Melons, branched polymers and beyond

James Ryan
Albert Einstein Institute
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Summary

Melons are branched polymers

Hausdorff and spectral dimensions coincide

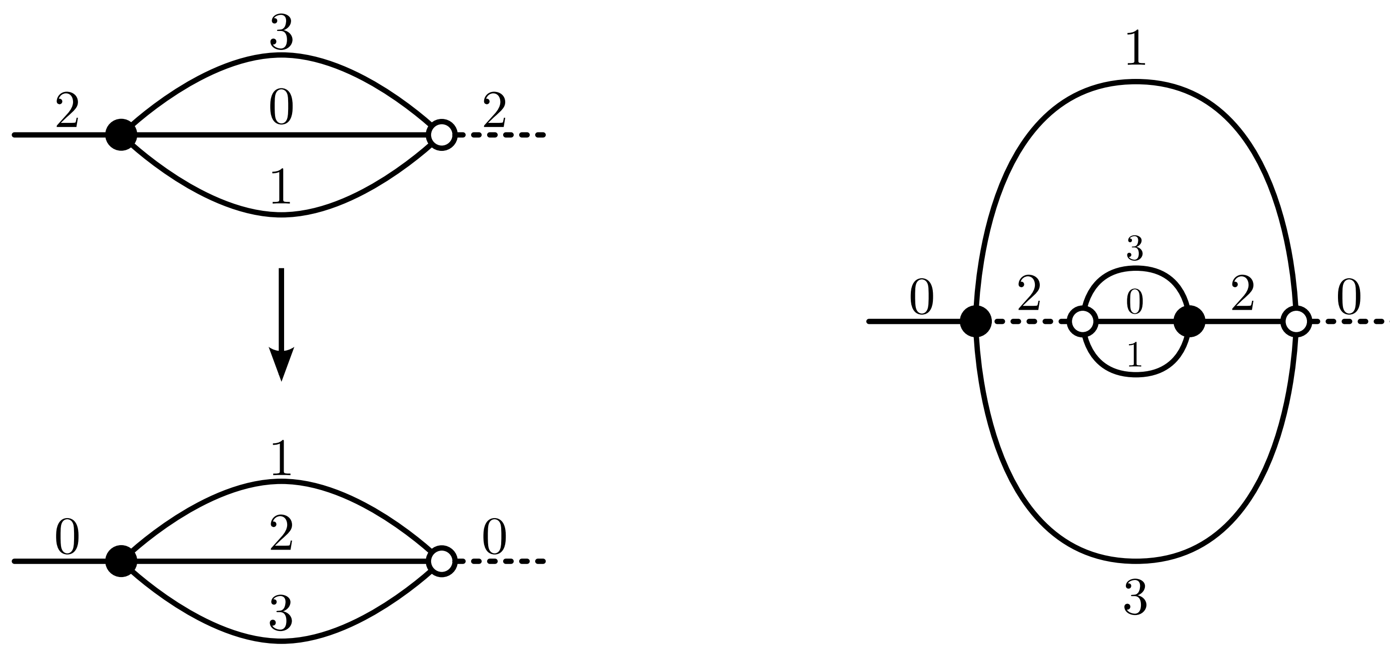
[Gurau, JR: 13]

Towards a double scaling limit

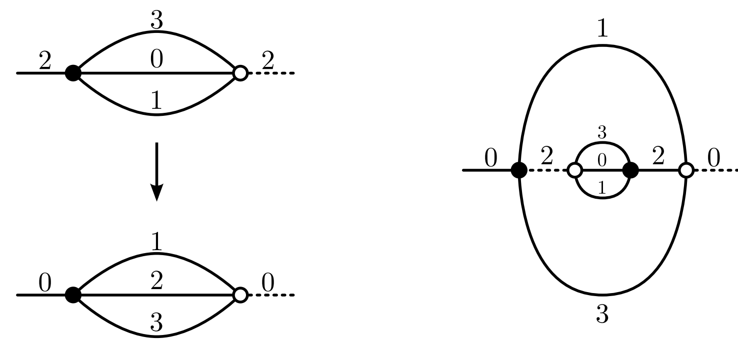
next-to-leading order behaviour

[Kaminski, Oriti, JR: to appear very soon!]

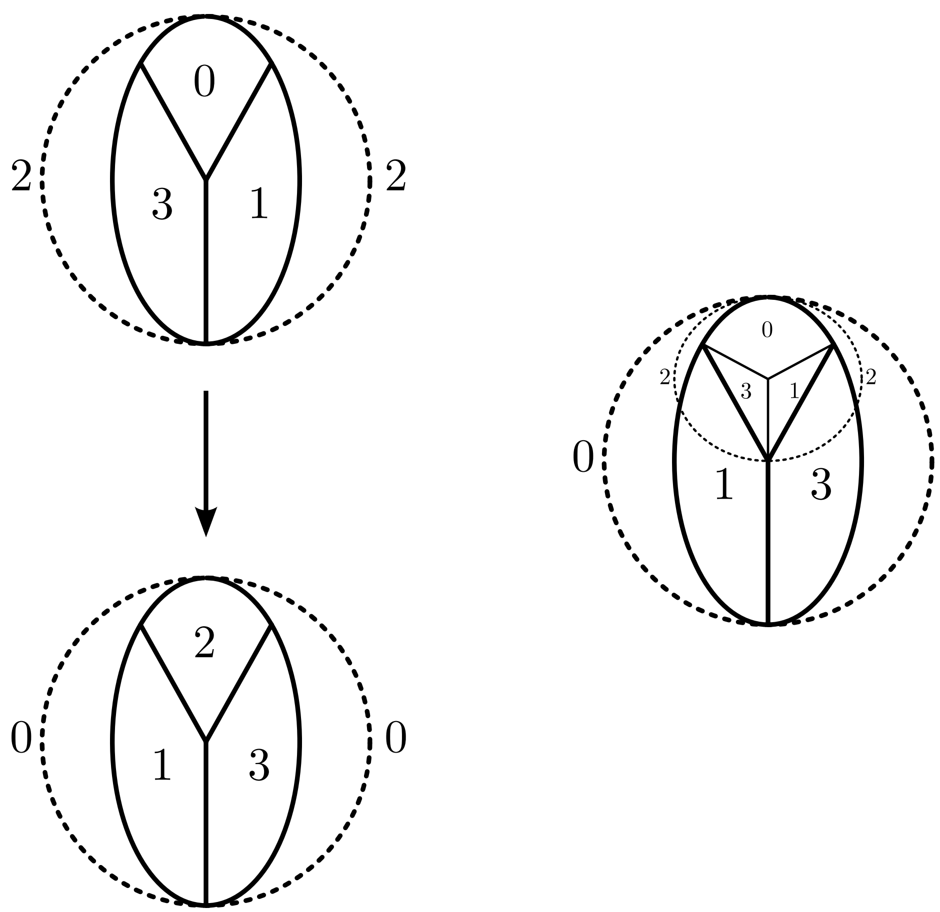
Rooted melonic graphs



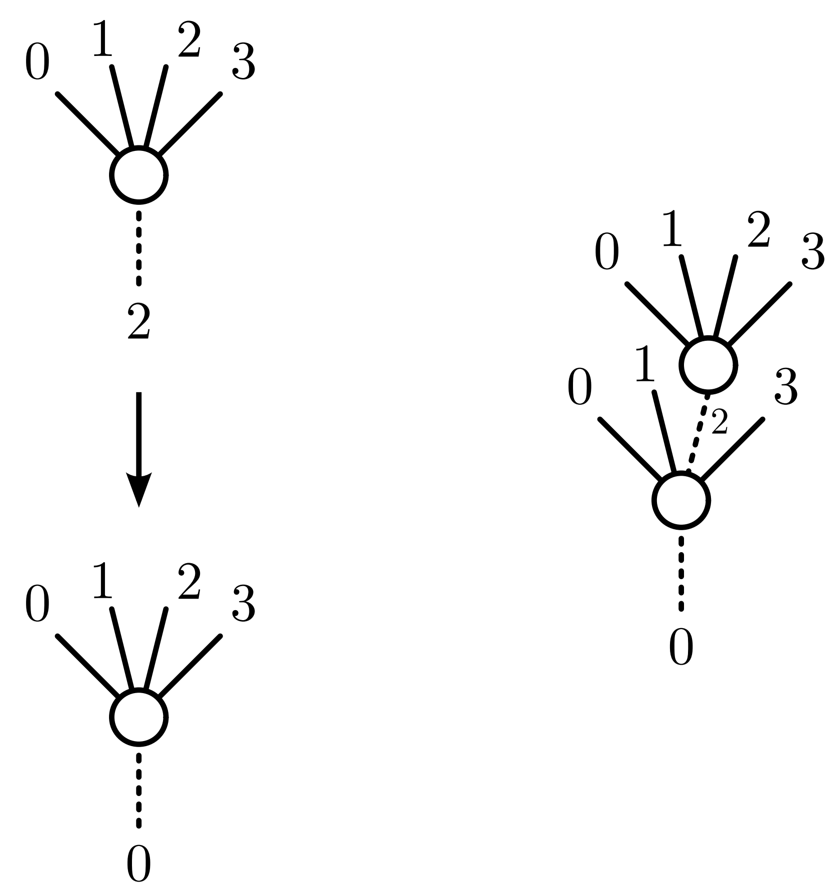
From melons to trees



melonic D-balls



(D+1)-ary trees



Why are melons interesting?

i.i.d model:

[Gurau: 10]
[Bonzom, Gurau, Rivasseau: 12]

$$Z_{N,\{t_{\mathcal{B}}\}} = \int [dT\, d\overline{T}]\; e^{-S_{N,\{t_{\mathcal{B}}\}}(T,\overline{T})}$$

$$S_{N,\{t_{\mathcal{B}}\}} = \mathrm{tr}_{\mathcal{B}_1}(T,\overline{T}) + \sum_{\mathcal{B}} \frac{t_{\mathcal{B}}}{N^{\frac{2}{(D-2)!}\omega(\mathcal{B})}} \mathrm{tr}_{\mathcal{B}}(T,\overline{T})$$

$$Z_{N,\{t_{\mathcal{B}}\}} = N^D \sum_{\mathcal{G}} \frac{(-1)^{|\rho|}}{\mathrm{SYM}(\mathcal{G})} \Big(\prod_{\rho} t_{\mathcal{B}_{(\rho)}}\Big) N^{-\frac{2}{(D-1)!}\omega(\mathcal{G})}$$

$$\omega(\mathcal{G}) \geq 0$$

$$\omega(\mathcal{G}) = 0 \quad \longrightarrow \quad \text{melonic sector}$$

[Gurau: 10]
[Gurau, Rivasseau: 11]

Critical Exponents

vanilla: $\gamma = 1/2$ [Bonzom, Gurau, Riello, Rivasseau: 11]

Ising

[Bonzom, Gurau, Rivasseau: 12]

hard dimer

[Bonzom: 12]
[Bonzom, Erbin: 12]

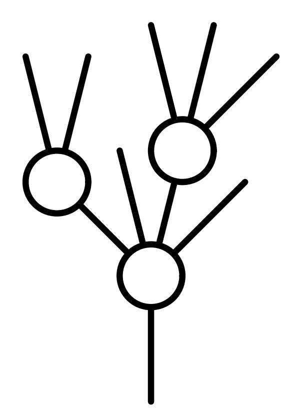
dually weighted

[Benedetti, Gurau: 12]

reminiscent of branched polymer phase

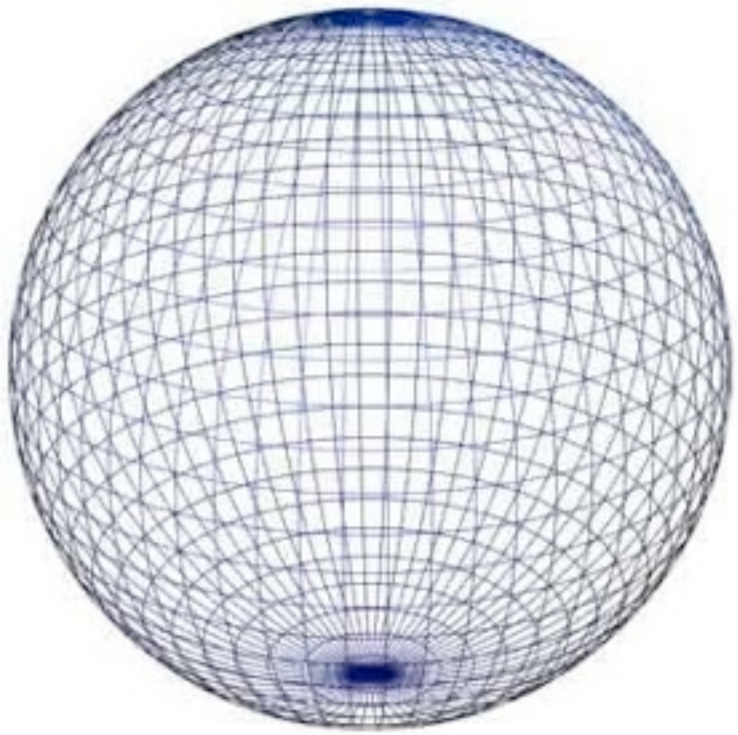
[Ambjorn, Durhuss, Fröhlich: 85]
[Ambjorn, Durhuus, Jonsson: 90]
[Bialas, Zurda: 96]

rooted branched polymer



The Hausdorff dimension

[Albenque, Marckert: 08]



$$V \sim r^{d_H}$$

Branched polymers: $d_H = 2$

1st Major component of our proof:

[Aldous: 91]
[Marckert, Mokkadem: 03]
[Le Gall: 05]

Galton-Watson trees

$\mu = 1$ (a.k.a. critical)

σ

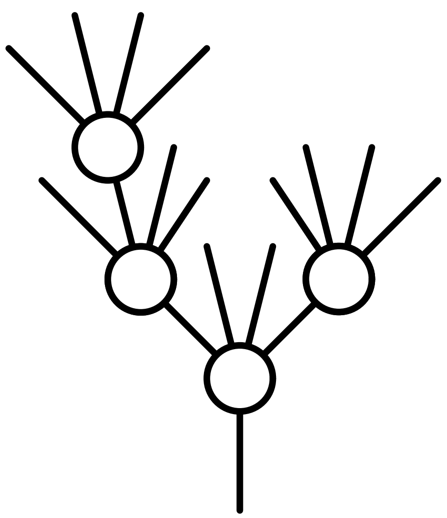
Continuum Random Trees (CRTs)

$$\left(T_n, \frac{d_{T_n}}{\sqrt{n/\sigma}}\right) \xrightarrow[n\to\infty]{} (\mathcal{T}_{2e}, d_{2e})$$

converge under distribution
w.r.t.
Gromov-Hausdorff topology

(D+1)-ary trees are GW trees

(D+1)-ary trees: T_n



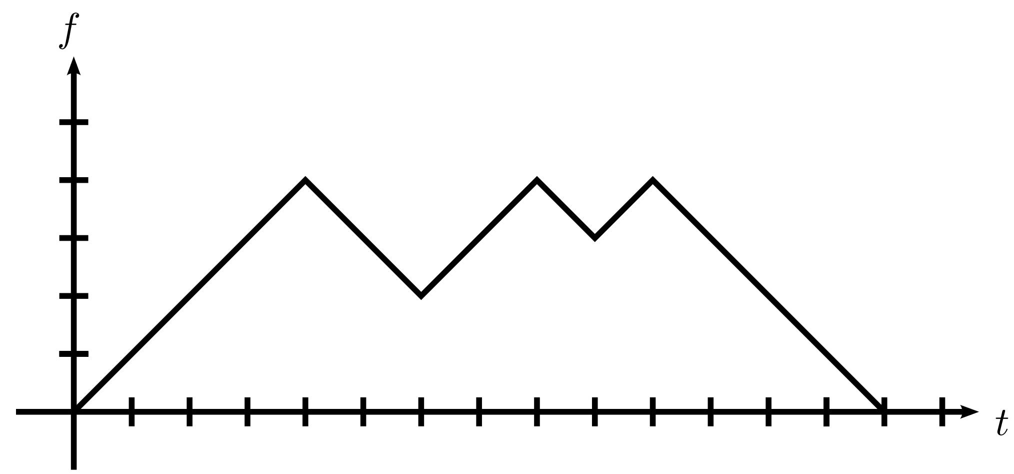
offspring distribution: ξ

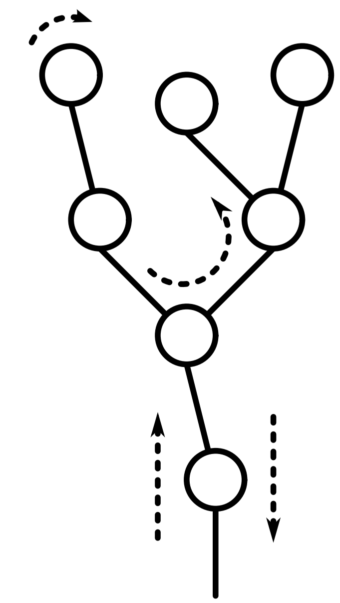
$$\xi_0 = \frac{D}{D+1} \qquad \xi_{D+1} = \frac{1}{D+1}$$

induces uniform distribution trees

Continuum Random Trees

Real tree coded by f:





CRT: $\mathcal{T}_e$

coded by the Brownian excursion e

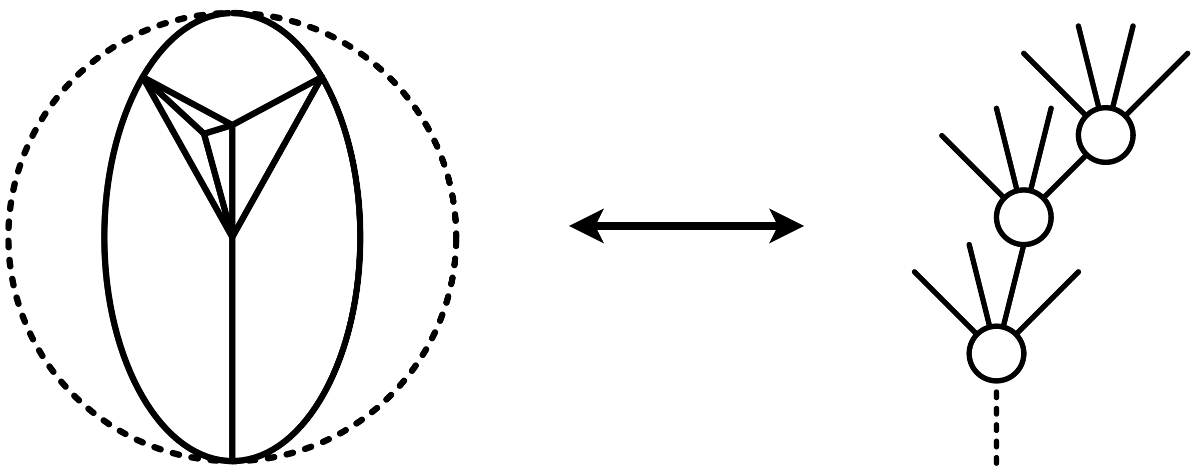
$$d\mu_e = \frac{1}{Z}\big[dq(t)\big]\Bigg|_{\substack{q(0)=q(1)=0 \\ q(t)>0}} \; e^{-\frac{1}{2}\int_0^1 [\dot{q}(t)]^2 dt}$$

D+1-ary trees converges to the CRT

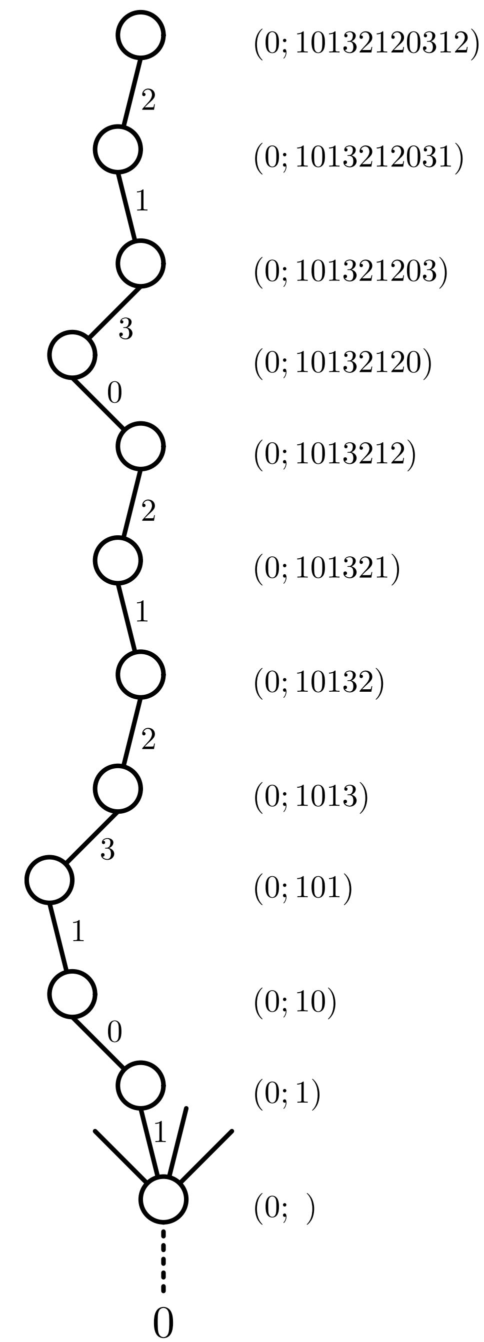
$$\left(T_n, \frac{d_{T_n}}{\sqrt{\frac{(D+1)n}{D}}}\right) \xrightarrow[n\to\infty]{} (\mathcal{T}_{2e}, d_{2e})$$

Depth:

simple example:



words on trees



tree distance = length of word

depth in D-ball = number of subwords

Λ

2nd major component of our proof:

average depth/tree-distance

$$\frac{1}{n}\Lambda(w) \xrightarrow[n\to\infty]{} \Lambda_\Delta$$

$$\Lambda_\Delta^{-1} = (D+1)\sum_{0\le r\le D}(-1)^{D-r}\binom{D}{r}\frac{r}{(D+1-r)^2}$$

3rd major component of our proof:

Convergence of metrics [Marckert: 06]

$$\left| \frac{d_{m_n}}{\Lambda_\Delta \sqrt{\frac{(D+1)n}{D}}} - \frac{d_{T_n}}{\sqrt{\frac{(D+1)n}{D}}} \right| \xrightarrow[n\to\infty]{(p)} 0$$

Result [Albenque, Marckert: 08]

$$\left(m_n, \frac{d_{m_n}}{\Lambda_\Delta \sqrt{\frac{(D+1)n}{D}}} \right) \longrightarrow_{n\to\infty} (\mathcal{T}_{2e}, d_{2e})$$

$$d_H = 2$$

The spectral dimension

[Jonsson, Wheater: 97]

Return probability:

$$P(t) \sim t^{-d_S/2} \qquad\qquad d_S = -2\frac{d\log P(t)}{d\log t}$$

Branched polymers: $d_S = \frac{4}{3}$

Rough argument:

$$\lim_{t\to\infty} P_V(t) = \frac{1}{V} \qquad t >> V^{\Delta} \qquad \lim_{V\to\infty} P_V(t) \sim \frac{1}{t^{d_S/2}}$$

$$P_V(t) = \frac{1}{V} + \frac{a}{t^{d_S/2}} \exp\left(-\frac{t}{N^{\Delta}}\right)$$

The spectral dimension

Generating functions:

$$P_V(y) = \sum y^t P_V(y) \sim \frac{1}{V}\frac{1}{1-y} + \frac{a}{(1-y+V^{-\Delta})^{1-d_S/2}}$$

$$\widetilde{P}_V(y) \sim \frac{a}{(1-y+V^{-\Delta})^{1-d_S/2}}$$

$$\widetilde{Q}(z,y) = \sum_{\mathcal{G}} z^V \widetilde{P}_V(y)$$

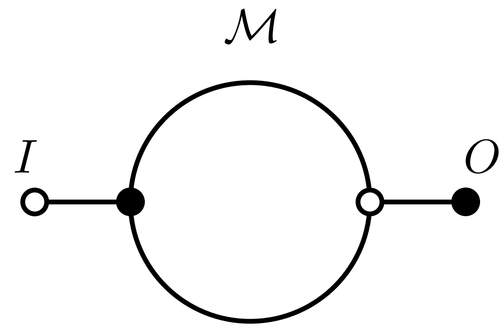
$$\widetilde{Q}(z,y) = \left(1-\frac{z}{z_0}\right)^{\beta} \; \widetilde{\Phi}\left(\frac{1-y}{\left(1-\frac{z}{z_0}\right)^{\Delta}}\right)$$

$$\beta = 1 - \gamma_{\text{LO}} + \Delta\left(\frac{d_S}{2}-1\right)$$

Random walks on a melon

return/transit

$$P_{\mathcal{M}}(t) = \begin{pmatrix} P^{II}_{\mathcal{M}}(t) & P^{IO}_{\mathcal{M}}(t) \\ P^{OI}_{\mathcal{M}}(t) & P^{OO}_{\mathcal{M}}(t) \end{pmatrix}$$



1st-return/1st-transit $P^1_{\mathcal{M}}(t)$

$$P^{XY}_{\mathcal{M}}(t) = \delta^{XY}\delta_{t,0} + P^{1;XY}_{\mathcal{M}}(t)$$

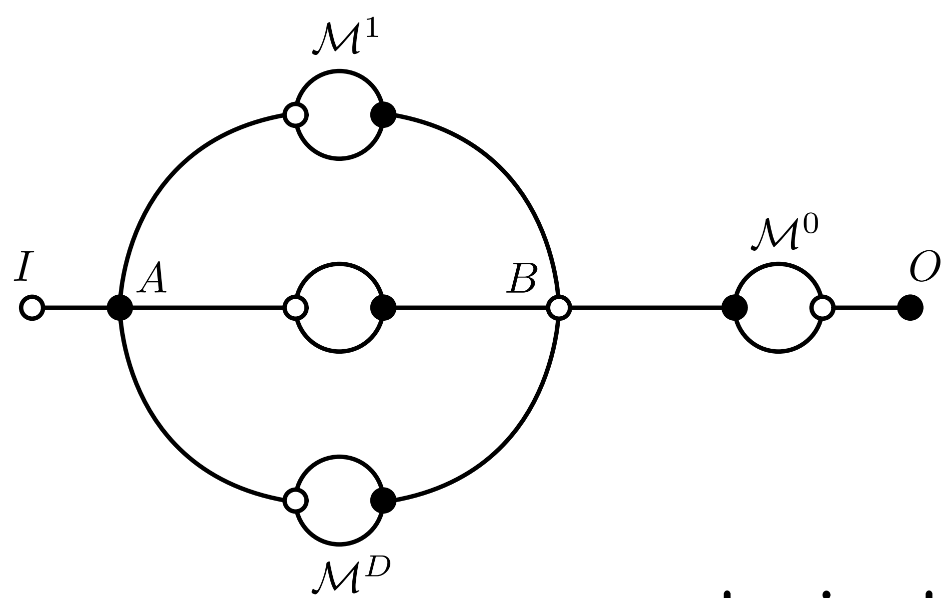
trajectory example: $IOOI$

$$+\sum_{q=1}^{\infty}\sum_{w_q}\sum_{t_0+\cdots+t_q=t} P^{1;Xw_q(1)}_{\mathcal{M}}(t_0)P^{1;w_q(1)w_q(2)}_{\mathcal{M}}(t_1)\dots P^{1;w_q(q-1)w_q(q)}_{\mathcal{M}}(t_{q-1})P^{1;w_q(q)Y}_{\mathcal{M}}(t_q)\ .$$

Relation $$P_{\mathcal{M}}(y) = 1 + P^1_{\mathcal{M}}(y) + [P^1_{\mathcal{M}}(y)]^2 + \cdots = \frac{1}{1-P^1_{\mathcal{M}}(y)}$$

$$P^{XY}_{\mathcal{M}}(y) = \sum_t y^t P^{XY}_{\mathcal{M}}(t)$$

Decomposition and iteration

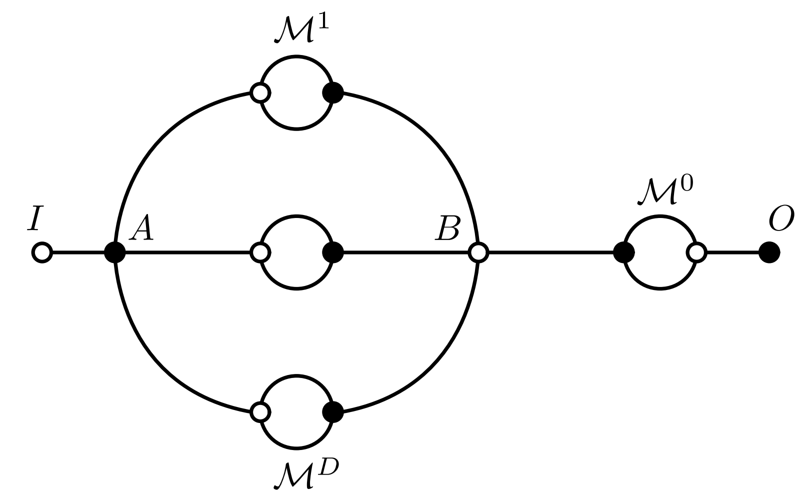


trajectory example: $ABBA$

$$P^{1,II}_{\mathcal{M}}(t) = \frac{1}{D+1}\delta_{t,2} + \frac{1}{D+1}P^{1;AA}(t-2)$$

$$+\frac{1}{D+1}\sum_{q=1}^{\infty}\sum_{w_q}\sum_{t_0+\cdots+t_q=t-2}\Big(P^{1;Aw_q(1)}(t_0)P^{1;w_q(1)w_q(2)}(t_1)\dots P^{1;w_q(q-1)w_q(q)}(t_{q-1})P^{1;w_q(q)A}(t_q)\Big)\,.$$

Recursive equation



$$P^1_{\mathcal{M}} = E^{22} P^1_{\mathcal{M}^0} E^{22} + \left(E^{12} y + E^{22} P^1_{\mathcal{M}^0} E^{11}\right)$$

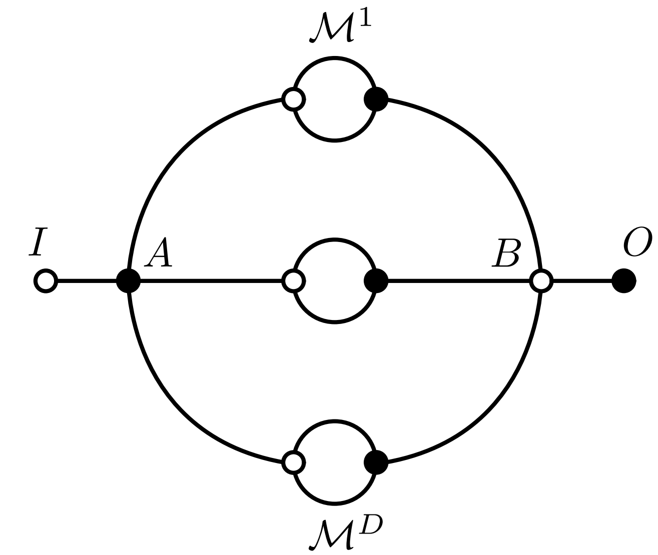
$$\times \frac{1}{D + 1 - \sum_{i=1}^{D} P^1_{\mathcal{M}^i} - E^{11} P^1_{\mathcal{M}^0} E^{11}} \left(y E^{21} + E^{11} P^1_{\mathcal{M}^0} E^{22}\right)$$

$$P^1_{\mathcal{M}_{(0)}} = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \equiv y\,\sigma \ .$$

$$E^{ab}_{\alpha\beta} = \delta^a_\alpha \, \delta^b_\beta \qquad \text{where} \qquad a, b, \alpha, \beta \in \{1, 2\} \ .$$

Simple melon case

$$P^1_{\mathcal{M}} = y^2 \frac{1}{D+1-\sum_{i=1}^{D} P^1_{\mathcal{M}^i}}$$



simultaneously diagonalize

$$\lambda_{\mathcal{M}}(y) = \frac{y^2}{D+1-\sum_i \lambda_{\mathcal{M}^i}(y)} \quad \text{and} \quad \lambda_{\mathcal{M}_{(0)}}(y) = \pm y$$

generating function

$$Q(z,y) = \sum_{\mathcal{M}} z^p \frac{1}{1-\lambda_{\mathcal{M}}(y)}$$

pole at y=1

Simple melon case

non-pole contribution

$$\widetilde{Q}(z,y) = -\frac{d}{dy}(1-y)Q(z,y)$$

$$\Big\downarrow$$

$$\frac{\partial\widetilde{Q}}{\partial z}(z,y) = \frac{1}{1-z/z_0}\;\widetilde{\Phi}\left(\frac{1-y}{(1-z/z_0)^{\frac{3}{2}}}\right)$$

extract spectral dimension

$$d_S = \frac{4}{3}$$

Towards a double scaling limit

Aim: double scaling limit for tensors

Requirements: control [David: 90]

$$E \sim \sum_h N^{\chi(h)} \left(1 - \frac{g}{g_c}\right)^{\frac{(2-\gamma_s)}{2}\chi(h)}$$

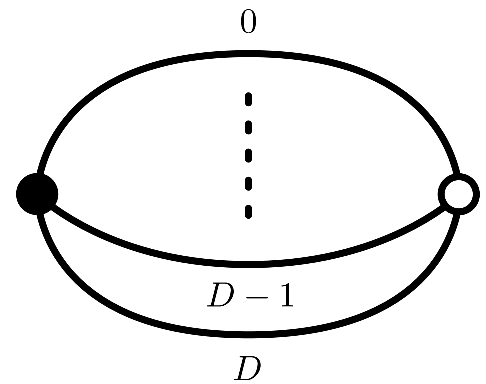
$$\gamma_s = -\frac{1}{2}$$

Why: access EDT at finite G

At the moment: next-to-leading order + friends

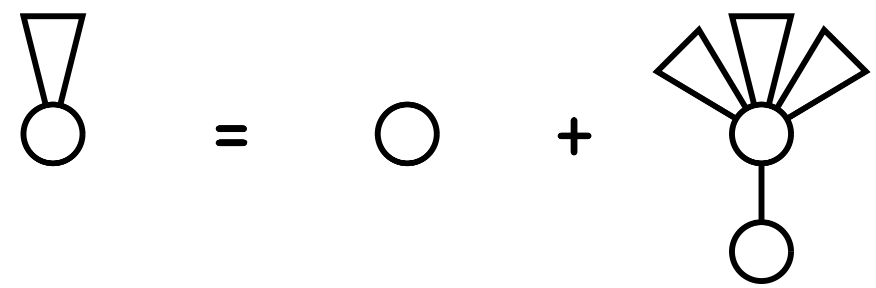
LO recap

Supermelon graph:



$$\omega(\mathcal{G}) = 0$$

Recursion relation for rooted melonic graphs:

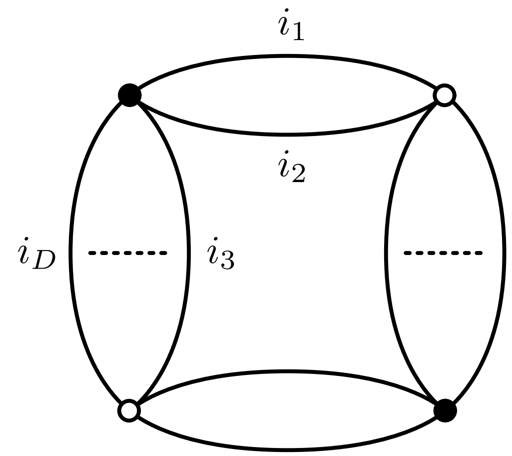


2-point function: $$G_{\mathrm{LO}} = 1 + g G_{\mathrm{LO}}^{D+1}$$

free energy: $$E_{\mathrm{LO}} \sim \left(1 - \frac{g}{g_c}\right)^{2-\gamma_{\mathrm{LO}}} \qquad \gamma_{\mathrm{LO}} = \frac{1}{2}$$

NLO

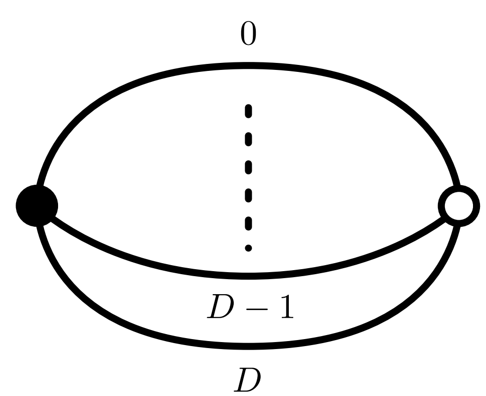
Core graphs:

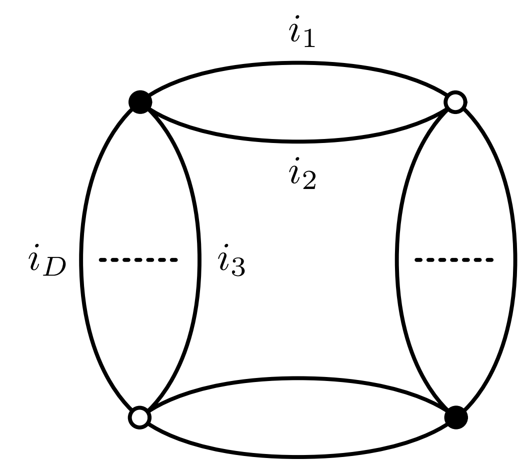


$$\omega(\mathcal{G}) = \frac{(D-1)!}{2}(D-2)$$

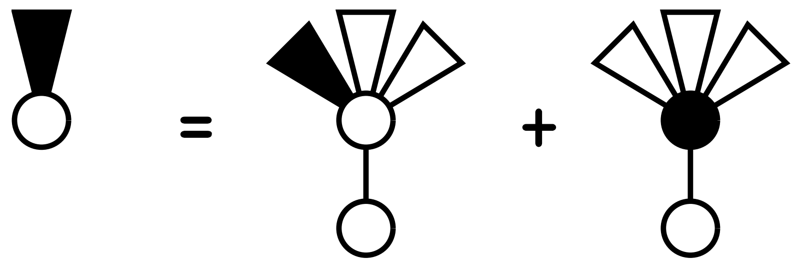
Enumerate graphs: cannot just add melons to core graphs

2-dipole move:





Critical behavior



$$G_{\text{NLO},g} = g\,(D+1)\left[G_{\text{LO},g}\right]^{D} G_{\text{NLO},g} + g\,\frac{D(D+1)}{2}\left[G_{\text{LO},g}\right]^{2D+2}$$

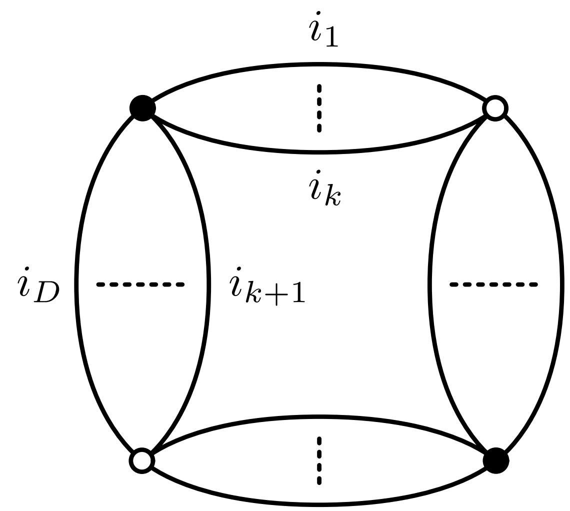
2-point function: $$G_{\text{NLO},g} = \frac{g\,\frac{D(D+1)}{2}\left[G_{\text{LO},g}\right]^{2D+2}}{1-g\,(D+1)\left[G_{\text{LO},g}\right]^{D}}$$

free energy: $$E_{\text{NLO},g} \sim \left(1-\frac{g}{g_c}\right)^{2-\gamma_{\text{NLO}}} \qquad \gamma_{\text{NLO}} = \frac{3}{2}$$

Further sectors

sum up so far: $$E \sim N^0 \left(1 - \frac{g}{g_c}\right)^{\frac{3}{2}} + N^{2-D}\left(1 - \frac{g}{g_c}\right)^{\frac{1}{2}} + \ldots\ldots$$

k-dipoles:



$$N^{(k-1)(k-D)}\left(1 - \frac{g}{g_c}\right)^{\frac{1}{2}}$$

washed away

Conjecture: only (multiple) 2-dipole sectors survive

double-scaling weakens as dimension increases

Outlook

Part 1:

Tensor models $\longleftrightarrow$ Brownian map

Part 2:

Complete the double scaling limit