Exact solution of the quartic matrix model

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Introduction

axiomatic settings for rigorous quantum field theories by

1. Wightman [1956]
2. Haag-Kastler [1964]
3. Osterwalder-Schrader [1974]

today: numerous examples in dimension 1,2,3;
not a single non-trivial example in 4 dimensions

this talk:

- We are concerned with a toy model for 4D QFT.
- We have advanced much further than expected.
  There are good chances that the model satisfies Osterwalder-Schrader.
- As by-product we find structures which could be relevant for quantum gravity.

A toy model for 4D non-linear QFT

- start with Euclidean QFT on 4D noncommutative geometry
- essentially a matrix model, with infinite number of Ward identities from action of $U(\infty)$ group
- These Ward identities, and the theory of singular integral equations, turn the Schwinger-Dyson equations into a fixed point problem for which we prove existence of a solution.

1. find numerical evidence for phase structure, phase transitions and critical phenomena
2. construct Schwinger functions on $\mathbb{R}^4$ satisfying (OS0) boundedness, (OS1) invariance and (OS3) symmetry
3. numerical evidence for (OS2) reflection positivity of the 2-point function in one of the phases

Field-theoretical matrix models

- A matrix is for us a compact (Hilbert-Schmidt) operator on Hilbert space $H = L^2(I, \mu)$.
- realise as integral kernel operators: $\phi = (\phi_{ab}) \in L^2(I \times I, \mu \times \mu)$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V \,\mathrm{tr}(E\phi^2 + P[\phi])$$

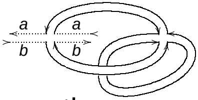
E – unbounded positive selfadjoint op. with compact resolvent,
$P[\phi]$ – polynomial in ϕ with scalar coefficients

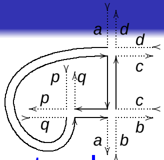
Euclidean quantum field theory

- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\phi] \; \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$
- For $P[\phi] \equiv 0$, $\mathcal{D}[\phi] e^{-V \mathrm{tr}(E\phi^2)} / \mathcal{Z}[0]$ is Gaußian measure (of covariance determined by E) on random s.a. matrices.
- Our aim is to construct $\mathcal{D}[\phi] e^{-V \mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)} / \mathcal{Z}[0]$.

Topological expansion

- Feynman graphs in matrix models are ribbon graphs.





- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p^1_1 \dots p^1_{N_1}| \dots |p^B_1 \dots p^B_{N_B}|} \prod_{\beta=1}^{B} J^{N_\beta}_{p^\beta_1 \dots p^\beta_{N_\beta}}$ according to the cycle structure.
- The $G_{|p^1_1 \dots p^1_{N_1}| \dots |p^B_1 \dots p^B_{N_B}|}$ become (smeared) Schwinger functions.
- QFT of matrix models determines the weights of Riemann surfaces with decorated boundary components compatible with (1) gluing and (2) symmetry.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J\right] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

...choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2007]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

We turn the Ward identity for E injective into formula for $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I}\frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} + \sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big) + V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J] + \frac{V}{E_p - E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

How to use the Ward identity

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E}\,, \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n}$$

Example: $G_{|ab|}$ (for $a \neq b$)

$$\begin{aligned} G_{|ab|} &= \frac{1}{V\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \\ &= \frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}} e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]} \frac{\partial}{\partial J_{ab}} e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0} \\ &= \frac{1}{E_a + E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\Big(\phi_{ab}\frac{\partial(-VS_{int})}{\partial\phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0} \end{aligned}$$

$\frac{\partial(-VS_{int})}{\partial\phi_{ab}}$ contains, for any $P[\phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

Ward identity and reality $\mathcal{Z} = \overline{\mathcal{Z}}$ let the usually infinite tower of Schwinger-Dyson equations collapse.

In a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{p\in I}$ finite, we have

1. A closed non-linear equation for $G^{(0)}_{|ab|}$

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p\in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G^{(0)}_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{|b_0 b_1 \dots b_{2l-1}|} G^{(0)}_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G^{(0)}_{|b_{2l} b_1 \dots b_{2l-1}|} G^{(0)}_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

(corresponds to restriction to genus $g = 0$)

Renormalisation theorem

Theorem (2013)

Given a real quartic matrix model $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ with E of compact resolvent.

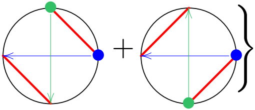
Assume that the selfconsistency equation for $G^{(0)}_{|ab|}$ has a finite solution after affine renormalisation $E \mapsto Z(E + C\mathbb{1})$ and $\lambda \mapsto Z^2\lambda$. Then

- All higher functions $G^{(0)}_{|b_0 \dots b_{N-1}|}$ with $N \geq 4$ are automatically finite without further need of a renormalisation of λ.
- All quartic matrix models (with renormalisable $G^{(0)}_{|ab|}$) have vanishing β-function (i.e. are almost scale-invariant).
- The perturbative observation $\beta = 0$ for Moyal [Disertori-Gurau-Magnen-Rivasseau, 2007] is generic!

(Similar statements hold for $B \geq 2$)

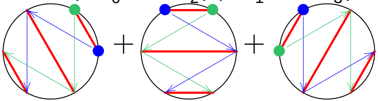
Graphical realisation for $N \geq 4$ (and $B = 1$)

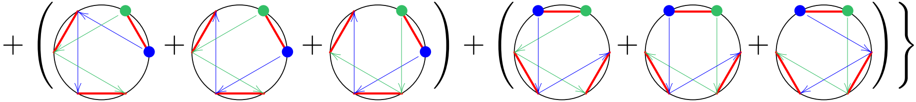
$$G_{|b_0 b_1 b_2 b_3|} = (-\lambda)\frac{G_{|b_0 b_1|}G_{|b_2 b_3|} - G_{|b_0 b_3|}G_{|b_2 b_1|}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_3})} = -\lambda \Big\{$$



$$\Big\}$$

$$G_{|b_0 \dots b_5|} = \lambda^2 \Big\{$$





$$\Big\}$$

b_i ——— b_j $= G_{|b_i b_j|}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

b_i ———→ b_j $= \frac{1}{E_{b_i} - E_{b_j}}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

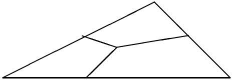
Open Problem: Which trees arise for a given chord diagram?

not unique: $\frac{1}{(E_a - E_b)(E_b - E_c)} + \frac{1}{(E_b - E_c)(E_c - E_a)} + \frac{1}{(E_c - E_a)(E_a - E_b)} = 0$

The Kontsevich model in 2D quantum gravity

$$\mathcal{Z}[E] = \frac{\int d\phi \; \exp\left(\mathrm{tr}(-\frac{1}{2}E\phi^2 + \frac{\mathrm{i}}{6}\phi^3)\right)}{\int d\phi \; \exp\left(\mathrm{tr}(-\frac{1}{2}E\phi^2)\right)}$$

- exact solution related to Korteweg-de Vries evolution equation
- resulting ribbon graphs dual to triangulations counted in other versions of 2D quantum gravity
- 2D quantum gravity might have equivalent descriptions as cubic and quartic matrix model (which we proved is also exactly solvable)



- possible advantages: positivity and boundedness from below

Coloured tensor models

- extend these methods to quantum gravity in $D \geq 3$
- Schwinger-Dyson equations and action of $U(\infty)$ group
- also exactly solvable? [Ousmane-Samary, 2014]

ϕ^4-QFT on a 4D noncommutative geometry

- has formulation as matrix model with eigenvalues $E_n = c_1 + c_2 n$, $n \in \mathbb{N}$, and multiplicity $\dim\ker(E - E_n\,\mathrm{id}) = n+1$
- suitable limits and rescaling $G_{|nm|} \mapsto G_{ab}$, with $a, b \in \mathbb{R}_+$, lead to Carleman-type singular integral equation for $G_{ab} - G_{a0}$

Theorem (2012/13) (for $\lambda < 0$)

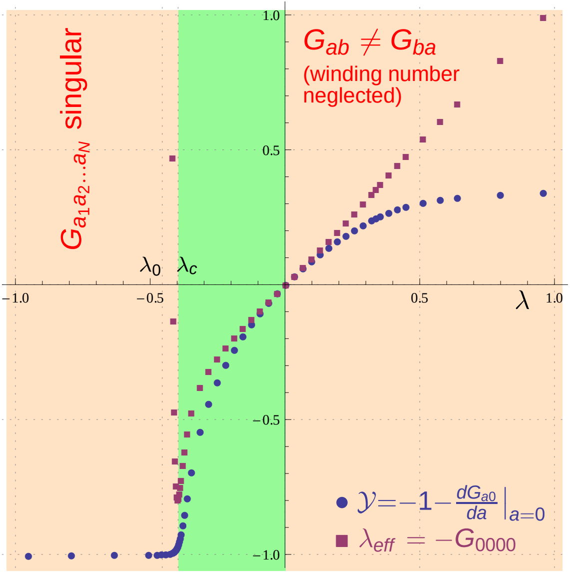
Let $\mathcal{H}_a(f) = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{f(p)\,dp}{p-a}$ be one-sided Hilbert transform. Then

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)] - \mathcal{H}_a[\tau_b(\bullet)])}$$

where $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$ and G_{b0} solution of

$$G_{b0} = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\lambda\pi p\mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Computer simulation: evidence for phase transitions



- G_{ab} for $\Lambda^2=10^7$ with 2000 sample points
- $\mathcal{Y}'$ discontinuous at $\lambda_c=-0.396$
- λ_{eff} singular at $\lambda_0=-0.455$ where $\mathcal{Y}=-1$
- Nothing particular at pole $\lambda_b=-\frac{1}{72}=0.014$ of Borel resummation
- A key property for Schwinger functions is realised in subinterval of $[\lambda_c, 0]$, not outside!

Osterwalder-Schrader reconstruction theorem (1975)

Assume for Schwinger functions $S(x_1, \dots, x_N)$:

- S0 growth rate: $\left| \int dx\, f(x_1, \dots, x_N) S(x_1, \dots, x_N) \right| \leq c_1 (N!)^{c_2} |f|_{Nc_3}$
- S1 Euclidean invariance: $S(x_1, \dots, x_N) = S(Rx_1 + a, \dots, Rx_N + a)$
- S2 reflection positivity: for each $(f_0, \dots, f_K)$ with $f_N \in \mathcal{S}(\mathbb{R}^{Nd})$,

$$\sum_{M,N=0}^{K} \int dxdy\; S(x_N, \dots, x_1, y_1, \dots y_M) \overline{f_N({}^r x_1, \dots, {}^r x_N)} f_M(y_1, \dots, y_M) \geq 0$$

where ${}^r(x^0, x^1, \dots x^{d-1}) := (-x^0, x^1, \dots x^{d-1})$

- S3 permutation symmetry: $S(x_1, \dots, x_N) = S(x_{\sigma(1)}, \dots, x_{\sigma(N)})$

Then the $S(\xi_1, \dots, \xi_{N-1})\big|_{\xi_i^0 > 0}$, with $\xi_i = x_i - x_{i+1}$, are inverse Laplace-Fourier transforms of FT $\hat{W}(q_1, \dots, q_{N-1})$ of Wightman distributions in a relativistic QFT.

If in addition the $S(x_1, \dots, x_N)$ satisfy

- S4 clustering

then the Wightman QFT has a unique vacuum state

From matrix model to Schwinger functions on $\mathbb{R}^4$

folding $G_{\dots}$ with eigenbasis of 4D harmonic oscillator:

Proposition (2013)

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4p_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle}\Bigg)$$

$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric (S3) and invariant under full Euclidean group (S1) (completely unexpected for NCQFT)
- growth conditions (S0) established
- clustering (S4) is violated: The $(N_1+\dots+N_B)$-point functions are insensitive to the distance of different boundaries.
- remains: reflection positivity (S2)

Osterwalder-Schrader reflection positivity

- Reflection positivity S2 gives spectrum condition which guarantees representation as Laplace transform in ξ^0, hence analyticity in $\mathrm{Re}(\xi^0) > 0$.

Theorem (2013)

$S(x_1, x_2)$ is reflection positive iff $a \mapsto G_{aa}$ is a Stieltjes function,

$$G_{aa} = \int_0^\infty \frac{d(\rho(t))}{a+t}$$

with ρ positive and non-decreasing.

Theorem [Widder, 1938]

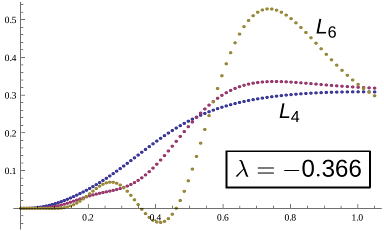
$f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff $\mathcal{C}^\infty$, positive and $L_{k,t}[f(\bullet)] \geq 0$, where

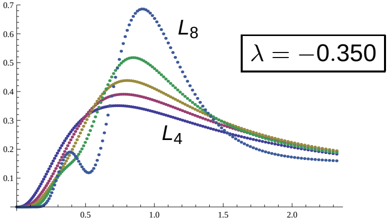
$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big) , \qquad c_1 = 1,\; c_{k>1} = k!(k-2)!$$

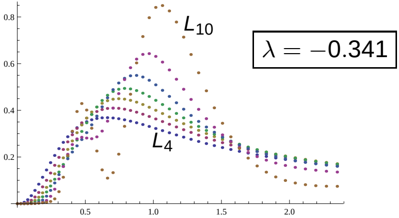
- In this case $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$ (weakly and a.e.).

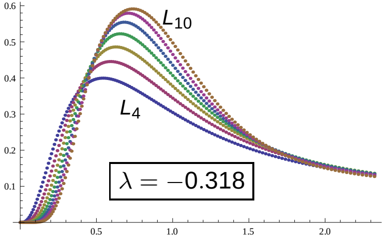
Widder's $L_{k,t}[G_{\bullet\bullet}]$

key step: integral formula for $\frac{\partial^{n+\ell} G_{ab}}{\partial a^n \partial^\ell b}$

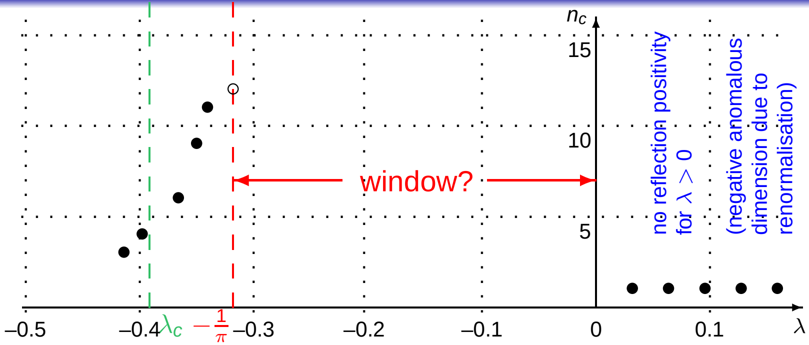








Is there a window of reflection positivity?



- If there is such a window, and if this extends to higher Schwinger functions, then this model defines the first 4D QFT.
- If so, is it non-trivial?
- preferrence of $\lambda < 0$ reminiscent of planar wrong-sign $\lambda\phi^4_4$-model [t'Hooft, 1983], [Rivasseau, 1983]

Summary

1. The quartic matrix model $\mathcal{Z} = \int dM \exp(\mathrm{tr}(JM - EM^2 - \frac{\lambda}{4}M^4))$ is exactly solvable in terms of solution of a non-linear equation.
2. Similar to Kontsevich model in 2D quantum gravity. Is there a deeper connection to quantum gravity?
3. ϕ^4_4-theory on a noncommutative geometry is of that type. In scaling limit, non-linear eq. reduced to fixed-point problem
   - unique non-perturbative and non-trivial solution for $\lambda < 0$
   - phase transitions and critical phenomena, hence interesting statistical physics model
4. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$:
   - S3 automatic, full Euclidean symmetry S1, control about S0
   - no clustering S4, no momentum exchange (close to triviality)
5. Reflection positivity S2 does not fail immediately. Why? Needs verification and extension to higher correlation functions

(Non)-triviality?

Projection to diagonal matrices brings the non-trivial intermediate matrix model close to triviality. This is more subtle:

- suppose we can prove S2, then reconstruct Hilbert space H, field operators $\varphi(f)$, unitaries $U(a, L)$ and some vacuum Ω
- uniqueness of Ω cannot be proved without clustering S4
- main problem: characterise set of Poincaré-invariant unit vectors of H, and find its extremal points Ω_e
- each restricted Hilbert space H_e, generated by its cyclic vector Ω_e, admits collision states (Haag-Ruelle theory) and (if asymptotically complete) an S-matrix
- involves new Wightman distributions

  $$W_e(x_1, ..., x_N) = \langle\Omega_e, \varphi(x_1) \cdots \varphi(x_N)\Omega_e\rangle$$

  expected to differ from $W(x_1, ..., x_N) = \langle\Omega, \varphi(x_1) \cdots \varphi(x_N)\Omega\rangle$

Consequently, a non-trivial $S \neq \mathbb{1}$ is not impossible.