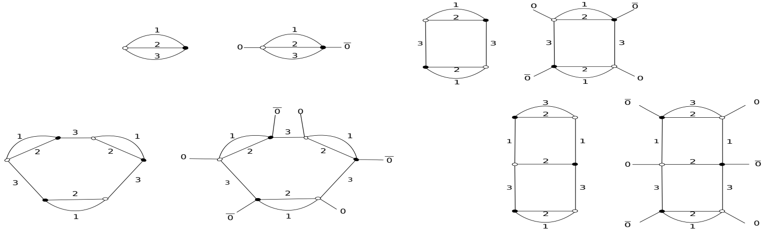
A graph can be closed or open



Other topological ingredients: Faces and bubbles

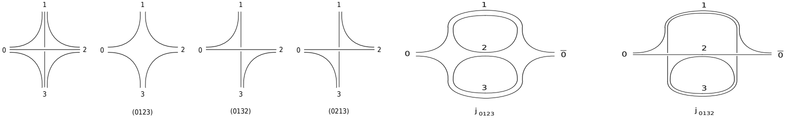


Back to tensor invariants: Labelled by colored bubbles + Interpretation as effective vertices.



Colored Tensor Graphs II

Jacket: Ribbon indexed by a cyclic permutation $(0, a_1, \dots, a_d)$ of $a_i \in [\![1, d]\!]$



Boundary graph $\partial\mathcal{G}$: Rank $d-1$ graph encoding the boundary of the simplicial complex associated with $\mathcal{G}$.

Large N limit and critical behavior

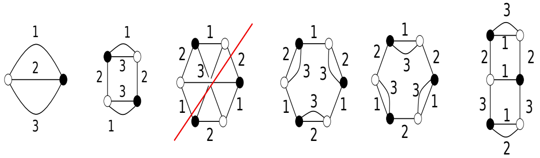
• Precise action: "Single" trace-like with weights allows to obtain, for a connected graph $\mathcal{G}$,

$$A(\mathcal{G}) = N^{d - \frac{2}{(d-1)!}\omega(\mathcal{G})}, \qquad \omega(\mathcal{G}) = \sum_{J \subset \mathcal{G}} g_J \tag{5}$$

• $\omega(\mathcal{G})$ called the degree of $\mathcal{G}$. [Gurau '10, Bonzom, Gurau, Riello, Rivasseau, '12]

Model Type	Matrix	Rank d Colored Tensor
Expansion in	$g(\mathcal{G})$ genus	$\omega(\mathcal{G})$ degree
Amplitudes	$A(\mathcal{G}) = N^{2-2g(\mathcal{G})}$	$A(\mathcal{G}) = N^{d - \frac{2}{(d-1)!}\omega(\mathcal{G})}$
Leading graphs	Planar $\equiv S^2$	Melon $\equiv S^d$
$F_{\rm Lead} \sim (\lambda_c - \lambda)^{2-\gamma}$	$\gamma_{\rm planar} = -\frac{1}{2}$	$\gamma_{\rm melonic} = \frac{1}{2}$

• "Melons" ?



• Phase transition $\rightsquigarrow$ Branched Polymer phase [Gurau & Ryan, '13]: The new phase of Colored TM defines a Continuous Random Tree in the sense of Aldous (same (Hausdorf, Spectral dimension) $= (2,4/3)$).

More results

• The Ising model in dimension D [Bonzom et. al, '12] on random tensor graphs (lattices): No phase transition at finite coupling (agrees with numerics); To force a phase transition modify the models;

• Universality [Gurau, '12]: Measures on random tensors converge (in distribution, in a precise sense) to the Gaussian measure at the large N limit.

• Multi-critical behavior on random (spherical) lattices and (hard) dimers [Bonzom, '12; Bonzom & Erbin, '12]: Coupling tensors interpreted as dimer activities; Phase transition, $\gamma_m = 1 - \frac{1}{m}$, $m \geq 2$, $\rightsquigarrow$ hard dimers on branched polymers.

• Double scaling in a T^4 model [Dartois et al. '13; Gurau & Schaeffer, '13]:

New double scaling limit for TM $$N \to \infty\,, \qquad \lambda \to \lambda_c\,, \qquad N^{\alpha}(\lambda - \lambda_c) = \text{const.} \quad \rightsquigarrow \text{ Polymers} \tag{6}$$

Using constructive techniques (loop vertex expansion), the series is Borel summable if $d < 6$.

• Beyond perturbations/Constructive techniques [Gurau, '13; Delpouve et al., '14]: The meaning of the $1/N$-expansion, reaching the critical point by analytic continuation $\lambda \to \lambda_c < 0$; Borel summability of T^4 quartic models.

Outline

Defining Tensor-like QFTs

"... one can define TGFT's with tensorial interactions and a soft breaking of the tensorial invariance of their propagator. ... + desirable features + ... + ..." (V. Rivasseau, The Tensor Track III, 1311.1461)

• Simple TM: A complex tensor $T_{p_1,\dots,p_d}$ with "Tensor Invariance" for defining the interactions $S^{\rm int}_{\mathbf{b}}(T,\bar T) = \mathrm{Tr}_{\mathbf{b}}(T \dots T) = S^{\rm int}_{\mathbf{b}}(T^U,\bar T^U)$ and Kinetic term

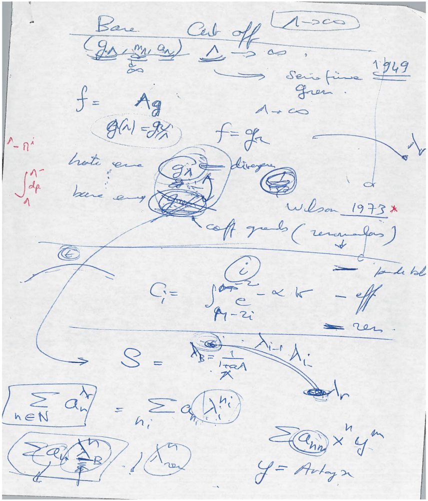
$$S^{\rm kin}(T,\bar T) = \sum_{p_s} \bar T_{p_1,\dots,p_d} \Big(\sum_s (p_s)^{2a} + \mu\Big) T_{p_1,\dots,p_d} \tag{7}$$

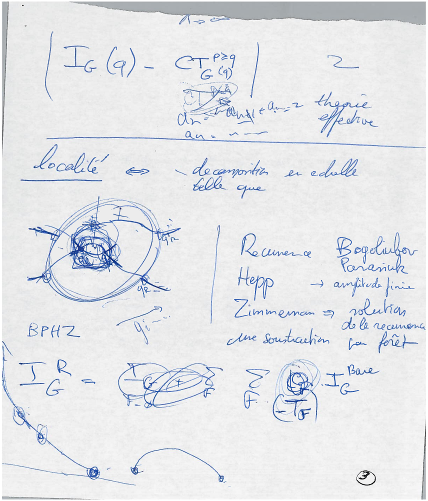
$0 < a \leq 1$; $(p_s^2)^a \equiv (\Delta)^a$ and $a \leq 1$ might be useful to recover O.S. positivity axiom (Rivasseau, Tensor Track III, '13).

• Gauge invariant models: Imposing constraints on T (Carrozza, Oriti, Ousmane Samary, Rivasseau, Vignes-Tourneret).

• Summing over arbitrary high momenta may imply divergent amplitudes.

Renormalization *à la* V. Rivasseau





Renormalization of TGFTs

in other words: QFT Renormalization is the "intricate" combination of a Multi-Scale Analysis, a Power-Counting Theorem and a Locality Principle.

- Consider $T : U(1)^d \to \mathbb{C}$ and its Fourier components $T_{p_1,\dots,p_d}$

$$Z = \int d\nu_C(T,\bar{T})\, e^{-S^{\rm int}(T,\bar{T})}\,, \quad S^{\rm int}(T,\bar{T}) = \sum_{\mathbf{b}} \lambda_{\mathbf{b}} {\rm Tr}_{\mathbf{b}}(T\dots T) \sim \sum_n \lambda_n \phi^{2n} \tag{8}$$

1. Multi-Scale Analysis: Slice decomposition: $C = \sum_{i=0}^{\infty} C_i$

$$\begin{aligned} C_i &= \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha\, e^{-\alpha(\sum_{s=1}^{d} p_s^{2a}+\mu)}\,, \ \ \forall i \geq 0\,, M>1 \\ &\leq KM^{-2i} e^{-\delta M^{-i}(\sum_{s=1}^{d} p_s^{a}+\mu^2)} \quad K>0, \delta>0 \end{aligned} \tag{9}$$

   High i probes high p; Cut-off $C^{\Lambda} = \sum_{i=0}^{\Lambda} C_i$;

2. Power-Counting Theorem: $|A_{\mathcal{G},\boldsymbol{\mu}}| \leq K^n M^{\omega_d(\mathcal{G})}$ where the divergence degree of a graph $\mathcal{G}$ is given by

$$\omega_d(\mathcal{G}) = -A_d(\omega(\mathcal{G}_{\rm color}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) - B_d\Big[N_{\rm ext} - C_d\Big] - \frac{1}{2}D_d \cdot V\,. \tag{10}$$

3. Locality Principle: Melons with melonic boundary are dominant $\omega(\mathcal{G}_{\rm color}) = 0; \quad \omega(\partial\mathcal{G}) = 0$

Results: Several Renormalizable Models

TGFT (type)	G_D	$\Phi^{k_{\max}}$	d	a	Renormalizability	UV behavior
	$U(1)$	Φ^6	4	1	Just-	AF
	$U(1)$	Φ^3	3	$\frac{1}{2}$	Just-	AF
	$U(1)$	Φ^6	3	$\frac{2}{3}$	Just-	AF
	$U(1)$	Φ^4	4	$\frac{3}{4}$	Just-	AF
	$U(1)$	Φ^4	5	1	Just-	AF
	$U(1)^2$	Φ^4	4	1	Just-	AF
	$U(1)$	Φ^{2k}	3	1	Super-	-
gi-	$U(1)$	Φ^4	6	1	Just-	AF
gi-	$U(1)$	Φ^6	5	1	Just-	AF
gi-	$SU(2)^3$	Φ^6	3	1	Just-	AF
gi-	$U(1)$	Φ^{2k}	4	1	Super-	-
gi-	$U(1)$	Φ^4	5	1	Super-	-
Matrix	$U(1)$	Φ^{2k}	2	$\frac{1}{2}(1-\frac{1}{k})$	Just-	($k=2$, AS$^{(\infty)}$); ($k=3$, LG)
Matrix	$U(1)^2$	Φ^{2k}	2	$1-\frac{1}{k}$	Just-	($k=2$, AS$^{(1)}$); ($k=3$, LG)
Matrix	$U(1)^3$ or $SU(2)$	Φ^6	2	1	Just-	LG
Matrix	$U(1)^3$ or $SU(2)$	Φ^4	2	$\frac{3}{4}$	Just-	AS$^{(1)}$
Matrix	$U(1)^4$	Φ^4	2	1	Just-	AS$^{(1)}$
Matrix	$U(1)$	Φ^{2k}	2	$\frac{1}{2}$	Super-	-
Matrix	$U(1)^2$	Φ^{2k}	2	1	Super-	-

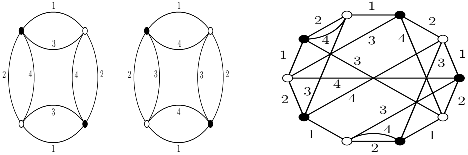
Table: Updated list of tensorial renormalizable models and their features (AF $\equiv$ asymptotically free; LG $\equiv$ existence of a Landau ghost; AS$^{(\ell)}$ asymptotically safe at ℓ-loops).

Outline

Meanders [Bonzom & Combes, '13]

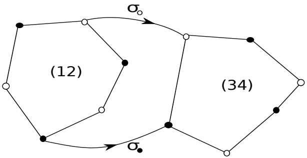
• Question: Computation of polynomials $P_{\mathbf{b}}(T, \bar{T})$ expectation values for the Gaussian measure, in the large N but including as much corrections;

• $\exists$ nice answer: In $d = 4$, if $\mathbf{b}$ has two unique faces labelled by a couple of colors, for instance, (12) and (34).



courtesy of Bonzom

• $\mathbf{b}_{\sigma_\circ, \sigma_\bullet}$ labelled by 2 permutations such that



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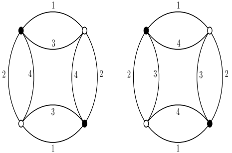

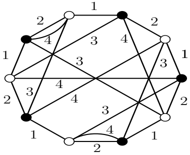

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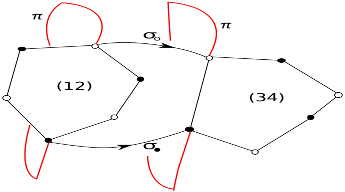

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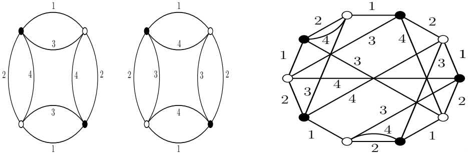

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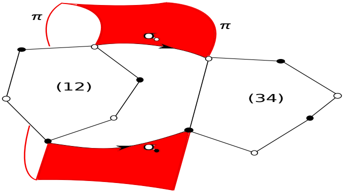
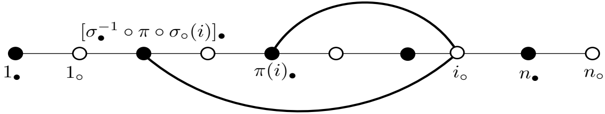


Meanders

• Using a nice labelling of vertices of both cycles $\rightsquigarrow$ Meanders



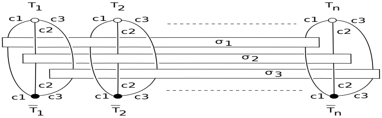
• $\mathbf{b}_{\sigma_\circ,\sigma_\bullet}$ such that

$$\langle P_{\sigma_\circ,\sigma_\bullet}\rangle = \frac{1}{Z}\int e^{-N^2 T\cdot\bar{T}} P_{\sigma_\circ,\sigma_\bullet}\ dTd\bar{T} \,, = \sum_{\text{paring } \pi} N^{\Omega(\sigma_\circ,\sigma_\bullet,\pi)}$$
$$= N^2 \sum_\pi N^{-2g_{012;\pi}-2g_{034;\pi}} =_{large\ N} N^2\,|\mathcal{M}_{\sigma_\circ,\sigma_\bullet}| \qquad (11)$$

• Subleading contributions can be evaluated in terms of irreducible meanders systems associated with polynomials labelled by stabilized-interval-free permutations.

Counting TM Observables/Permutation-TFT [JBG & Rangoolam, '13]

- Rank $d = 3$: Determination of possible graphs $\mathrm{Tr}_{\mathbf{b}}(T, \bar{T})$ amounts to count 3-uples



$$(S_n \times S_n \times S_n) \ni (\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1\gamma_2, \gamma_1\sigma_2\gamma_2, \gamma_1\sigma_3\gamma_2) \qquad (12)$$

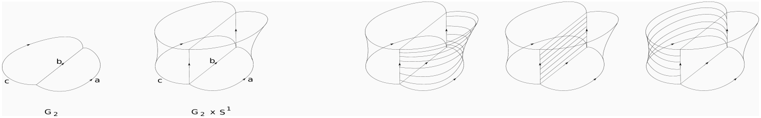
- Counting points in the double coset

$$\mathcal{S}_3(n) = \mathrm{Diag}(S_n)\backslash(S_n \times S_n \times S_n)/\mathrm{Diag}(S_n)\,. \qquad (13)$$

- Burnside's lemma: Permutation-TFT formulation !

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\gamma_{1,2}\in S_n} \sum_{\sigma_i \in S_n} \Big(\delta(\gamma_1\sigma_1\gamma_2\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2\sigma_3^{-1}) \Big) \qquad (14)$$

$a \to \sigma_1, \;\; b \to \sigma_2, \;\; c \to \sigma_3,$



- Solve one delta, say σ_1 (fixing the gauge), $\sigma_1^{-1}\sigma_i = \tau_{i-1}$, inserting $\tau_0 /\ \tau_0\tau_1\tau_2 = 1$ and get

$$Z_3(n) = \frac{1}{n!} \textstyle\sum_{\gamma,\gamma_0\in S_n} \sum_{\tau_{1,2}\in S_n} \Big(\delta(\gamma\tau_1\gamma^{-1}\tau_1^{-1})\delta(\gamma\tau_2\gamma^{-1}\tau_2^{-1})\delta(\tau_0\tau_1\tau_2) \Big) \qquad (15)$$

counts equivalent classes of branched covers of $S^2\backslash\{3\,\bullet\}$! (I dunno yet why but this is important ☺)

Outline

Open questions/Next Challenges

- Tensor models generalize matrix models and have, in a sense (after restrictions) better behavior: The double scaling limit is possible for T^4 models and the series is Borel summable whereas the BP phase persists.

  $\sim$ Open question: Determine another summable family of subleading graphs which may change this phase (triple/multiple scaling). Rinse, Repeat....

  $\sim$ Challenge: The new phase should be not model dependent (universality).

- Field Theory: There exist several models which can be renormalizable and AF.

  $\sim$ Open questions: • Determine conditions to truncate this space of model (Renormalizability and AF seems to be too weak). • Prove the OS positivity axiom and first steps towards a 4D Minkowskian spacetime; • There exist certainly different types of TGFT than the one present here and Renormalization analysis works as well: Start by a matrix-vector coupling with model covariance:

$$S[M,\phi] = \mathrm{Tr}\Big(M_{ab}(a+b+\mu)M_{ab} + \phi^a(\mu_\phi + a)\phi^a + M_{ab}(a+b+\mu)\phi^a\phi^b \Big) + V(M,\phi) \quad (16)$$

  $\sim$ Challenge: • Hunting nonperturbative and universal effects (FRG methods, recovering phase diagrams) • Make the rank d of tensors or the group dimension $\dim G$ dynamical.

- Combinatorics: • Elucidate the correspondence between equivalent classes of S^2 branched covers and the counting TM observables; • Generic formula for the TM correlation functions ? • How the enumeration of TI are useful for other conjectures in Math (e.g. Gromov's on sphere triangulations) ? • Polynomials Invariant generalizing Tutte/Bollobas Riordan: Universality theorem for a 3D polynomial invariant. New polynomial invariants issued from Tensor Invariants FT (wip with R. Toriumi).