Quantum Gravity

on a

Lattice

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Outline

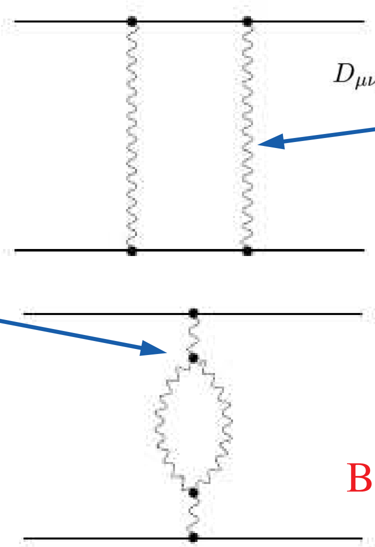
- *Motivation: Perturbative Quantum Gravity*
  - Failure of perturbative renormalization in d=4
  - Quantum Gravity in 2+ε dimensions
  - Non-renormalizable theories: Sigma Model
- *Methods: Formulation of Lattice Quantum Gravity*
  - Simplicial lattice-regularized formulation
  - Matter fields, Observables
  - Methods for determining non-trivial scaling dimensions
- *Outlook: Possible non-perturbative ground state scenarios*
  - Running of Newton's G
  - Effective non-local covariant relativistic field equations

Perturbative Quantum Gravity

UV Divergences: Compute QM amplitudes by Feynman diagram perturbation theory:

'*t* Hooft. & Veltman, 1974

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$



$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2}\,\frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}}{p^2 + i\epsilon}$$

$$\begin{aligned}
&U(q_1, q_2, q_3)_{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3} = \\
&-i\frac{\kappa}{2}\Big[q^2_{(\alpha_1}q^3_{\beta_1)}\left(2\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - \tfrac{2}{d-2}\eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\right) \\
&\quad + q^1_{(\alpha_2}q^3_{\beta_2)}\left(2\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1} - \tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_3\beta_3}\right) \\
&\quad + q^1_{(\alpha_3}q^2_{\beta_3)}\left(2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1} - \tfrac{2}{d-2}\eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\right) \\
&\quad + 2q^3_{(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}q^2_{\beta_3)} + 2q^1_{(\alpha_3}\eta_{\beta_3)(\alpha_2}\eta_{\beta_2)(\alpha_1}q^3_{\beta_1)} + 2q^2_{(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)(\alpha_2}q^1_{\beta_2)} \\
&\quad + q^2\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}\eta_{\alpha_3\beta_3} + \tfrac{2}{d-2}\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}\eta_{\alpha_2\beta_2} - 2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1}\right) \\
&\quad + q^1\cdot q^3\left(\tfrac{2}{d-2}\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)\beta_2}\eta_{\alpha_3\beta_3} + \tfrac{2}{d-2}\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)\beta_2}\right) \\
&\quad + q^1\cdot q^2\left(\tfrac{2}{d-2}\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)\beta_3}\eta_{\alpha_2\beta_2} + \tfrac{2}{d-2}\eta_{\alpha_3(\alpha_2}\eta_{\beta_2)\beta_3}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)(\alpha_2}\eta_{\beta_2)\beta_3}\right)\Big]
\end{aligned}$$

Bad high energy behavior

$$\frac{G(k^2)}{G} = 1 + \text{const}\; G\,k^2 + \ldots$$

Non-Renormalizability in Four Dimensions

$$I \;=\; \lambda \int d^d x \sqrt{g} - \frac{1}{16\,\pi\,G} \int d^d x \sqrt{g}\, R$$

Radiative corrections generate a host of new interactions…

$$\Gamma^{(1)}_{div} = \frac{1}{4-d} \frac{\hbar}{16\pi^2} \int d^4x \sqrt{g} \left(\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right)$$

$$\Gamma^{(2)}_{div} = \frac{1}{4-d} \frac{209}{2880} \frac{\hbar^2\, G}{(16\pi^2)^2} \int d^4x \sqrt{g}\, R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu\nu}$$

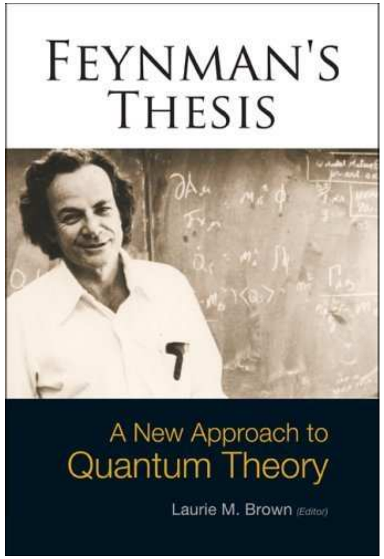
$$I \;\longrightarrow\; \lambda \int d^d x \sqrt{g} - \frac{1}{16\,\pi\,G} \int d^d x \sqrt{g}\, R + \frac{\alpha_0}{\Lambda^{4-d}} \int d^d x \sqrt{g}\, R_{\mu\nu} R^{\mu\nu} + \frac{\beta_0}{\Lambda^{4-d}} \int d^d x \sqrt{g}\, R^2 + \cdots$$

- 4-d perturbation theory in (ordinary) gravity seemingly leads to a dead end…
- Non-perturbative methods ? $\Rightarrow$ non-perturbative regularization, search for a new vacuum …

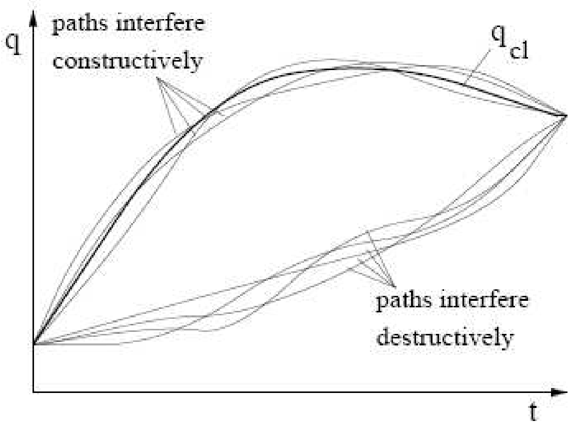
Feynman Path Integral

Reformulate QM amplitudes in terms of discrete

Sum over Paths



- non-commuting operators P,Q replaced by random Wiener paths.
- In complex time $t = -i\tau$ probabilities are *real* (as in statistical mechanics: $KT \rightarrow \hbar$).



$$K(q'', q'; T) = \sum_{all\ paths} A\, e^{iS(q'', q'; T)/\hbar}$$

$$\Longrightarrow \quad K = \int \mathcal{D}q(t) e^{iS[q(t)]}$$

Path Integral for Quantum Gravitation

$$||\delta g||^2 \equiv \int d^d x \, G^{\mu\nu,\alpha\beta}[g(x)] \, \delta g_{\mu\nu}(x) \, \delta g_{\alpha\beta}(x)$$

DeWitt approach to measure: Volume element in function space obtained from *super-metric* over metric deformations.

$$G^{\mu\nu,\alpha\beta}[g(x)] = \tfrac{1}{2}\sqrt{g(x)} \left[g^{\mu\alpha}(x) g^{\nu\beta}(x) + g^{\mu\beta}(x) g^{\nu\alpha}(x) + \lambda \, g^{\mu\nu}(x) g^{\alpha\beta}(x) \right]$$

$$\int d\mu[g] = \int \prod_x \left(\det[G(g(x))] \right)^{\frac{1}{2}} \prod dg_{\mu\nu}(x)$$

$$\int d\mu[g] = \int \prod_x [g(x)]^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \underset{d=4}{\rightarrow} \int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

$$Z_{cont} = \int [d \, g_{\mu\nu}] \, e^{-\lambda \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g} \, R}$$

Euclidean E-H action *unbounded below* (conformal instability).

Only One Coupling

Pure gravity path integral:

$$Z = \int [d\, g_{\mu\nu}]\, e^{-I_E[g]}$$

$$I_E[g] = \lambda_0\, \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi\, G_0} \Lambda^{d-2} \int dx \sqrt{g}\, R$$

Rescale metric (edge lengths):

$$g'_{\mu\nu} = \lambda_0^{2/d}\, g_{\mu\nu} \qquad g'^{\mu\nu} = \lambda_0^{-2/d}\, g^{\mu\nu}$$

$$I_E[g] = \Lambda^d \int dx \sqrt{g'} - \frac{1}{16\pi\, G_0\, \lambda_0^{\frac{d-2}{d}}} \Lambda^{d-2} \int dx \sqrt{g'}\, R'$$

- In the absence of matter, only *one* dimensionless coupling:

$$\tilde{G} \equiv G_0\, \lambda_0^{(d-2)/d}$$

Similar to the g of Y.M. !

Functional Measure cont'd

Add volume term to functional measure (Misner 1955) ;
coordinate transformation $x^\mu + \epsilon^\mu(x)$

$$\prod_x [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \rightarrow \prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

$$\prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma = \prod_x [\det(\delta_\alpha{}^\beta + \partial_\alpha \epsilon^\beta)]^\gamma = \exp\left\{ \gamma \, \delta^d(0) \int d^d x \, \partial_\alpha \epsilon^\alpha \right\} = 1$$

[Faddeev & Popov, 1973]

Skeptics should systematically investigate (on the lattice) effects due to the addition of an ultra-local term of the type

$$\prod_x [g(x)]^{\sigma/2} = \exp\left\{ \tfrac{1}{2} \sigma \, \delta^d(0) \int d^d x \ln g(x) \right\}$$

Due to it's ultra-local nature, such a term would not be expected to affect the propagation properties of gravitons (which are det. by R-term).

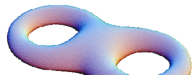
Perturbatively Non-Renorm. Interactions

Some early work :

- K.G. Wilson, *Quantum Field Theory Models in D < 4,* PRD 1973.
- K. Symanzik, *Renormalization of Nonrenormalizable Massless* $\boldsymbol{\varphi^4}$ *Theory,* CMP 1975.
- G. Parisi, *Renormalizability of not Renormalizable Theories,* LNC 1973.
- G. Parisi, *Theory Of Nonrenormalizable Interactions - Large N,* NPB 1975.
- E. Brézin and J. Zinn-Justin, *Nonlinear* $\boldsymbol{\sigma}$ *Model in* $\boldsymbol{2+\varepsilon}$ *Dimensions,* PRL 1976.
- D. Gross and A. Neveu …

Gravity in 2.000001 Dimensions

- Wilson expansion: formulate in 2+ε dimensions…



$$\chi = \frac{1}{4\pi}\int d^2x\sqrt{g}\,R$$

G becomes dimensionless in d = 2 ... “Kinematic singularities” as d → 2 make limit *very delicate*.

$$D_{\mu\nu\rho\sigma}(p) \;=\; \frac{i}{2}\,\frac{\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}}{p^2+i\epsilon}$$

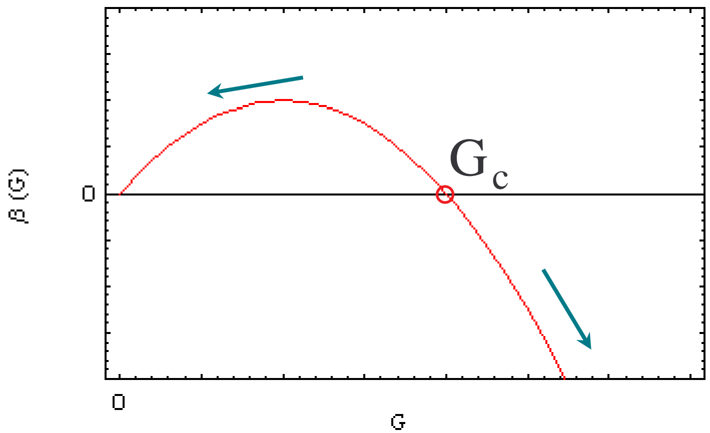
But G is dim-less and theory is pert. renormalizable,

Weinberg 1977
Kawai, Ninomiya 1995
Kitazawa, Aida 1998

$$\mu\,\frac{\partial}{\partial\,\mu}\,G(\mu) \;=\; \beta(G(\mu))$$

$$\beta(G) \;=\; (d-2)\,G \;-\; \frac{2}{3}\,(25-n_s)\,G^2 \;-\; \frac{20}{3}\,(25-n_s)\,G^3 \;+\;\ldots$$

$$\left\{\begin{aligned} G_c \;&=\; \frac{3}{2\,(25-n_s)}\,(d-2) - \frac{45}{2(25-n_s)^2}\,(d-2)^2 + \ldots \\ \nu^{-1} \;&=\; -\,\beta'(G_c) \;=\; (d-2) \;+\; \frac{15}{25-n_s}\,(d-2)^2 \;+\;\ldots \end{aligned}\right.$$



(two loops, manifestly covariant, gauge independent)

A phase transition…

More on 2.000001 dim's …

Graviton loops

Graviton-ghost loops

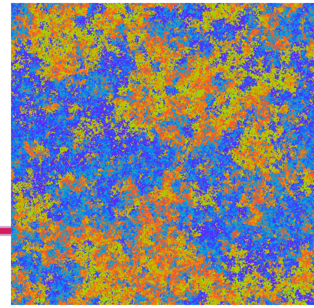
- Singularity structure in *d > 2* unclear (Borel)…
- But analytical control of UV fixed point at G_c .

$$G(k^2) = \frac{G_c}{1 \pm (m^2/k^2)^{(d-2)/2}}$$

Nontrivial scaling determined by UV FP.

$$m \sim \Lambda \exp\left(- \int^G \frac{dG'}{\beta(G')} \right) \underset{G \to G_c}{\sim} \Lambda \, | \, G - G_c |^{-1/\beta'(G_c)}$$

Detour : Non-linear Sigma model



- Field theory description [*O(N)* Heisenberg model] :

$$Z = \int [d\sigma] \prod_x \delta\left(\sigma^a(x)\,\sigma^a(x) - 1\right) \exp\left(-\frac{\Lambda^{d-2}}{g^2} \int d^d x\, \partial_\mu \sigma^a(x)\, \partial_\mu \sigma^a(x) + \int d^d x\, j^a(x) \sigma^a(x) \right)$$

Coupling *g* becomes *dimensionless* in *d = 2.*

For *d > 2* theory is not perturbatively renormalizable,

but in the 2+ ε expansion one finds:

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
N.A. Kivel et al, 1994
E. Brezin and S. Hikami, 1996

$$\Lambda \frac{\partial g^2}{\partial \Lambda} \equiv \beta(g^2) = (d-2)\, g^2 - \frac{N-2}{2\pi} g^4 + O\left(g^6, (d-2) g^4\right)$$

Phase Transition = non-trivial UV fixed point; new non-perturbative mass scale.

$$\nu^{-1}(\epsilon) = \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} - \left[30 - 14 + n^2 + (54 - 18n)\zeta(3)\right] \frac{\epsilon^4}{4(n-2)^3} + \ldots$$

$$\xi(g^2) \equiv m^{-1}(g^2) \simeq c_d \Lambda \left(\frac{1}{g_c^2} - \frac{1}{g^2} \right)^{\nu}$$

$$\langle \vec{S}(\mathbf{x}) \cdot \vec{S}(\mathbf{0}) \rangle \sim \exp\{-|\mathbf{x}|/\xi\}$$

Renormalization Group Equations

In the framework of the *double* (*g and* $2+\varepsilon$) expansion the model looks just *like any other renormalizable theory, to every order…*

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}-\frac{n}{2}\zeta(g)+\rho(g)\,h\,\frac{\partial}{\partial h}\right]\Gamma^{(n)}(p_i,g,h,\Lambda)=0$$

Callan-Symanzik Eq.

$$\Lambda\frac{\partial}{\partial\Lambda}|_{\text{ren.fixed}}\; g \;=\; \beta(g)$$

$$\Lambda\frac{\partial}{\partial\Lambda}|_{\text{ren.fixed}}\,(-\ln Z) \;=\; \zeta(g)$$

$$2-d+\tfrac{1}{2}\zeta(g)+\frac{\beta(g)}{g} \;=\; \rho(g)\;.$$

$$g \;=\; (\Lambda/\mu)^{d-2}Z_g g_r \qquad \pi(x)=Z^{-1/2}\pi(x)$$

$$h \;=\; Z_h h_r \qquad Z_h=Z_g/\sqrt{Z}$$

$$\Gamma_r^{(n)}(p_i,g_r,h_r,\mu)=Z^{n/2}(\Lambda/\mu,g)\,\Gamma^{(n)}(p_i,g,h,\Lambda)$$

… but the price one pays is that now one needs $\varepsilon \longrightarrow 1$!

Similar result are obtained in large N limit.

But is it correct ?

Experimental test: *O(2)* non-linear sigma model describes the phase transition of *superfluid Helium*

Space Shuttle experiment (2003)

High precision measurement of specific heat of superfluid Helium He4 (zero momentum energy-energy correlation at FP)

J.A. Lipa et al, Phys Rev 2003: $\alpha = 2 - 3\nu = -0.0127(3)$

4-ε expansion to four loops, & to six loops in d=3: $\alpha = 2 - 3\nu \approx -0.0125(4)$

LIPA *et al.*

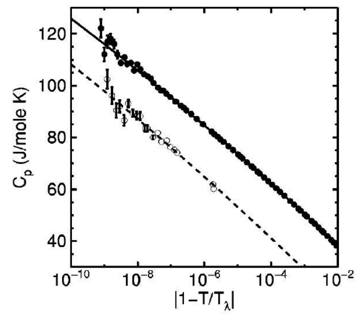


FIG. 15. Semilogarithmic plot of the specific heat vs reduced temperature over the full range measured. Below the transition the data (closed symbols) were binned with a density of 10 bins per decade, and above (open symbols) with a density of 8 bins per decade. Lines show best fits to the data.

One of the most accurate predictions of QFT.
Theory value reviewed in J. Zinn-Justin, 2007

The non-linear sigma model in 3d provides an explicit example of a field theory which :

- ✓ Is *not* perturbatively renormalizable in d=3 .

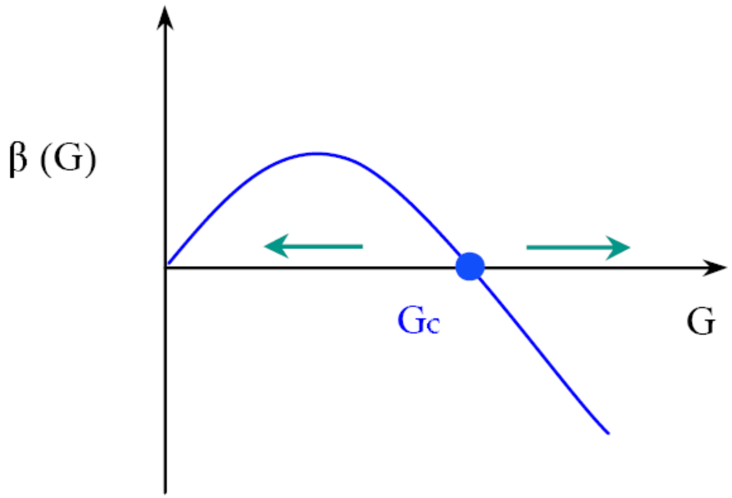
- ✓ Nevertheless leads to *detailed, calculable* predictions in the scaling limit r » a ($q^2 \ll \Lambda^2$) .

- ✓ Involves a new non-perturbative scale ξ, essential in determining the scaling behavior in the vicinity of the FP.

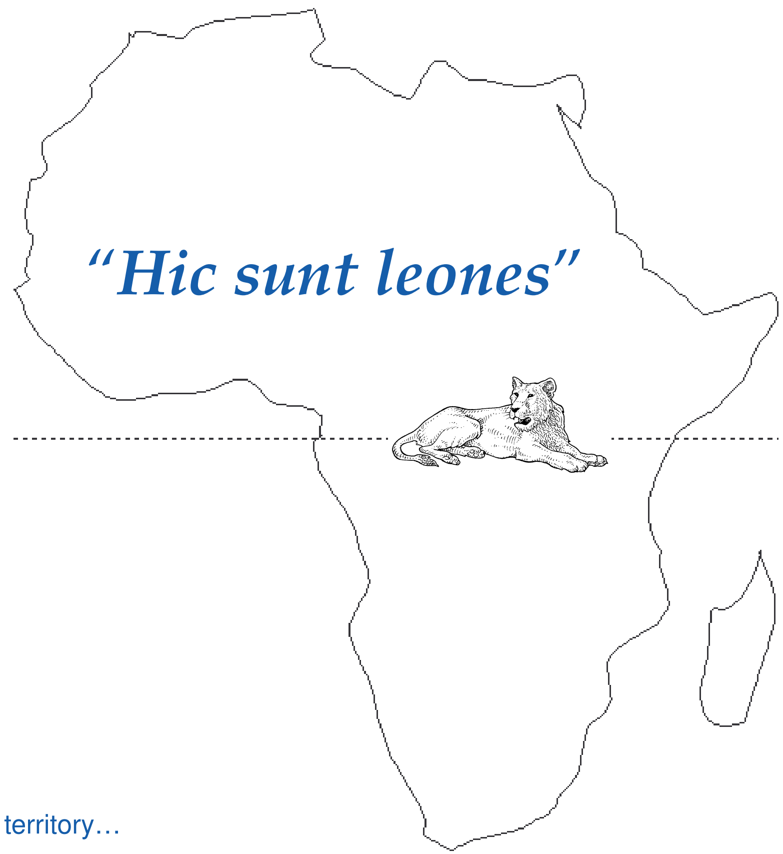
- ✓ Whose non-trivial, universal predictions *agree* with experiments.

Key question:

What is left of the above q. gravity scenario in 4 dimensions?



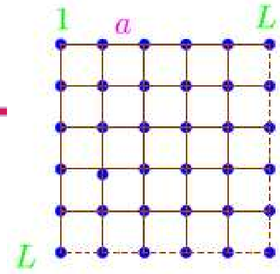
Strongly coupled gravity



The Roman's description of unknown territory…

Lattice Theory

Lattice Quantum Gravity

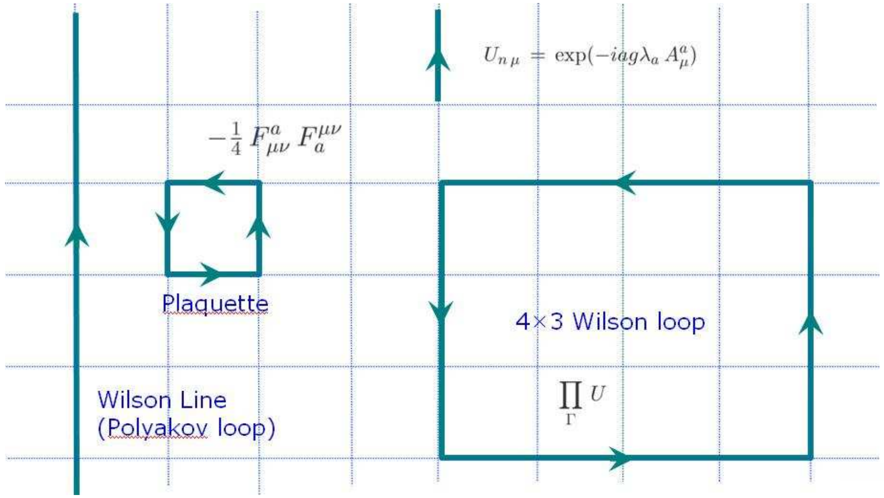


Lattice regularization provides explicit short distance cutoff.

- Regularized theory is finite, allows non-perturbative treatment.
- Methods of statistical field theory.
- Multi-year experience with lattice QCD.
- Numerical evaluation feasible.
- Continuum limit requires UV fixed point.

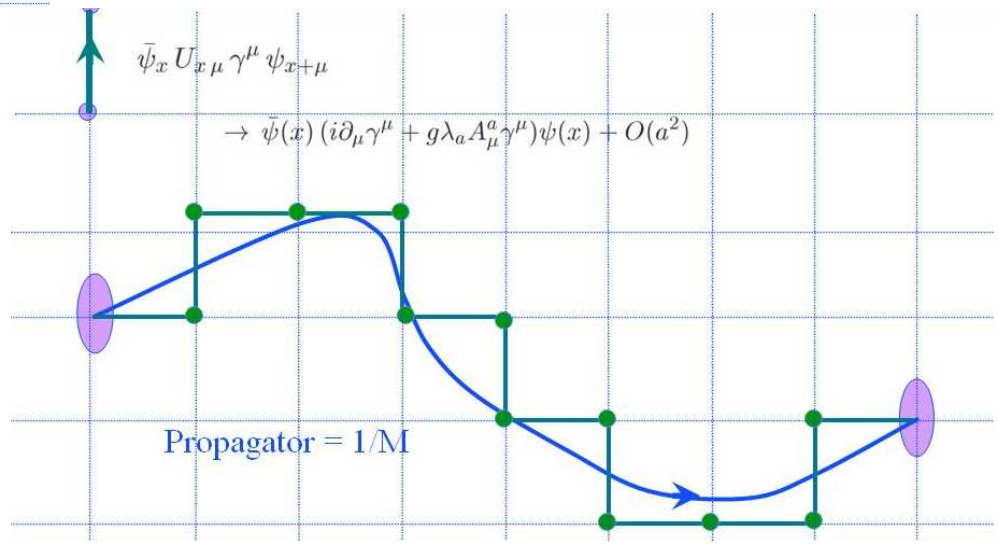
statistical mechanics	$\longleftrightarrow$	quantum field theory
ensemble	$\longleftrightarrow$	phase space
ensemble average	$\longleftrightarrow$	path integral
$exp\{-\beta_k H\}$	$\longleftrightarrow$	$exp\{-S^E\}$
$\beta_k \int d^3x \mathcal{H}$	$\longleftrightarrow$	$\int dx_4 \int d^3x \mathcal{L}$
finite β_k	$\longleftrightarrow$	finite $\int dx_4 = T$
zero temperature	$\longleftrightarrow$	infinite time extent T

Proto: Wilson' Lattice Gauge Theory



Local gauge invariance

→ exact lattice Ward identities



Lattice Gauge Theory Works

$$L_{\mathrm{QCD}} = -\frac{1}{4} F^{(a)}_{\mu\nu}\, F^{(a)\mu\nu} + i \sum_q \overline{\psi}^i_q\, \gamma^\mu\, (D_\mu)_{ij}\, \psi^j_q - \sum_q m_q\, \overline{\psi}^i_q\, \psi_{qi}\ ,$$

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln\left[\ln(\mu^2/\Lambda^2)\right]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4 \ln^2(\mu^2/\Lambda^2)} \times \left(\left(\ln\left[\ln(\mu^2/\Lambda^2)\right] - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right) \right] .$$

Lattice gauge theory provides (so far) the only convincing evidence for *confinement* and *chiral symmetry breaking* in QCD.

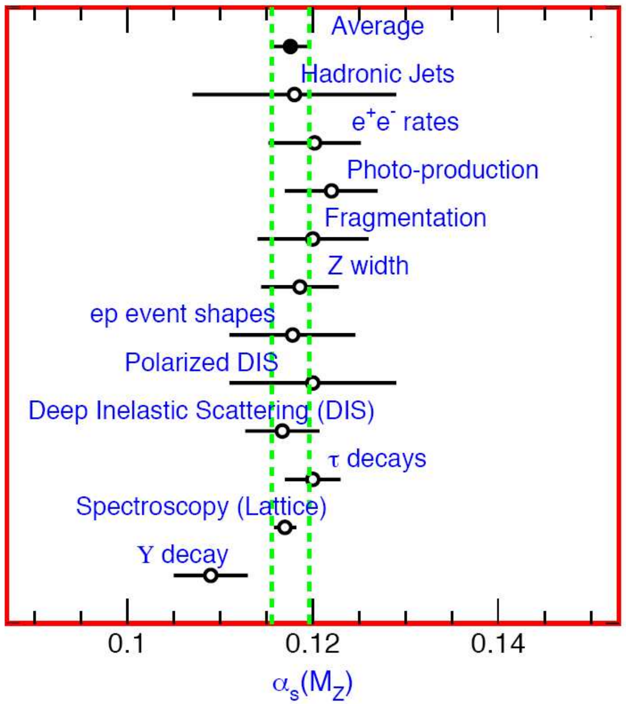


Figure 9.1: Summary of the value of $\alpha_s(M_Z)$ from various processes. The values shown indicate the process and the measured value of α_s extrapolated to $\mu = M_Z$. The error shown is the *total* error including theoretical uncertainties. The average quoted in this report which comes from these measurements is also shown. See text for discussion of errors.

[Particle Data Group LBL, 2008]

Quantum Continuum Limit

- Naïve continuum limit : $a \to 0 \quad (\Lambda = \pi/a \to \infty)$
- Quantum continuum limit (based on RG) :

$$a \to 0 \quad g(a) \to 0$$

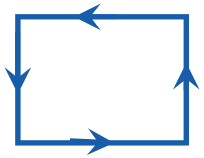
$$\xi = \frac{1}{m_{\text{phys}}} = \text{const.} \times a \exp\left\{\frac{1}{2\beta_0 g^2(a)}\right\} \quad \text{fixed}$$

or simply: $\boxed{\xi/a \to \infty}$

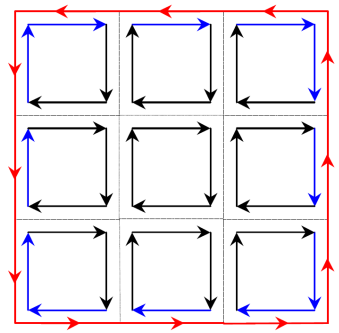
A *phase transition* (UV fixed point) is required for the existence of a non-trivial continuum limit [Wilson, 1974].

Wilson Loop in *SU(N)* Gauge Theories

- Wilson loop in Lattice Gauge Theories,



$$W(C) = \left\langle \mathrm{tr}\,\mathcal{P}\exp\left\{ ig \oint_C A_\mu(x)dx^\mu \right\} \right\rangle, \quad \sim_{A\to\infty} \exp(-A(C)/\xi^2),$$



Gives linear confinement [textbook result, Peskin & Schroeder p. 783]

ξ = gauge correlation length

$$G_\square(x) = \left\langle \mathrm{tr}\,\mathcal{P}\exp\left\{ ig \oint_{C'_\epsilon} A_\mu(x')dx'^\mu \right\} \times (x)\,\mathrm{tr}\,\mathcal{P}\exp\left\{ ig \oint_{C''_\epsilon} A_\mu(x'')dx''^\mu \right\}(0) \right\rangle_c . \sim_{|x|\to\infty} \exp(-|x|/\xi).$$

… Both results are essentially geometric in nature.
They follow (almost trivially) from the use of the *SU(N)* Haar measure.

Simplicial Lattice Formulation

"General Relativity without coordinates" *(T.Regge)*

MTW ch 42.

- Based on a <u>dynamical lattice.</u>
- Incorporates <u>continuous local</u> invariance.
- Puts within the reach of <u>computation</u> problems which in practical terms are beyond the power of normal analytical methods.
- It affords any desired level of <u>accuracy</u> by a sufficiently fine subdivision of the space-time region under consideration.

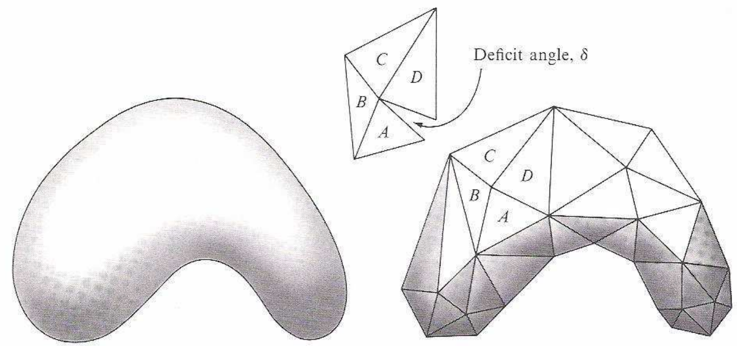
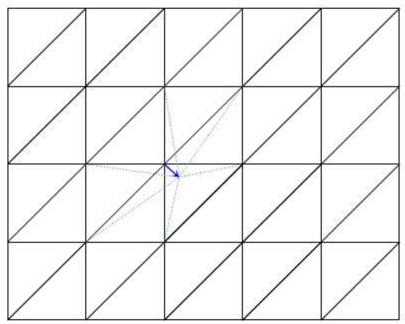
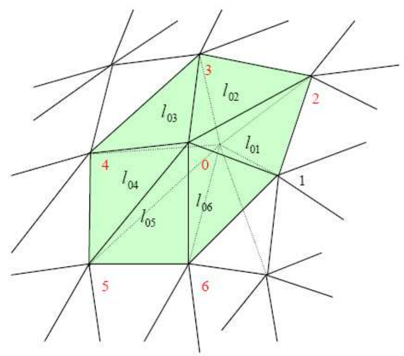
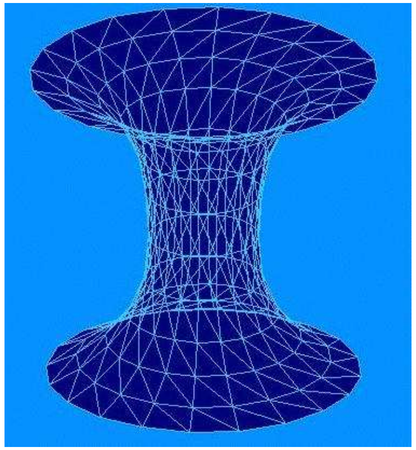


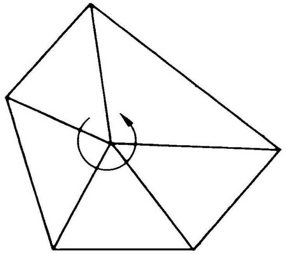
Figure 42.1.

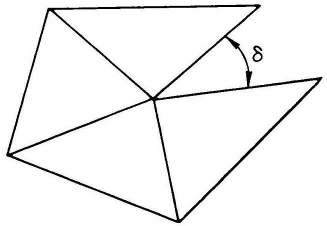


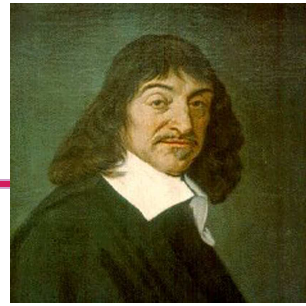


Curvature - Described by Angles









$d = 2$

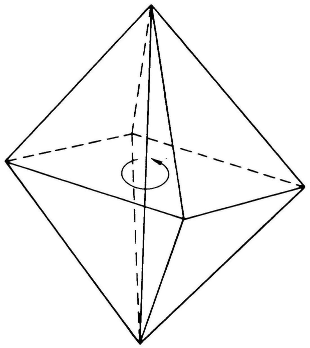
$$\chi = \frac{1}{2\pi}\sum_h \delta_h$$

$$g_{ij} = \frac{1}{2}\left(l^2_{1,i+1} + l^2_{1,j+1} - l^2_{i+1,j+1}\right)$$

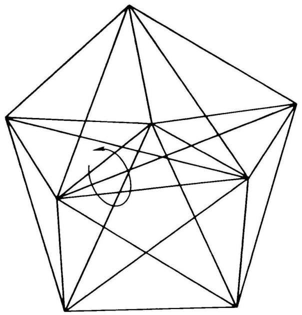
$$V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$\sin\theta_d = \frac{d}{d-1}\,\frac{V_d\,V_{d-2}}{V_{d-1}\,V'_{d-1}}$$

$$\delta_h = 2\pi - \sum_{\substack{\text{d-simplices}\\ \text{meeting on h}}} \theta_d$$



$d = 3$



$d = 4$

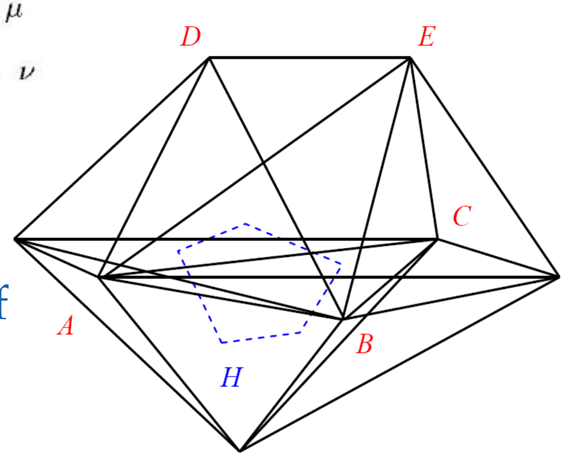
Curvature determined by edge lengths

T. Regge 1961

J.A. Wheeler 1964

Lattice Rotations

$$\phi^\mu(s_{n+1}) = R^\mu{}_\nu(P)\,\phi^\nu(s_1) \qquad R^\mu{}_\nu = \left[P\, e^{\int_{\substack{\text{path}\\ \text{between simplices}}} \Gamma_\lambda dx^\lambda}\right]^\mu{}_\nu$$



Elementary polygonal path around a hinge (triangle) in four dimensions.

$$\mathbf{R}(C) = \mathbf{R}(s_1, s_n) \cdots \mathbf{R}(s_2, s_1)$$

Due to the hinge's intrinsic orientation, only components of the vector in the plane *perpendicular to the hinge* are rotated:

$$U_{\mu\nu}(h) = \mathcal{N}\,\epsilon_{\mu\nu\alpha_1\alpha_{d-2}}\, l^{\alpha_1}_{(1)} \ldots l^{\alpha_{d-2}}_{(d-2)}$$

$$R^\mu{}_\nu(C) = \left(e^{\delta U}\right)^\mu{}_\nu$$

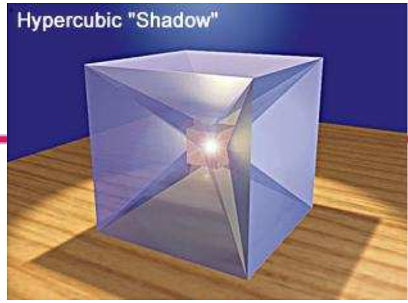
$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)}\, U_{\mu\nu}(h)\, U_{\lambda\sigma}(h)$$

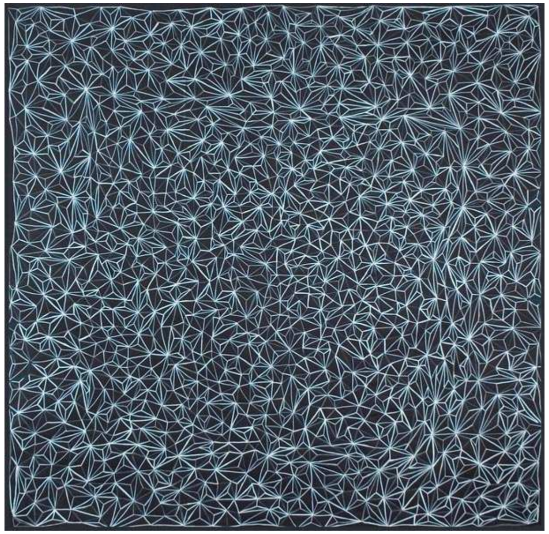
$$R(h) = 2\,\frac{\delta(h)}{A_C(h)}$$

Exact lattice Bianchi identity,

$$\prod_{\substack{\text{hinges h}\\ \text{meeting on edge p}}} \left[e^{\delta(h) U(h)}\right]^\mu{}_\nu = 1$$

Choice of Lattice Structure

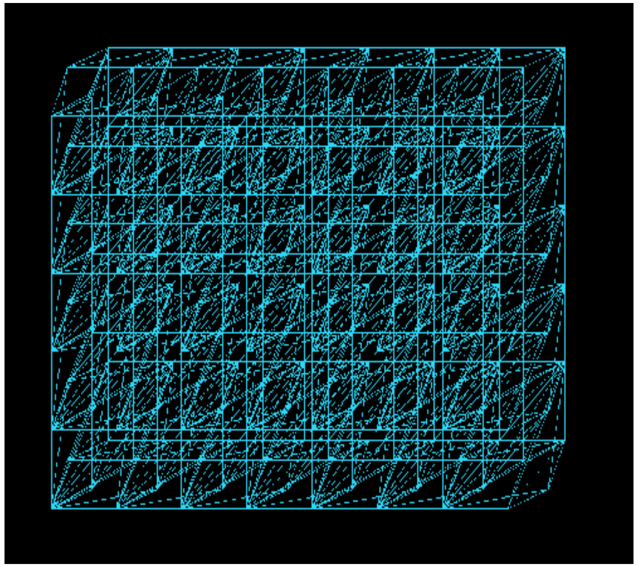




A not so regular lattice …

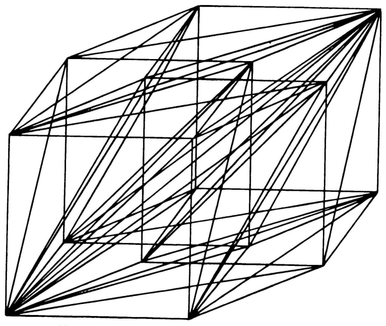
Timothy Nolan,

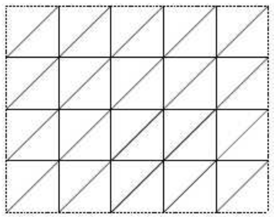
Carl Berg Gallery, Los Angeles

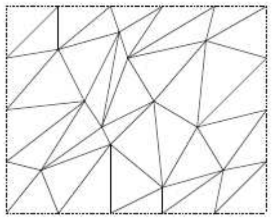


... and a more regular one:

Regular geometric objects (hypercubes) can be *stacked* - to form a regularly coordinated lattice of infinite extent.



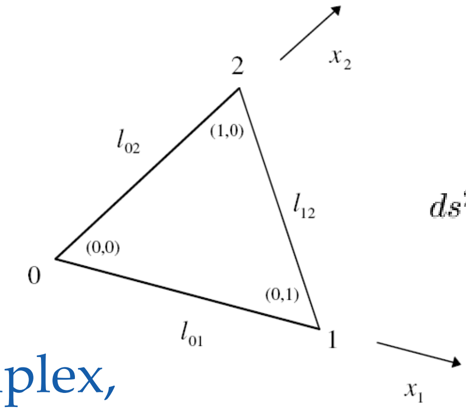




Lattice Measure

Metric deformations linearly related to
squared edge lengths

$$\delta g_{ij}(l^2) \;=\; \tfrac{1}{2}\left(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2\right)$$



$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Jacobian from g's to l's is constant within a simplex,

$$\left(\frac{1}{d!}\sqrt{\det g_{ij}(s)}\right)^{\sigma} \prod_{i\geq j} dg_{ij}(s) \;=\; \left(-\tfrac{1}{2}\right)^{\frac{d(d-1)}{2}} \left[V_d(l^2)\right]^{\sigma} \prod_{k=1}^{d(d+1)/2} dl_k^2$$

CMS, 1982 ; T.D.Lee, 1982
J.Hartle, 1984 ;
H. & Williams, 1984 ; 1998
B. Berg, 1985 .

$$\Longrightarrow \quad \int [d\,l^2] \;=\; \int_0^\infty \prod_s \left(V_d(s)\right)^{\sigma} \prod_{ij} dl_{ij}^2 \; \Theta[l_{ij}^2]$$

Alternatively, can construct the discrete analog of DeWitt's (super) metric over metric deformations, and obtain same result [CMS]…

$$\|\,\delta g(s)\,\|^2 \;=\; \sum_s G^{ijkl}\left(g(s)\right)\; \delta g_{ij}(s)\, \delta g_{kl}(s)$$

Lattice Measure is Non-Trivial

There are important nontrivial constraints on the lattice gravitational measure,

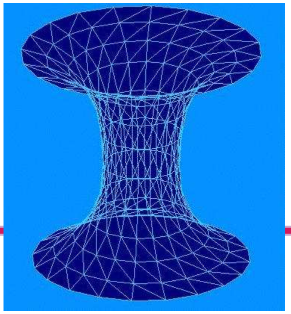
$$\int [d\, l^2] \; = \; \int_0^\infty \prod_s \, (V_d(s))^\sigma \, \prod_{ij} dl_{ij}^2 \; \Theta[l_{ij}^2]$$

which is generally subject to the *"triangle inequality constraints"* :

$$\begin{cases} l_{ij}^2 > 0 \\ V_k^2 = \left(\frac{1}{k!} \right)^2 \det g_{ij}^{(k)}(s) > 0 \qquad k = 2 \ldots d \end{cases}$$

Generally these are implied in the *continuum* functional measure as well, but are normally not spelled out in detail …

Lattice Path Integral



Lattice path integral follows from edge assignments,

$$g_{ij} = \frac{1}{2}\left(l^2_{1,i+1} + l^2_{1,j+1} - l^2_{i+1,j+1}\right) \qquad V_d = \frac{1}{d!}\sqrt{\det g_{ij}}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g} R \longrightarrow I_L = \lambda_0 \sum_h V_h(l^2) - 2\kappa_0 \sum_h \delta_h(l^2) A_h(l^2)$$

$$Z = \int [d\, g_{\mu\nu}]\; e^{-\lambda_0 \int d^d x \sqrt{g} + \frac{1}{16\pi G} \int d^d x \sqrt{g} R} \longrightarrow Z_L = \int [d\, l^2]\; e^{-I_L[l^2]}$$

$$\int [d\, g_{\mu\nu}] = \int \prod_x [g(x)]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \longrightarrow \int [d\, l^2] \equiv \int_0^\infty \prod_{ij} dl^2_{ij} \prod_s [V_d(s)]^\sigma\; \Theta(l^2_{ij})$$

Without loss of generality, one can set bare $\lambda_0 = 1$;

Besides the cutoff, the only relevant coupling is κ (or G).

Alternate Lattice Actions

$$\sqrt{g}\,(x) \quad \to \quad \sum_{\text{hinges } h \supset x} V_h$$

$$\sqrt{g}\,R(x) \quad \to \quad 2 \sum_{\text{hinges } h \supset x} \delta_h A_h$$

$$\sqrt{g}\,R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}(x) \quad \to \quad 4 \sum_{\text{hinges } h \supset x} (\delta_h A_h)^2 / V_h$$

More than one way to *finite-difference* a continuum expression…

- Alternate actions can be a useful device for analytical estimates (i.e. large d)
- Should exhibit same continuum limit (universality)

$$I_R(l^2) \;=\; -\,k \sum_{\text{hinges h}} \delta(h)\, V^{(d-2)}(h) \qquad\qquad k = 1/(8\pi G)$$

$$I_{\text{com}}(l^2) \;=\; -\,k \sum_{\text{hinges h}} \tfrac{1}{2}\,\omega_{\alpha\beta}(h)\, R^{\alpha\beta}(h) \;\to\; \sin\delta_p$$

hinge bivector

rotation matrix

J. Fröhlich 1980

T.D. Lee 1984

Caselle, d'Adda Magnea 1989

Lattice Higher Derivative Terms

- **HDQG** is perturbatively renormalizable, asymptotically free, but contains *s=0* and *s=2* ghosts.

$$\int d^4x \sqrt{g}\, R^2$$

$$\int d^4x \sqrt{g}\, R_{\mu\nu} R^{\mu\nu}$$

$$\int d^4x \sqrt{g}\, R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$$

$$\int d^4x \sqrt{g}\, C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma}$$

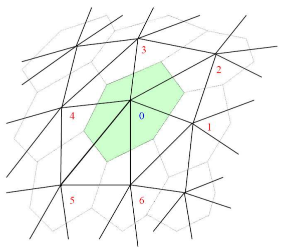
$$\int d^4x \sqrt{g}\, \epsilon^{\mu\nu\kappa\lambda} \epsilon^{\rho\sigma\omega\tau}\, R_{\mu\nu\rho\sigma} R^{\kappa\lambda\omega\tau} = 128\pi^2 \chi$$

$$\int d^4x \sqrt{g}\, \epsilon^{\rho\sigma\kappa\lambda}\, R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\kappa\lambda} = 96\pi^2 \tau$$

$$k < h_{\mu\nu}(q)\, h_{\rho\sigma}(-q) > = \; 2P^{(2)}_{\mu\nu\rho\sigma} \left[\frac{1}{q^2} - \frac{1}{q^2 + \frac{k}{a}} \right] + P^{(0)}_{\mu\nu\rho\sigma} \left[-\frac{1}{q^2} + \frac{1}{q^2 + \frac{k}{2b}} \right]$$

- *Lattice* higher derivative terms

  … involve deficit angles *squared*, as well as coupling between hinges,

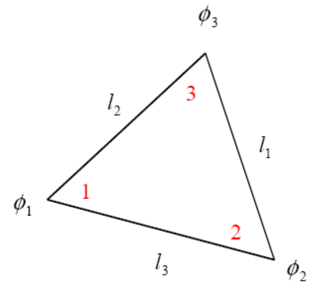


$$\frac{1}{4} \int d^d x \sqrt{g}\, R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \;\to\; \sum_{\text{hinges h}} V_h \left(\frac{\delta_h}{A_{C_h}} \right)^2$$

$$\int d^d x \sqrt{g}\, C_{\mu\nu\lambda\sigma} C^{\mu\nu\lambda\sigma} \sim \qquad \frac{2}{3} \sum_s V_s \sum_{h,h' \subset s} \epsilon_{h,h'} \left(\omega_h \left[\frac{\delta}{A_C} \right]_h - \omega_{h'} \left[\frac{\delta}{A_C} \right]_{h'} \right)^2$$

Scalar Matter

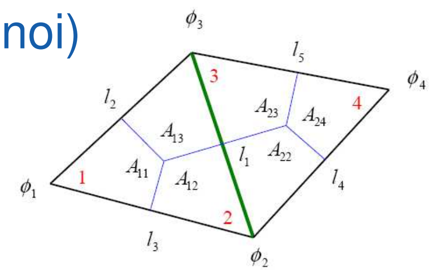
Make use of *lattice metric* to correctly define lattice field derivatives [Itzykson & Drouffe 1984; Ninomiya 1985] …



$$\begin{aligned} g_{\mu\nu}(x) &\longrightarrow g_{ij}(\Delta) \\ \det g_{\mu\nu}(x) &\longrightarrow \det g_{ij}(\Delta) \\ g^{\mu\nu}(x) &\longrightarrow g^{ij}(\Delta) \\ \partial_\mu\phi\,\partial_\nu\phi &\longrightarrow \Delta_i\phi\,\Delta_j\phi \end{aligned}$$

$$g_{ij}(\Delta) = \begin{pmatrix} l_3^2 & \frac{1}{2}(-l_1^2+l_2^2+l_3^2) \\ \frac{1}{2}(-l_1^2+l_2^2+l_3^2) & l_2^2 \end{pmatrix}$$

… and obtain a simple geometric form, involving dual (Voronoi) volumes



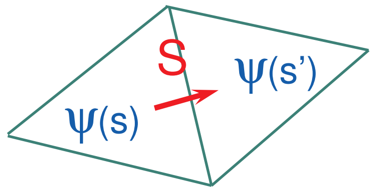
$$I(l^2,\phi) = \frac{1}{2}\sum_{<ij>} V_{ij}^{(d)}\left(\frac{\phi_i-\phi_j}{l_{ij}}\right)^2$$

$$\longrightarrow \qquad I[g,\phi] = \frac{1}{2}\int dx\,\sqrt{g}\,[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + (m^2+\xi R)\phi^2] + \ldots$$

…which also allows correct definition of *lattice Laplacian*: $G_{ij}(l^2) = \left[\frac{1}{-\Delta(l^2)+m^2}\right]_{ij}$

Fermionic Matter

Start from continuum Dirac action



$$I = \int dx \sqrt{g}\, \bar{\psi}(x)\, \gamma^{\mu}\, D_{\mu}\, \psi(x)$$

$$\{\gamma^{\mu}(s), \gamma^{\nu}(s)\} = 2\, g^{\mu\nu}(s)$$

$$D_{\mu} = \partial_{\mu} + \tfrac{1}{2}\omega_{\mu a b}\sigma^{ab} \qquad \sigma_{ab} = \frac{1}{2i}[\gamma_a, \gamma_b]$$

Discrete action *[Drummond 1986]* involves *lattice spin connection :*

$$I = \tfrac{1}{2} \sum_{\text{faces f(ss}')} V(f(s,s'))\, \bar{\psi}_s\, \mathbf{S}(\mathbf{R}(s,s'))\, \gamma^{\mu}(s')\, n_{\mu}(s,s')\, \psi_{s'}$$

Potential problems with fermion doubling (as in ordinary LGT)…

Lattice Weak Field Expansion

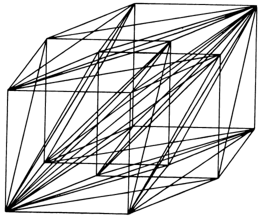
- Exhibits correct nature of gravitational degrees of freedoms in the *lattice* weak field limit.
- Allows *clear connection between* ***lattice*** *and* ***continuum*** *operators.*

… start from Regge lattice action

Roĉek and Williams, PLB 1981
CMS 1983, T.D. Lee 1984

$$- 2\,\kappa_0 \sum_h \delta_h(l^2)\, A_h(l^2)$$

… call small edge fluctuations "*e*" :



$$I_R = \frac{1}{2} \sum_{ij} e_i\, M_{ij}\, e_j$$

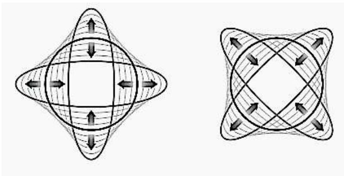
… then Fourier transform, and express result in terms of metric

deformations :

$$\delta g_{ij}(l^2) \;=\; \tfrac{1}{2}\left(\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2\right)$$

… obtaining in the vacuum gauge precisely the familiar *TT form* in k→0 limit:

$$\frac{1}{4}\mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k})\; h_{ij}^{TT}(\mathbf{k})$$



Wilson Loop vs. Loop correlations

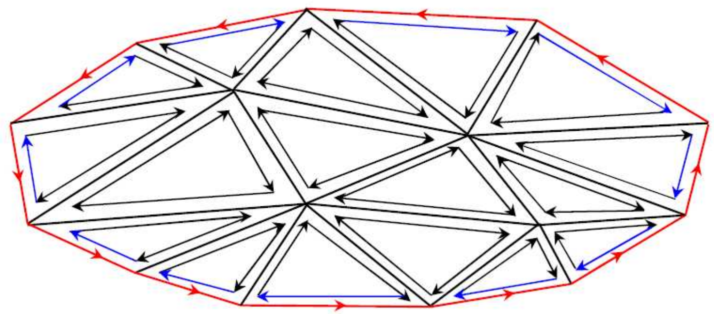


FIG. 2 (color online). Gravitational analog of the Wilson loop.

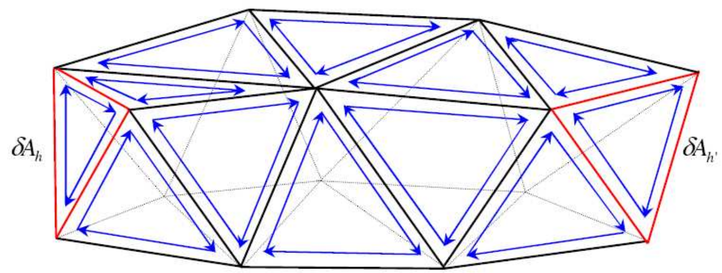


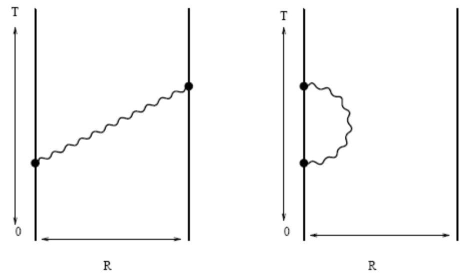
FIG. 6 (color online). Correlations between action contributions on hinge h and hinge h' arise to lowest order in the strong

G..Modanese PRD,NPB 1995

$$R^{\alpha}{}_{\beta}(C) = \left[\mathcal{P} \exp\left\{ \oint_{\textbf{path C}} \Gamma^{\cdot}_{\lambda \cdot} dx^{\lambda} \right\} \right]^{\alpha}{}_{\beta} .$$

$$G_R(d) \sim < \sqrt{g}\, R(x)\, \sqrt{g}\, R(y)\, \delta(|x-y|-d) >_c$$

Wilson Loop does *not* give Potential



In ordinary LGT, Wilson loop gives ***V(r)***

$$W(\Gamma) = < \exp\left\{ ie \oint_\Gamma A_\mu(x) dx^\mu \right\} >$$

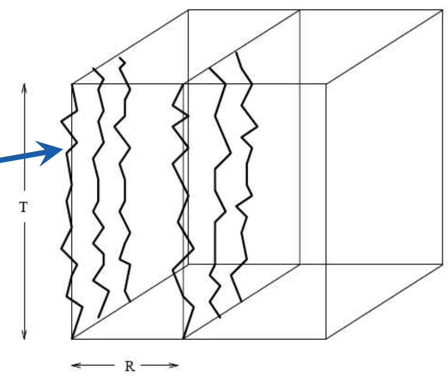
$$V(R) = -\lim_{T \to \infty} \frac{1}{T} \log < \exp\left\{ ie \oint_\Gamma A_\mu dx^\mu \right\} >$$

In lattice regularized gravity, potential is computed from the correlation of geodesic *line segments, associated with the particle's world line:*

G. Modanese, PRD 1994; NPB 1995

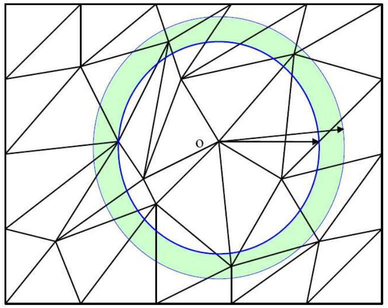
$$\mu \int_{\tau(a)}^{\tau(b)} d\tau \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

$$L(0;\ \mu_1) = \exp\left\{ -\mu_1 \int d\tau \sqrt{g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \right\}$$



Correlations

… of invariant operators at fixed *geodesic* distance.



Distance is a function of metric, which fluctuates:

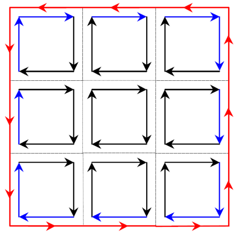
$$d(x,y\,|\,g) \;=\; \min_{\xi} \int_{\tau(x)}^{\tau(y)} d\tau\, \sqrt{g_{\mu\nu}(\xi)\frac{d\xi^\mu}{d\tau}\frac{d\xi^\nu}{d\tau}}$$

$$< \int dx \int dy\, \sqrt{g}R(x)\, \sqrt{g}R(y)\, \delta(|x-y|-d) >$$

$$\longrightarrow \quad G_R(d) \equiv < \sum_{h\supset x} \delta_h A_h \sum_{h'\supset y} \delta_{h'} A_{h'}\, \delta(|x-y|-d) >_c$$

$$G(x,y|g) = < x\,|\frac{1}{-\Delta(g)+m^2}|\,y > \underset{d(x,y)\to\infty}{\sim} d^{-(d-1)/2}(x,y)\exp\{-m\,d(x,y)\}$$

Hypercubic Lattice Gravity



- Flat hypercubic lattice - geometric features not manifest
  e.g. Mannion &Taylor PLB 1982 ; see also Smolin 1978; Das Kaku Townsend 1982.
- Lattice discretization of the Cartan theory based on SL(2,C) ➔ SO(3,1) ➔ SO(4)

$$U_\mu(n) = \Big[U_{-\mu}(n+\mu)\Big]^{-1} = \exp[i\,B_\mu(n)] \qquad B_\mu = \tfrac{1}{2} a B_\mu^{ab}(n) J_{ba} \qquad \sigma_{ab} = \frac{1}{2i}[\gamma_a, \gamma_b]$$

$$E_\mu(n) = a\, e_\mu{}^a \gamma_a$$

Local *gauge* invariance:

$$U_\mu \to \Lambda(n)\, U_\mu(n)\, \Lambda^{-1}(n+\mu) \qquad E_\mu(n) \to \Lambda(n)\, E_\mu(n)\, \Lambda^{-1}(n)$$

$$I = \frac{i}{16\kappa^2} \sum_{n,\mu,\nu,\lambda,\sigma} \mathrm{tr}[\gamma_5\, U_\mu(n) U_\nu(n+\mu) U_{-\mu}(n+\mu+\nu) U_{-\nu}(n+\nu)\, E_\sigma(n)\, E_\lambda(n)]$$

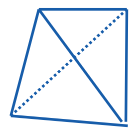
$$\rightarrow \quad I = \frac{1}{4\kappa^2} \int d^4x\, \epsilon^{\mu\nu\lambda\sigma}\, \epsilon_{abcd}\, R_{\mu\nu}{}^{ab}\, e_\lambda{}^c\, e_\sigma{}^d$$

Path integral over U's (Haar) and E's:

$$Z = \int \prod_{n,\mu} dB_\mu(n) \prod_{n,\sigma} dE_\sigma(n) \exp\Big\{-I(B,E)\Big\}$$

Dynamical Triangulations

- Simplified version of Regge Gravity
- Edge lengths *fixed to unity*, vary incidence matrix [David 1984, ...]



an integer

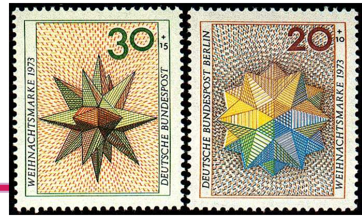
$$V_d = \frac{1}{d!}\sqrt{\frac{d+1}{2^d}} \qquad \cos\theta_d = \frac{1}{d} \qquad \delta(h) = 2\pi - n_s(h)\,\theta_d$$

- No immediate notion of continuous metric, or continuous diffeos.
- Curvature varies in discrete steps.
- No continuous metric deformations – hence no w.f.e., and no gravitons (at least not in an explicit way).

Constraints on functional measure unclear, since theory has no explicit metric.

Pathological behavior of Euclidean theory [Loll et al] ⟶ explore numerical Lorentzian path integral (with yet unresolved convergence issues).

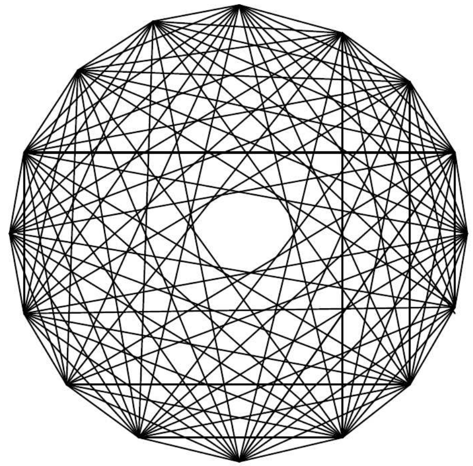
Large D Limit



Early work in *continuum* by A. Strominger (1984, *λ=0*), ...

On the lattice, phase transition persists at d = ∞.

$$
\left\{
\begin{aligned}
k_c &= \frac{\lambda_0^{\frac{d-2}{d}}}{d^3} \left[\frac{2}{d} \, \frac{d! \, 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \\
l_0^2 &= \frac{1}{\lambda_0^{2/d}} \left[\frac{2}{d} \, \frac{d! \, 2^{d/2}}{\sqrt{d+1}} \right]^{2/d}
\end{aligned}
\right.
$$



N-cross polytope, homeomorphic to a sphere

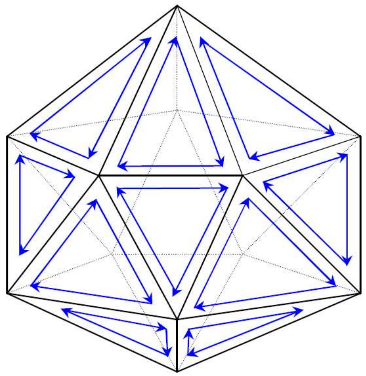
- Conformal mode instability disappears, *O(1/d).*
- At large d, partition function at large *G* dominated by *closed surfaces*, tiled with elementary parallel transport polygonal loops.

  Very large surfaces are important as k → k_c .

H & Williams, PRD 2006

Large D Limit - Exponent ν

- At large d, *characteristic size* ξ of random surface diverges *logarithmically* as $G \to G_c$ (D. Gross PLB 1984).
- Suggests universal correlation length exponent $\nu = 0$.



Known results from random surface theory then imply:

$$\xi \sim \sqrt{\log T} \underset{k \to k_c}{\sim} |\log(k_c - k)|^{1/2}$$

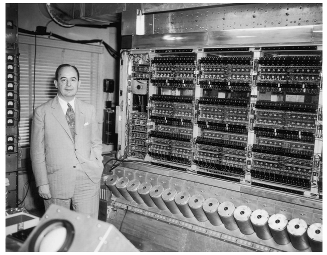
$$\nu = 1/(d-1) \longleftrightarrow \nu = 1/2d$$

Lattice

D. Litim PRL 2004, PLB 2007

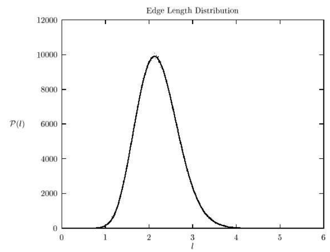
scalar field	$\nu = \frac{1}{2}$
lattice gauge field	$\nu = \frac{1}{4}$
lattice gravity	$\nu = 0$

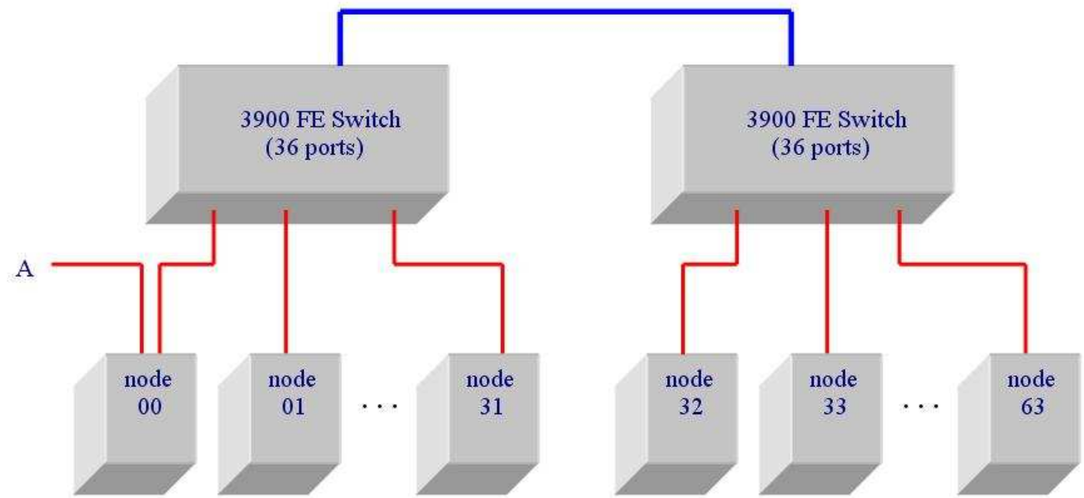
Numerical Evaluation of Z

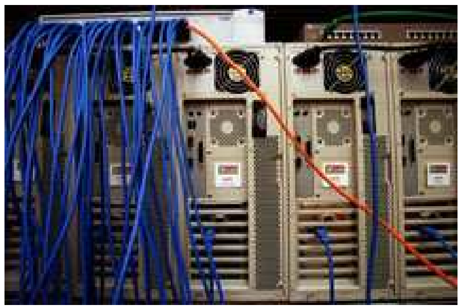


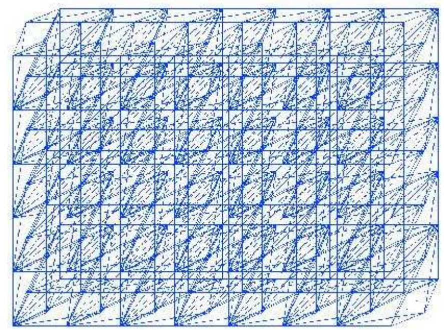
CM5 at NCSA, 512 processors

Dedicated Parallel Supercomputer

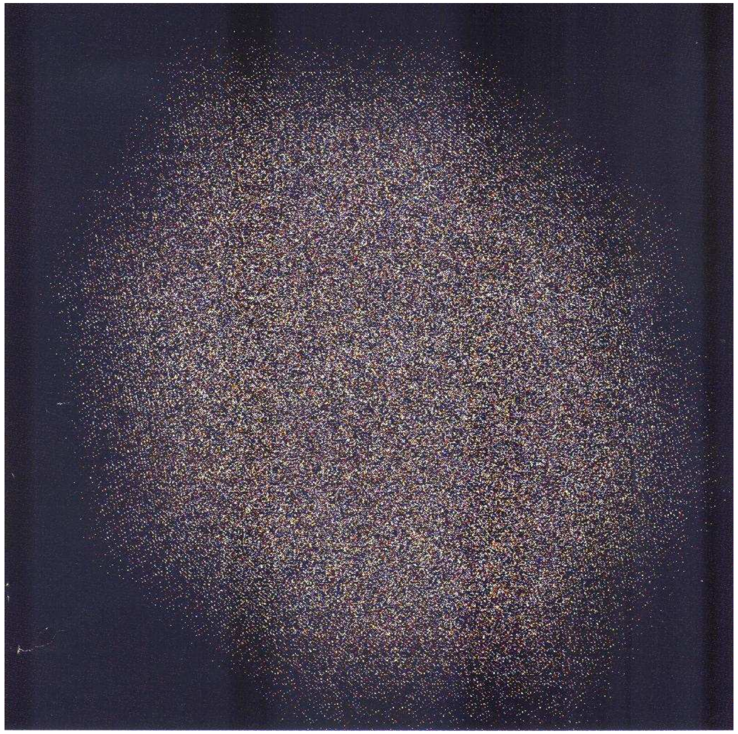




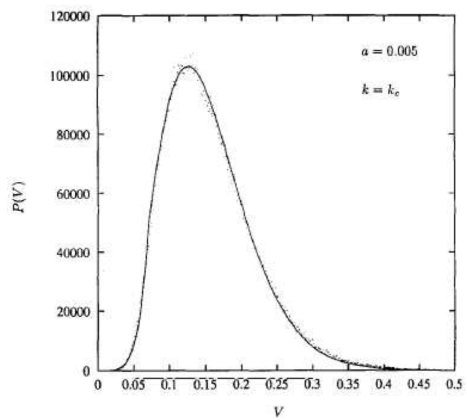


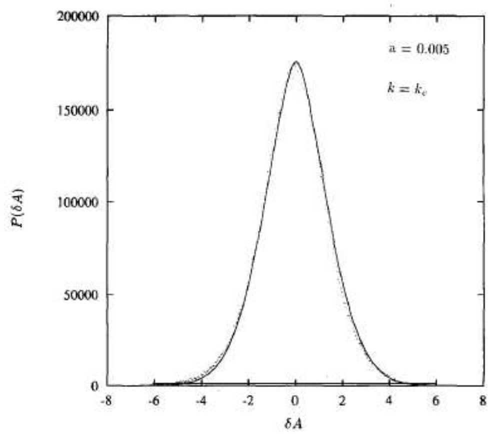


Edge length/metric distributions



- L=4 → 6,144 simplices
- L=8 → 98,304 simplices
- L=16 → 1,572,864 simplices
- L=32 → 25,165,824 simplices



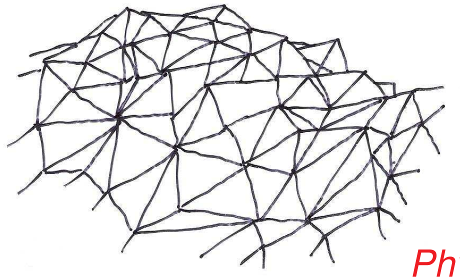


Two Phases of L. Quantum Gravity

Earliest studies of Regge lattice theories found evidence for :

$G > G_c$ Smooth phase: R ≈ 0

$$\langle g_{\mu\nu} \rangle \approx c\, \eta_{\mu\nu}$$

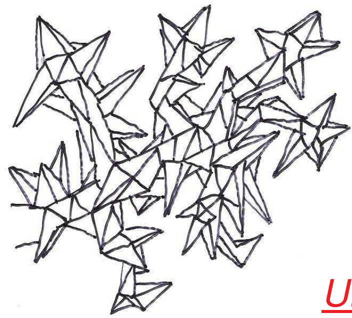


Physical

$$N(\tau) \underset{\tau \to \infty}{\sim} \tau^{\, d_v}$$

$G < G_c$ Rough phase : branched polymer, d ≈ 2

$$\langle g_{\mu\nu} \rangle = 0$$



Unphysical

Lattice manifestation of conformal instability

Similar two-phase structure also found later in some d=4 DTRS models [Migdal, …]

Invariant Averages

$$\mathcal{R}(k) \sim \frac{< \int d^4x \sqrt{g}\, R(x) >}{< \int d^4x \sqrt{g} >}$$

$$\chi_{\mathcal{R}}(k) \sim \frac{< (\int \sqrt{g}\, R)^2 > - < \int \sqrt{g}\, R >^2}{< \int \sqrt{g} >}$$

$$\mathcal{R}(k) \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z_L$$

$$\chi_{\mathcal{R}}(k) \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z_L$$

Singularities in the free energy *F* are determined from non-analiticities in invariant local averages.

- Divergent local averages provide information about non-trivial *exponents.*
- Finite Size Scaling (FSS) theory useful.

$$O(L,t) = L^{x_O/\nu} \left[\tilde{f_O}(L\, t^\nu) + \mathcal{O}(L^{-\omega}) \right]$$

- *Correlations* are harder to compute directly (geodesic distance).

$$F_{sing}(G) \sim \xi^{-d}$$

$$\xi \sim t^{-\nu}$$

"Scaling assumption" for F = ln Z

Determination of Scaling Exponents

$$\mathcal{R}(k) \underset{k \to k_c}{\sim} -A_{\mathcal{R}}\,(k_c - k)^{\delta} \qquad \nu = \frac{1+\delta}{d}$$

$$\chi_{\mathcal{R}}(k) \underset{k \to k_c}{\sim} -A_{\mathcal{R}}\,(k_c - k)^{-(1-\delta)}$$

Scaling assumption: $F_{sing}(G) \sim \xi^{-d}$

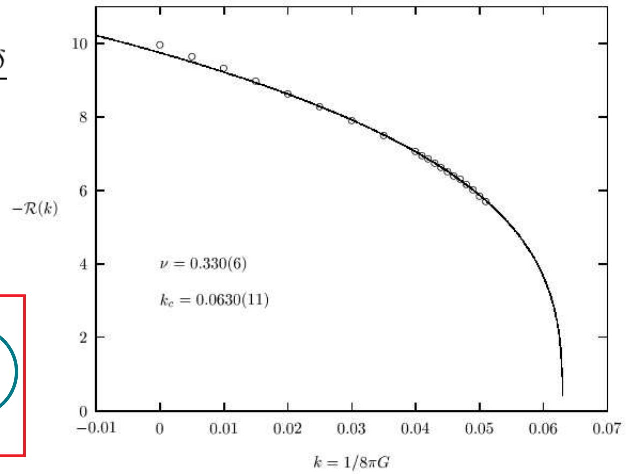
$$\xi(k) \equiv m(k)^{-1} \underset{k \to k_c}{\sim} A_{\xi}\,(k_c - k)^{-\nu}$$

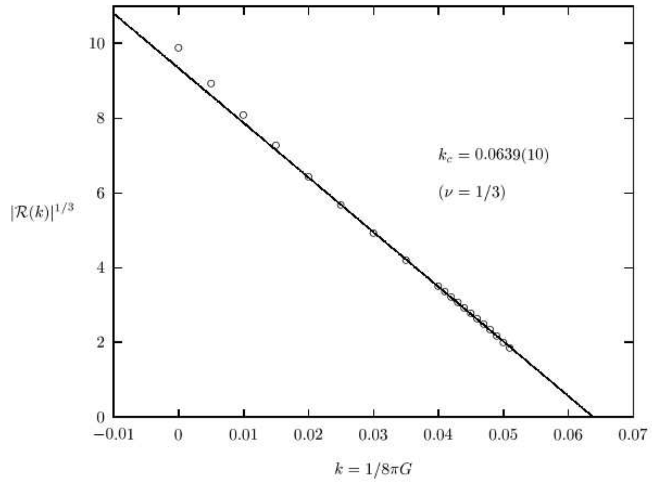
Find value close to 1/3:

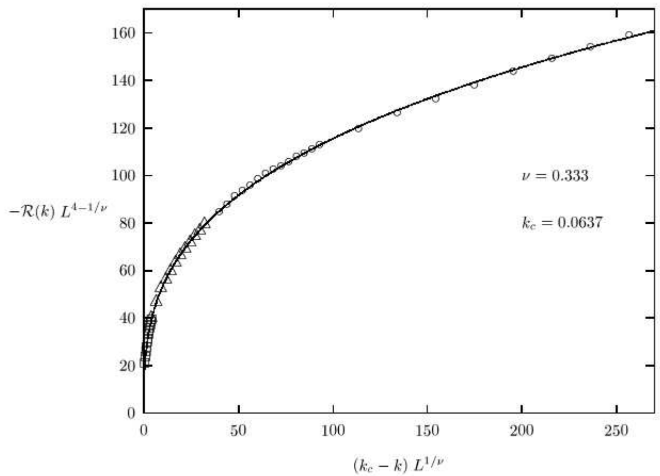
$$k_c = 0.0636(11) \qquad \nu = 0.335(9)$$

$$\mathcal{R}(\xi) \underset{k \to k_c}{\sim} \frac{1}{l_P\,\xi}$$

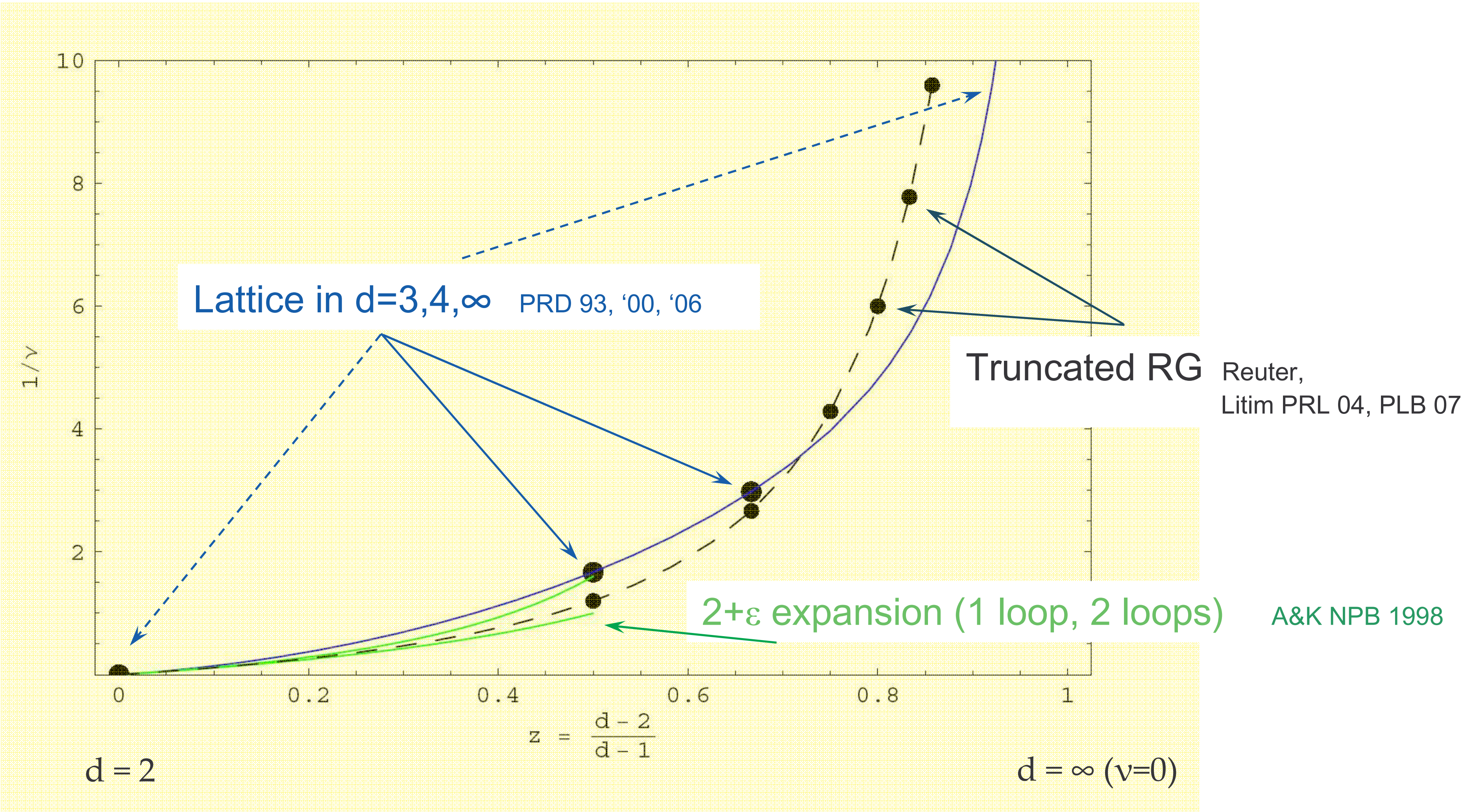
$$\nu \approx ⅓$$





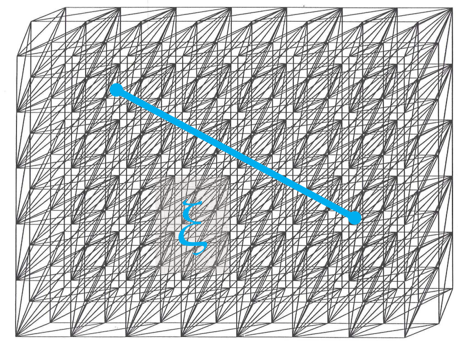


Exponent ν compared



(Lattice) Continuum Limit $\Lambda \to \infty$

Standard (Wilson) procedure in cutoff field theory:



$$\xi = 1/m \qquad\qquad m \underset{G(\Lambda)\to G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0 \, G_c} \right]^{\nu}$$

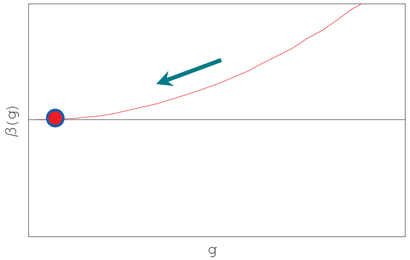
RG *invariant* correlation length ξ is kept fixed

UV cutoff $\Lambda \to \infty$ (average lattice spacing → 0)

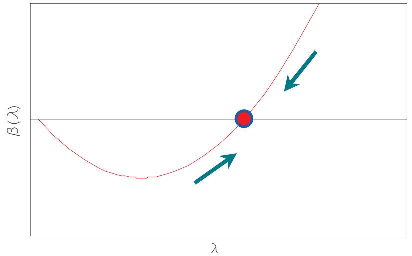
Bare G *must* approach UV fixed point at Gc .

The *very same* relation gives the RG running of G(μ) close to the FP.

RG Running Scenarios

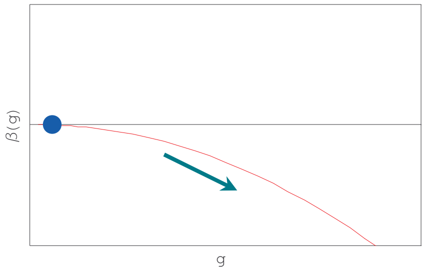


“Triviality” of lambda phi 4

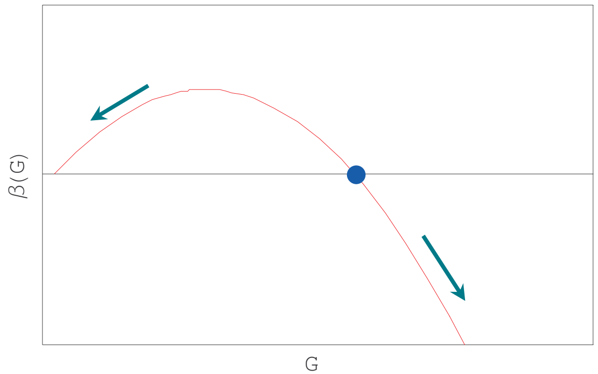


Wilson-Fisher FP in d<4

- Coupling gets *weaker at large r*
- … approaches an IR FP at large r.
- … gets *weaker at small r* : UV FP
- Both possibilities can coexist: nontrivial UV fixed point.



Asymptotic freedom of YM



Ising model, σ-model, Gravity (2+ε, lattice)

Callan-Symanzik. beta function(s):

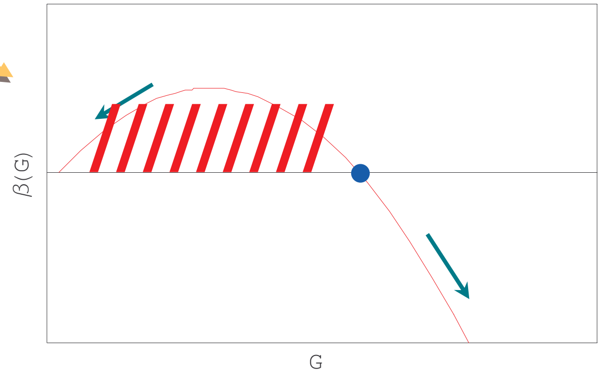
$$\mu \frac{\partial}{\partial \mu} G(\mu) \;=\; \beta(G(\mu))$$

Only One Phase?

Weak coupling phase is seemingly unphysical (branched polymer).

✓ Lattice results appear to exclude the weak coupling phase as physically relevant…

✓ Leads to a gravitational coupling G that *increases* with distance…

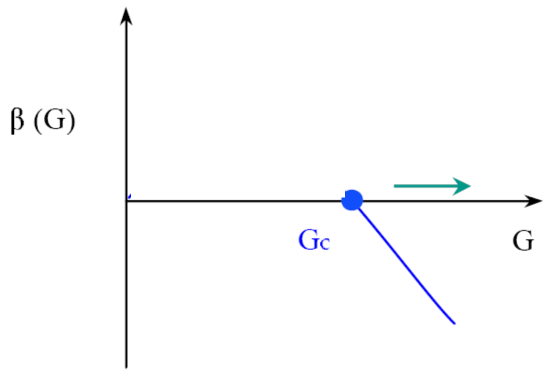


Anti-screening phase.

Gravity

New question then :

Is this new scenario physically acceptable?



Running Newton's G

- ξ is a *new invariant scale of gravity*. $m \equiv \xi^{-1} = \Lambda\, F(G)$
- Newton's constant *G must* run (as in 2+ε).
- *Cutoff* dependence determines β-function :

$$\Lambda \frac{d}{d\,\Lambda} m(\Lambda, G(\Lambda)) \; = \; 0 \quad \text{and} \quad \Lambda \frac{\partial}{\partial\,\Lambda} G(\Lambda) \; = \; \beta(G(\Lambda)) \longrightarrow \beta(G) \; = \; - \frac{F(G)}{\partial F(G) / \partial G}$$

[In fact, one can be quite specific …

$$\beta(G) \underset{G \to G_c}{\sim} -\frac{1}{\nu}(G - G_c) \quad \Longleftrightarrow \quad m \sim \Lambda \exp\left(- \int^G \frac{dG'}{\beta(G')} \right) \underset{G \to G_c}{\sim} \Lambda\, |\, G - G_c|^{-1/\beta'(G_c)}$$]

Running of *G* det. largely by ξ and ν :

$$\mu \frac{\partial}{\partial\,\mu} G(\mu) \; = \; \beta(G(\mu)) \Longrightarrow G(k^2) \; = \; G_c \left[1 \, + \, a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} \, + \, O((m^2/k^2)^{\frac{1}{\nu}}) \right]$$

So, what value to take for ξ ?

- ξ is an RG invariant.
- $m=1/\xi$ has dimensions of a mass.

In Yang-Mills m = glueball mass

Three Theories Compared

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = 4\pi e j_\nu$$

$$\partial^\mu \partial_\mu \phi + m^2 \phi = \frac{g}{3!} \phi^3$$

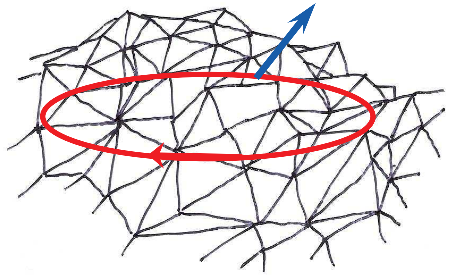
Suggests $\lambda_{phys} \simeq \frac{1}{\xi^2}$

RG invariants

Running couplings

$m = 1/\xi$

Gravitational Wilson Loops



- In gravity, Wilson loop not related to static potential
  [G. Modanese PRD 1993; PRD 1994]
- Parallel transport of a vector done via lattice rotation matrix

$$\mathbf{R}^{\alpha}{}_{\beta} = \left[\mathcal{P}\, e^{\int_{\text{between simplices}}^{\text{path}} \Gamma^{\lambda} dx_{\lambda}} \right]^{\alpha}_{\beta}$$

For a *large* closed circuit obtain *Wilson loop* - which can be computed at strong coupling using a first order formulation of Regge gravity [Caselle, d'Adda, Magnea PLB 1989]

$$W(\Gamma) \sim \operatorname{Tr} \mathcal{P} \exp \left[\int_C \Gamma^{\lambda}{}_{..}\, dx_{\lambda} \right] \sim \exp \left[\int_{S(C)} R^{\cdot}{}_{\cdot\,\mu\nu} A^{\mu\nu}_{C} \right] \sim \exp(-A/\xi^2)$$

- Stokes theorem -

- ξ related to curvature.
- ξ RG invariant.
- prediction of positive cosmological constant?

$$\lambda_{phys} \simeq \frac{1}{\xi^2}$$

"Area law"
would follows from loop tiling …
HH&R.M.Williams, PRD 76, 2007

Vacuum Condensate Picture of QG?

- Lattice Quantum Gravity: Curvature condensate *See also J.D.Bjorken, PRD '05*

$$\mathcal{R} \simeq (10^{-30} eV)^2 \sim \xi^{-2} \qquad \lambda_{phys} \simeq \frac{1}{\xi^2}$$

- Quantum Chromodynamics: Gluon and Fermion condensate

$$\alpha_S < F_{\mu\nu} \cdot F^{\mu\nu} > \simeq (250 MeV)^4 \sim \xi^{-4}$$

$$\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$$

$$(\alpha_S)^{4/\beta_0} < \bar{\psi} \psi > \simeq -(230 MeV)^3 \sim \xi^{-3}$$

- Electroweak Theory: Higgs condensate

Effective Theory

Graviton Vacuum Polarization Cloud

Picture: Source mass M surrounded by *virtual graviton cloud*

Need a *covariant* running of G.

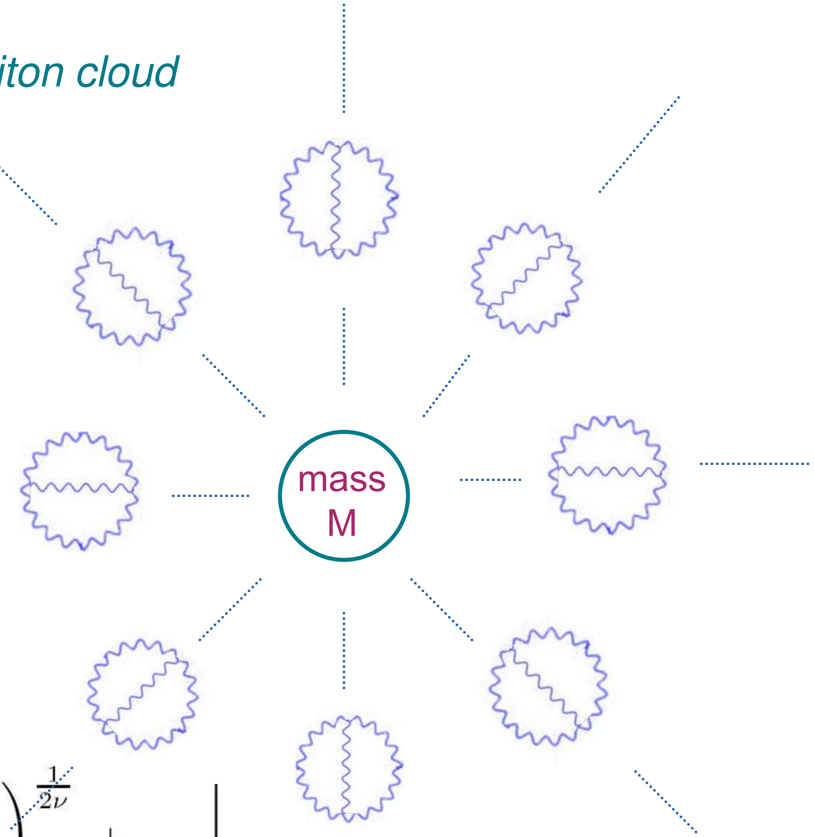
Effective field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu}$$

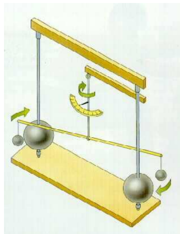
$$\lambda \simeq 1/\xi^2$$

$$G \rightarrow G(\Box) = G \left[1 + a_0 \left(\frac{m^2}{-\Box + m^2} \right)^{\frac{1}{2\nu}} + \dots \right]$$

$$\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$$



Relative Scales in the Cutoff Theory

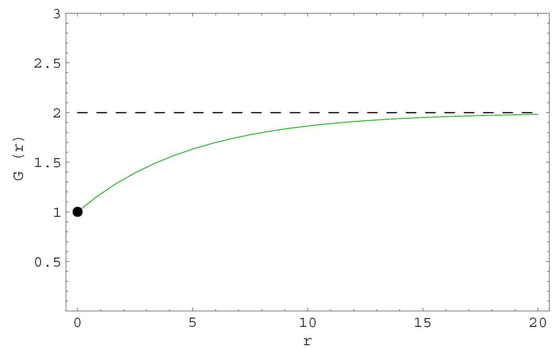


$\approx 10^{-33}\,cm$ $\approx 10^{28}\,cm$

l_P ξ

$$l_P << r << \xi$$

$$r >> \xi$$



At the Planck length new terms appear:

higher derivative terms, string corrections, conformal anomaly contributions…

Cosmological Solutions

Explore possible effective field equations…generally covariant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}$$

G. Veneziano
G.A. Vilkovisky ..

$$A(\Box) = c_\Box \left(\frac{1}{\xi^2 \Box}\right)^{1/2\nu}$$

$\Lambda = 0$ initially for simplicity

… for RW metric

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - k\, r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2 \right) \right\}$$

… and perfect fluid $p(t) = 0$

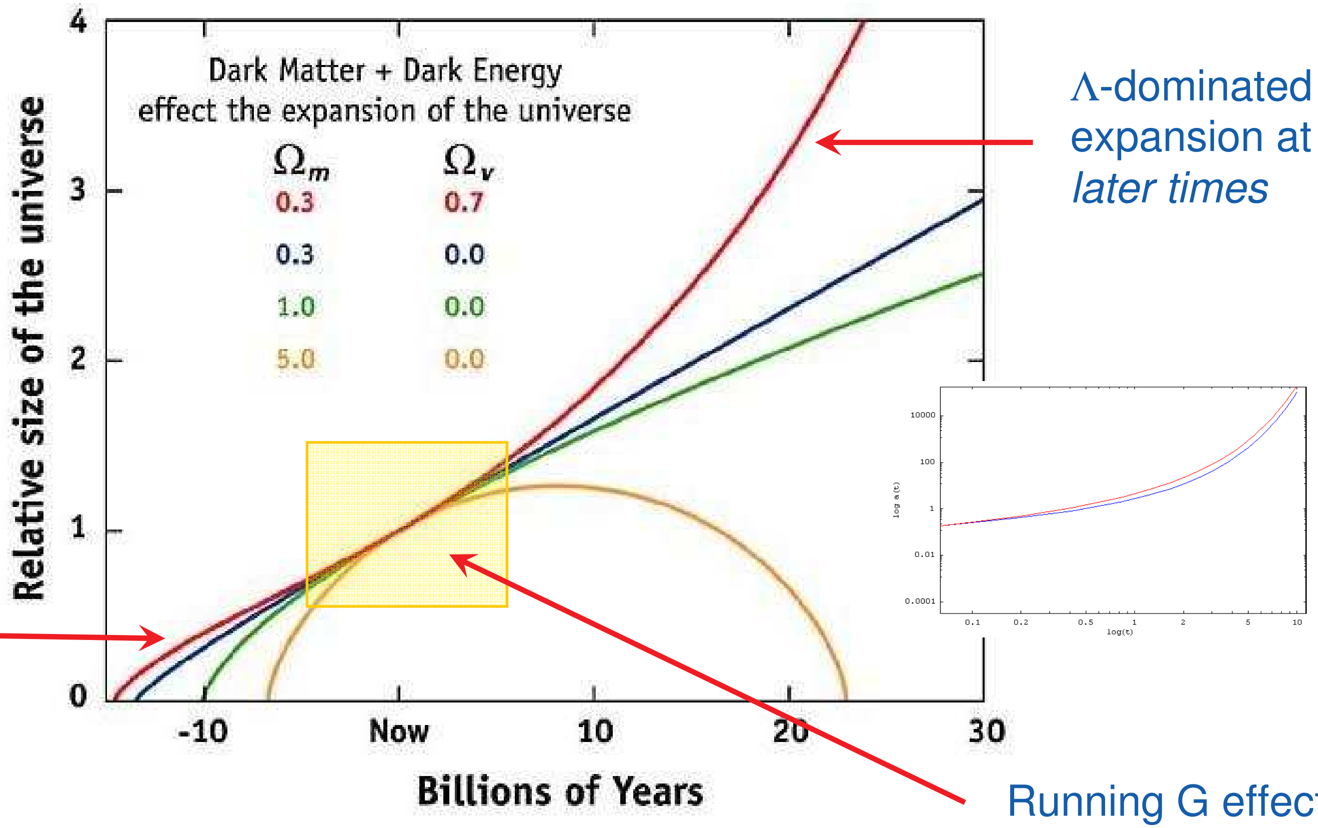
Consistency condition:

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu \left[\left(1 + A(\Box)\right) T_{\mu\nu} \right] = 0$$

$$\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \qquad \Box\, T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} = g^{\mu\nu} \nabla_\mu \left(\nabla_\nu\, T^{\alpha\beta\ldots}{}_{\gamma\delta\ldots} \right)$$

Form of D’Alembertian depends on object it acts on …

Modified cosmological expansion rate



Λ-dominated expansion at *later times*

Standard FRW expansion at *early times*

Running G effects are maximal *“now”*
[T. D’Amour 2007]

Static Isotropic Solution

Start again from *fully covariant* effective field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G \left(1 + A(\Box)\right) T_{\mu\nu}$$

$$A(\Box) = a_0 \left(\frac{m^2}{-\Box + m^2} \right)^{1/2\nu}$$

General *static isotropic* metric

$$\lambda \simeq 1/\xi^2 \longrightarrow 0$$

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$A(r)^{-1} = 1 - \frac{2MG}{r} + \frac{\sigma(r)}{r}$$

$$B(r) = 1 - \frac{2MG}{r} + \frac{\theta(r)}{r}$$

a_0 small

$r \gg 2MG$

Search solution for a point source, or *vacuum solution* for $r \neq 0$.

$$T_{\mu\nu} = \mathrm{diag}\left[B(r)\,\rho(r),\ A(r)\,p(r),\ r^2\,p(r),\ r^2 \sin^2\theta\, p(r) \right]$$

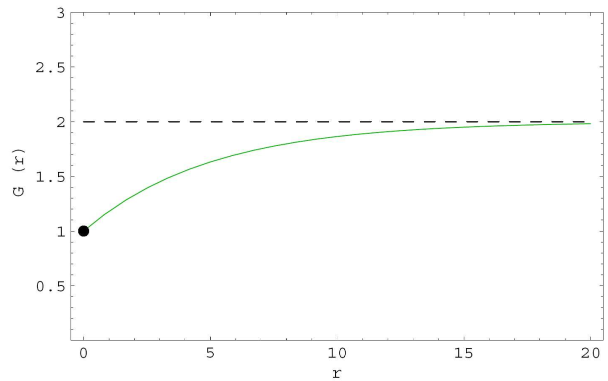
H. & Williams, PLB 2006; PRD 2007

Relativistic Fluid cont'd

And finally …

$$B(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + ..$$

$$A^{-1}(r) = 1 - \frac{2\,M\,G}{r} + \frac{4\,a_0\,M\,G\,m^3}{3\,\pi}\,r^2\,\ln(m\,r) + ..$$



…which can be *consistently* interpreted as a ***G(r):***

$$G \;\rightarrow\; G(r) \;=\; G\left(1 + \frac{a_0}{3\,\pi}\,m^3\,r^3\,\ln\frac{1}{m^2\,r^2} + \ldots\right)$$

$$m = 1/\xi$$

Reminiscent of QED (Uehling) answer:

$$Q(r) = 1 + \frac{\alpha}{3\,\pi}\,\ln\frac{1}{m^2\,r^2} + \ldots \qquad m\,r \ll 1$$

$a_0 \simeq 42.$

Outlook

- More Work is Needed
  - *2 + ε* expansion to three loops is a clear, feasible goal.
  - Careful investigation of 4d s. gravity should be pursued.
  - Status of weak coupling phase unclear.
  - Connection with other lattice models, eg hypercubic?
- Covariant Effective Field Equations
  - Formulation of fractional operators.
  - Further investigation on nature of solutions (horizons).
  - Possible Cosmological (observable) ramifications.

The End