

Superstring Perturbation Theory

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The Measure and Amplitudes

- The measure for the RNS superstring includes a sum over spin structure for the worldsheet fermions and an integral over moduli space.
- The prescription for the two-loop path-integral measure was given (D'Hoker, Phong, 2001-2007).
- Only NS four-point functions up to two loops have been computed.
- The three-loop measure was suggested (D'Hoker, Phong).
- Recently, the three-loop measure was shown to work (Cacciatori, Dalla Piazza, Bert van Geemen, 2008).
- Recently, progress was made in obtaining the genus four measure based on a suggestion of D'Hoker and Phong (Cacciatori, Dalla Piazza, van Geemen, 2008; Grushevsky, 2008).

Green-Schwarz Amplitude Status

- The Green-Schwarz superstring higher genus amplitude is complicated because of the need to add contact terms (Green, Schwarz, 1983; Mandelstam, 1974; Mandelstam, 1986; Greensite, Klinkhamer, 1988).
- These contact terms have been derived from first principles by requiring target-space supersymmetry (and Lorentz invariance) at every point of moduli space (Green, Seiberg, 1988).
- Only tree-level and one-loop four-point functions have been computed in the light-cone gauge.

The matter sigma-model

The action

- The matter sigma-model is given by (Berkovits, 2000)

$$S_m = \int d^2z \left(\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial} \theta^\alpha \right) .$$

x_m are the bosonic target-space coordinates, θ^α are the weight $(0, 0)$ target-space fermionic coordinates forming a 10d Weyl spinor of the appropriate representation and p_α are their weight $(1, 0)$ conjugate momenta.

- The supersymmetric momentum is

$$\Pi^m = \partial x^m + \theta \gamma^m \partial \theta .$$

- The supersymmetric covariant derivative reads

$$d_\alpha = p_\alpha + (\gamma^m \theta)_\alpha \left(\partial x_m + \frac{1}{2} \theta \gamma_m \partial \theta \right) .$$

The matter sigma-model

OPEs and Central Charge

- The OPEs of the various operators are

$$x^m(z)x^n(0) \sim -\eta^{mn} \log |z|^2 ,$$

$$p_\alpha(z)\theta^\beta(0) \sim \frac{\delta_\alpha^\beta}{z} ,$$

$$d_\alpha(z)d_\beta(0) \sim \frac{2}{z}\gamma_{\alpha\beta}^m \Pi_m(0) .$$

- The matter energy-momentum tensor is

$$T_m = \frac{1}{2} \partial x^m \partial x_m + p_\alpha \partial \theta^\alpha .$$

- The matter central charge is

$$c_m = \underbrace{10}_x + \underbrace{16(-2)}_{p\theta} = -22 .$$

The Pure Spinor CFT

- Add the two worldsheet fields w_α and λ^α , which are target space Weyl spinors of the same representation as p_α and θ^α , respectively.
- These form holomorphic $\beta\gamma$ -system of weight $(1, 0)$.
- λ^α is subject to the pure spinor constraint

$$\lambda\gamma^m\lambda = 0 .$$

- λ and w have 11 independent components each.
- The central charge is $c_\lambda = 11 \cdot 2 = 22$.

The BRST Operator

- The BRST operator is

$$Q_B = \oint \frac{dz}{2\pi i} \lambda^\alpha d_\alpha .$$

- Nilpotence is assured by the pure spinor condition:

$$Q_B^2 = \oint \frac{dz}{2\pi i} \lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta \Pi_m(z) = 0 .$$

- The unintegrated physical states are given by the ghost number one cohomology of Q_B .

The Spectrum

- The unintegrated massless vertex operator is of the form (Berkovits, 2000)

$$U = \lambda^\alpha A_\alpha(x, \theta) .$$

- Requiring it to be BRST-closed and using the relation $\lambda^\alpha \lambda^\beta \propto \gamma_{mnpqr}^{\alpha\beta} (\lambda \gamma^{mnpqr} \lambda)$ one gets

$$\gamma_{mnpqr}^{\alpha\beta} D_\alpha A_\beta = 0 ,$$

where D_α is the covariant derivative.

- The gauge invariance $\delta U = Q_B \Omega(x, \theta)$, where Ω is a ghost number zero operator, reads

$$\delta A_\alpha = D_\alpha \Omega .$$

- These are the field equations and gauge transformation of $D = 10$ super-Maxwell theory.

Tree-level prescription

- Originally was deduced and not derived from the path-integral (Berkovits, 2000)
- The non-zero modes are integrated out by using their OPEs.
- Conformal symmetry on the sphere implies three fixed and $N - 3$ integrated vertex operators in N -point functions. Thus the saturation rule has ghost-number 3.
- Amplitudes should be BRST-invariant.
- Amplitudes should be $SO(9, 1)$ -invariant.
- There is only one such an operator involving just the θ and λ (the other worldsheet fields have no zero modes) yielding the saturation rule

$$\langle (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\lambda\gamma_{mnp}\theta) \rangle = 1 .$$

Tree-level Prescription

- The amplitudes have been proven to be cyclic and consistent with RNS amplitudes for massless scattering of up to four fermions (Berkovits, Vallilo, 2000).

Multi-loop Prescription

The Zero-mode mode measures

- Path integral measures for the zero-modes on a genus g surface now exist (Berkovits, 2004).
- Being a worldsheet scalar, there are eleven λ zero-modes on a genus g Riemann surface.
- The path integral measure $[\mathcal{D}\lambda]$ for the pure spinor zero-modes is defined by

$$(d^{11}\lambda)^{[\alpha_1\dots\alpha_{11}]} = [\mathcal{D}\lambda](\epsilon T)_{(\beta_1\beta_2\beta_3)}^{[\alpha_1\dots\alpha_{11}]} \lambda^{\beta_1} \lambda^{\beta_2} \lambda^{\beta_3},$$

where $(\epsilon T)_{(\beta_1\beta_2\beta_3)}^{[\alpha_1\dots\alpha_{11}]}$ = $\epsilon^{\alpha_1\dots\alpha_{16}} \mathcal{T}_{(\beta_1\beta_2\beta_3)[\alpha_{12}\dots\alpha_{16}]}$ and \mathcal{T} is a Lorentz tensor composed of Dirac matrices and determined by Lorentz covariance and the pure spinor constraint.

Multi-loop Prescription

The zero-mode measures

- The measure $[\mathcal{D}N]$ for the pure spinor Lorentz and ghost currents $N^{mn} \sim w\gamma\lambda$, $J \sim w\lambda$ is given by

$$(d^{10}N)^{[[m_1 n_1] \dots [m_{10} n_{10}]]} \wedge dJ = [\mathcal{D}N](\text{ghost no. 8 term}) .$$

- Hence, for a genus g surface

$$\mathcal{A} = \int [\mathcal{D}\lambda] \prod_{n=1}^g [\mathcal{D}N_n] f(\lambda, N_1, J_1, \dots, N_g, J_g) ,$$

where $f(\lambda, N_1, J_1, \dots, N_g, J_g)$ is the integrand obtained after integrating out the non-zero modes.

Multi-loop Prescription

Picture Changing Operators

- The existence of bosonic zero-modes requires the introduction of picture changing operators in order to absorb them.
- The picture changing operator for the pure spinor is

$$Y_C = C_\alpha \theta^\alpha \delta(C_\beta \lambda^\beta),$$

where C_α is a Lorentz spinor required because λ and θ are not invariant under Lorentz transformations.

- Y_C does not break the super-Poincaré invariance of the amplitude because its super-Poincaré transformation yields a BRST-exact term.
- The worldsheet derivative of Y_C is BRST-exact — the amplitude does not depend on its position.

Multi-loop Prescription

Picture Changing Operators

- The picture changing operators of the zero-modes of N and J are

$$Z_B = \frac{1}{2} B_{mn} (\lambda \gamma^{mn} d) \delta(B^{pq} N_{pq}) , \quad Z_J = (\lambda^\alpha d_\alpha) \delta(J) .$$

- The super-Poincaré variation and worldsheet derivative of these are again BRST-exact.
- For a genus g Riemann surface 11 insertions of Y_C , $10g$ insertions of Z_B and g insertions of Z_J are required.

Multi-loop Prescription

The b -ghost

- Genus g Riemann surfaces have $3g - 3$ moduli and hence insertions of the b -ghost are needed.
- Because of the first-class pure spinor constraint w_α can only appear in the gauge invariant ghost-number 0 combinations N and J so no natural the b -ghost satisfying $\{Q, b\} = T$ exists.
- However, a substitute non-local $\tilde{b}_B(u, z)$ can be constructed such that

$$\{Q, \tilde{b}_B(u, z)\} = T(u)Z_B(z) .$$

- It can provide the $3g - 3$ insertions of b -ghost as well as $3g - 3 Z_B$ insertions.

Multi-loop Prescription

The Amplitude

- Using these building blocks, the N -point closed string genus g amplitude is given by (Berkovits, 2004)

$$\mathcal{A} = \int \prod_{i=1}^{3g-3} d^2\tau_i \langle | \prod_{\rho=1}^{3g-3} \int d^2u_\rho \mu_\rho(u_\rho) \tilde{b}_{B_\rho}(u_\rho, z_\rho) \prod_{m=3g-2}^{10g} Z_{B_m}(z_m) \prod_{n=1}^g Z_J(v_n) \prod_{l=1}^{11} Y_{C_l}(y_l) |^2 \prod_{k=1}^N \int d^2t_k U_k(t_k) \rangle ,$$

where τ_i are the metric moduli, μ_ρ are the Beltrami differentials and U_k are the integrated vertex operators.

Status

- The derivation of the amplitude prescription is based on the path integral's non-vanishing and not on basic gauge fixing.
- The triviality of the cohomology of the non-zero picture ghost-number 1 non-zero weight vertex operators was needed for the existence of \tilde{b}_B but was not proven.
- Massless four-point functions were computed to one- and two-loops (Berkovits, 2004 and 2005).
- One-loop massless four-point amplitude equivalence to RNS and GS results has been shown (Anguelova, Grassi, Vanhove, 2000).
- The two-loop four-point amplitude of four NS states has been shown to match the RNS one (Berkovits, Mafra, 2005).

Results

- Identities relating the kinematic factors of tree-level and one- and two-loop massless four-point amplitude have been found (Mafra, 2008)
- It was proven that there are no perturbative corrections higher than one-loop to the R^4 and $\partial^2 R^4$ in the effective Type IIB supergravity (Berkovits, 2004).
- Proven that the massless N -point genus g amplitude vanishes for $N < 4$ and $g > 0$ (Berkovits, 2004). This with the assumption of factorization and absence of unphysical divergences inside the moduli space implies the finiteness of the amplitudes to all orders in perturbation theory (Berkovits, 2004 and refs. therein).

The Non-minimal Pure Spinor CFT

- In addition to the left-moving λ^α and w_α one adds the left-moving boson $\bar{\lambda}_\alpha$ and the fermionic r_α satisfying (Berkovits, 2005)

$$\bar{\lambda}_\alpha \gamma_m^{\alpha\beta} \bar{\lambda}_\beta = 0, \quad \bar{\lambda}_\alpha \gamma_m^{\alpha\beta} r_\beta = 0.$$

- The non-minimal superstring action is now

$$\int d^2z \left(\frac{1}{2} \partial x^m \bar{\partial} x_m + p_\alpha \bar{\theta}^\alpha - w_\alpha \bar{\lambda}^\alpha - \bar{w}^\alpha \bar{\partial} \bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha \right),$$

where \bar{w}^α and s^α are the conjugate momenta.

- The BRST operator is modified to

$$Q_{\text{nm}} = \int ds (\lambda^\alpha d_\alpha + \bar{w}^\alpha r_\alpha).$$

The cohomology remains unmodified due to the quartet mechanism.

The b -ghost

- Although there is a non-globally defined b -ghost in the minimal pure spinor, one can define such a ghost in the non-minimal version.
- In the Čech language it can be written as

$$b = (b_\alpha) + (b_{\alpha\beta}) + (b_{\alpha\beta\gamma}) + (b_{\alpha\beta\gamma\delta}) ,$$

where each term is defined locally in the pure spinor target space.

- They are given by

$$(b_\alpha) = \frac{G_\alpha}{\lambda^\alpha} , \quad (b_{\alpha\beta}) = \frac{H^{[\alpha\beta]}}{\lambda^\alpha \lambda^\beta} , \quad (b_{\alpha\beta\gamma}) = \frac{K^{[\alpha\beta\gamma]}}{\lambda^\alpha \lambda^\beta \lambda^\gamma} ,$$

$$(b_{\alpha\beta\gamma\delta}) = \frac{L^{[\alpha\beta\gamma\delta]}}{\lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta} ,$$

The b -ghost

- The non-minimal pure spinor corresponds to the Dolbeault picture in which

$$b = \frac{\bar{\lambda}_\alpha G^\alpha}{\bar{\lambda}\lambda} + \frac{\bar{\lambda}_\alpha r_\beta H^{[\alpha\beta]}}{(\bar{\lambda}\lambda)^2} - \frac{\bar{\lambda}_\alpha r^\beta r^\gamma K^{[\alpha\beta\gamma]}}{(\bar{\lambda}\lambda)^3} - \frac{\bar{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\bar{\lambda}\lambda)^4}$$

- And we have $T = \{Q_{\text{nm}}, b\}$. where T is the minimal energy-momentum tensor. By defining the non-minimal b -ghost $b_{\text{nm}} = b + s^\alpha \partial \bar{\lambda}_\alpha$ we get $\{Q_{\text{nm}}, b_{\text{nm}}\} = T_{\text{nm}}$, where $T_{\text{nm}} = T + \bar{w}^\alpha \partial \bar{\lambda}_\alpha - s^\alpha \partial r_\alpha$ is the energy-momentum tensor of the non-minimal superstring.

Multi-loop Prescription

- Ordinarily, the multi-loop prescription should be

$$A = \int_{\mathcal{M}_{g,N}} d^{3g-3} \tau \left\langle \prod_{j=1}^{3g-3} \left(\int dw_j \mu_j(w_j) b(w_j) \right) \prod_{r=1}^N dz_r U(z_r) \right\rangle$$

- However, there are two subtleties:
 - The integration over the $2g$ zero-modes of λ , $\bar{\lambda}$ and $2g$ zero modes of w , \bar{w} diverges.
 - The measure for λ and $\bar{\lambda}$ converges as $(\bar{\lambda}\lambda)^{11}$. Hence, if the poles in b combine to higher than $\frac{1}{(\bar{\lambda}\lambda)^{11}}$, divergences are expected.

Multi-loop Prescription

The Zero-Modes

- The zeros due to the fermionic zero-modes of θ^α , r_α , p_α , and s^α cancel the infinities generated from the bosonic zero-modes of λ^α , $\bar{\lambda}_\alpha$, w_α and \bar{w}^α .
- This can be obtained by regulating by $\mathcal{N} = e^{\{Q, \chi\}}$, which is the identity up to BRST-exact terms.
- A suitable χ was given (Berkovits, 2005):

$$\chi = \bar{\lambda}_\alpha \theta^\alpha - \sum_{l=1}^g \left(\frac{1}{2} N_{mn}^l S^{mnl} + J^l S^l \right),$$

where N_{mn}^l , S^{mnl} , J^l and S^l are the zero-modes of these currents.

Multi-loop Prescription

The Poles in the b -ghost

- If operators with up to $\frac{1}{(\lambda\lambda)^{11}}$ are allowed in the Hilbert space, the cohomology is rendered trivial (Berkovits, 2005).
- A BRST-invariant regulator for the b -ghost was suggested (Berkovits, Nekrasov, 2006)

$$b_\epsilon = \int d^{11}f d^{11}\bar{f} d^{11}g d^{11}\bar{g} e^{-(\bar{f}_\alpha f^\alpha + \bar{g}_\alpha g^\alpha)} b'(y)$$

where $b'(y)$ is the b -ghost after a similarity transformation.

- The regularized b -ghost satisfies

$$b_\epsilon = b + \{Q, \chi_\epsilon\}$$

so the regularization does not modify the amplitude.

Multi-loop Prescription

The Final Amplitude

- The final prescription for the amplitude is

$$\mathcal{A} = \lim_{\epsilon \rightarrow 0} \int d^{3g-3} \tau \left\langle \prod_{i=1}^{3g-3} \left(\int dw_i \mu_i(w_i) b_\epsilon(w_i) \right) \prod_{j=1}^N \int dz_j U_j(z_j) \mathcal{N} \right\rangle.$$

- Terms in which some of the θ s have to be taken from \mathcal{N} do not require ϵ -regularization.
- Other terms are more complicated requiring the ϵ -regularization in order to assure finiteness.

Status and Results

- It is easier to compute amplitudes in the minimal pure spinor superstring since there is a real b -ghost, there is no need for picture changing operators preserving manifest Lorentz invariance throughout the computation.
- The prescription seems to be finite. However, the consistency relies on requiring the cohomology to include no operators with poles higher than tenth-order.
- Massless open string four-point one- and two-loop as well as two-loop six-point amplitudes were computed (Berkovits, Mafra, 2006).
- It was proven that $\partial^n R^4$ effective action terms are not corrected above $n/2$ loops for $0 < n < 12$ and these terms are identical for type IIA and IIB supergravities for $n \leq 8$ (Berkovits, 2006).