Cosmological billiards

and hidden symmetries of gravity

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Motivations

Hidden symmetries (duality symmetries) are intriguing symmetries of gravitational theories that have a long history

Buchdal 1954

(static solutions)

$$g_{00} \longrightarrow \frac{1}{g_{00}}$$

$$h_{ij} = g_{00}\, g_{ij} = \text{fixed}$$

(T-duality before T-duality!)

($D = 4$ Einstein)

Ehlers 1959

($D = 4$ Einstein / one Killing vector)

$SL(2, \mathbb{R})$

$$Z' = \frac{aZ + b}{cZ + d}$$

$$Z = B + i\Delta$$

Δ: g_{33}

B: dual of $\frac{g_{\lambda 3}}{g_{33}} = A_\lambda$

(m, m_{TaubNUT})

$(\varepsilon_{\mu\nu\lambda}\, \partial^\lambda B \sim \Delta^2\, (\partial_\mu A_\nu - \partial_\nu A_\mu))$

(Pure gravity in D dimensions with $D-3$ commuting Killing vectors $\longrightarrow SL(D-2, \mathbb{R})$)

Spectacular example

$D = 11$ supergravity

Toroidal compactification

to $D = 4$ $\quad E_7$ (Cremmer, Julia)

$D = 3$ $\quad E_8$ (Marcus, Schwarz)

Many other examples

P. Breitenlohner, D. Maison and G. Gibbons, CMP <u>120</u> (1988) 295

E. Cremmer, B. Julia, H. Lu and C. Pope, hep-th/9909099

Supersymmetry not essential here (but gravity is)

Recently, it has been argued that these symmetries are part of a much bigger group (in fact, an infinite-dimensional group) that might, in fact, be a symmetry group of the theory already prior to dimensional reduction.

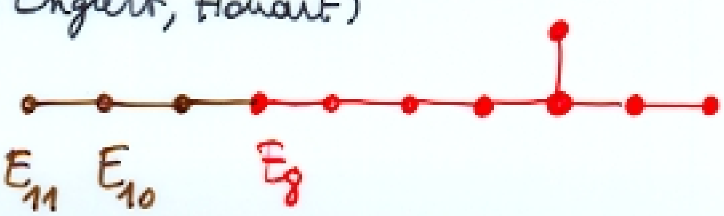
[Geroch, Julia // de Wit - Nicolai]

In the case of $D = 11$ supergravity, this symmetry group would be

$E_{10} = E_8^{++}$ ("cosmological billiards", Damour, M. H., Julia, Nicolai, de Buyl, Paulot, Schomblond)

or even

$E_{11} = E_8^{+++}$ (West, Englert, Houart)



Purpose of talk

* explain evidence from "cosmological billiards"

* explain attempt to make the infinite-dimensional symmetry manifest from the outset, by reformulating the theory as a non linear σ-model in 1+0 dimension on $G^{++}/K(G^{++})$

($G \longrightarrow G^{+}$ = untwisted affine extension
$\longrightarrow G^{++}$ = (standard) overextension)

B. Julia

P. West

T. Damour, M. H., H. Nicolai

F. Englert

L. Houart

A. Kleinschmidt

⋮

Theories considered here:

Gravity + dilatons + p-forms in $D \geq 4$ dimensions such that reduction to 3D gives (after dualization)

$$\mathcal{L}_E + \mathcal{L}_{G/H}$$

(H = maximal compact subgroup)

Special cases: * pure gravity in 4D

$G = SL(2,R)$
$H = SO(2)$

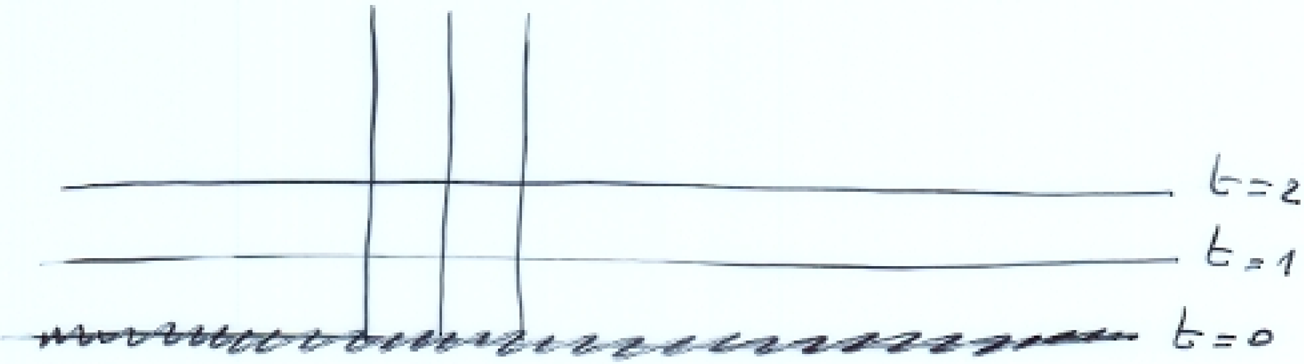
* bosonic sector of 11D sugra

$G = E_8$

$H = SO(16)$

1. Cosmological billiards and infinite symmetry

Belinskii, Lifshitz, Khalatnikov (BKL)



$g_{ij}(t, \vec{x})$

$\pi^{ij}(t, \vec{x})$

$+$ other fields

$t \to 0$

$x^0 = -\ln t \to +\infty$

Diagonalize metric (by "Iwasawa" change of frame)

$$ds^2 = -dt^2 + a^2(t,\vec{x})\,\ell^2 + b^2(t,\vec{x})\,m^2 + c^2(t,\vec{x})\,n^2$$

$\ell = \ell_i(t,\vec{x})\,dx^i$

etc

"scale factors"

As $t \to 0$:

* off-diagonal components freeze
  → dynamics carried out by diagonal components (scale factors)

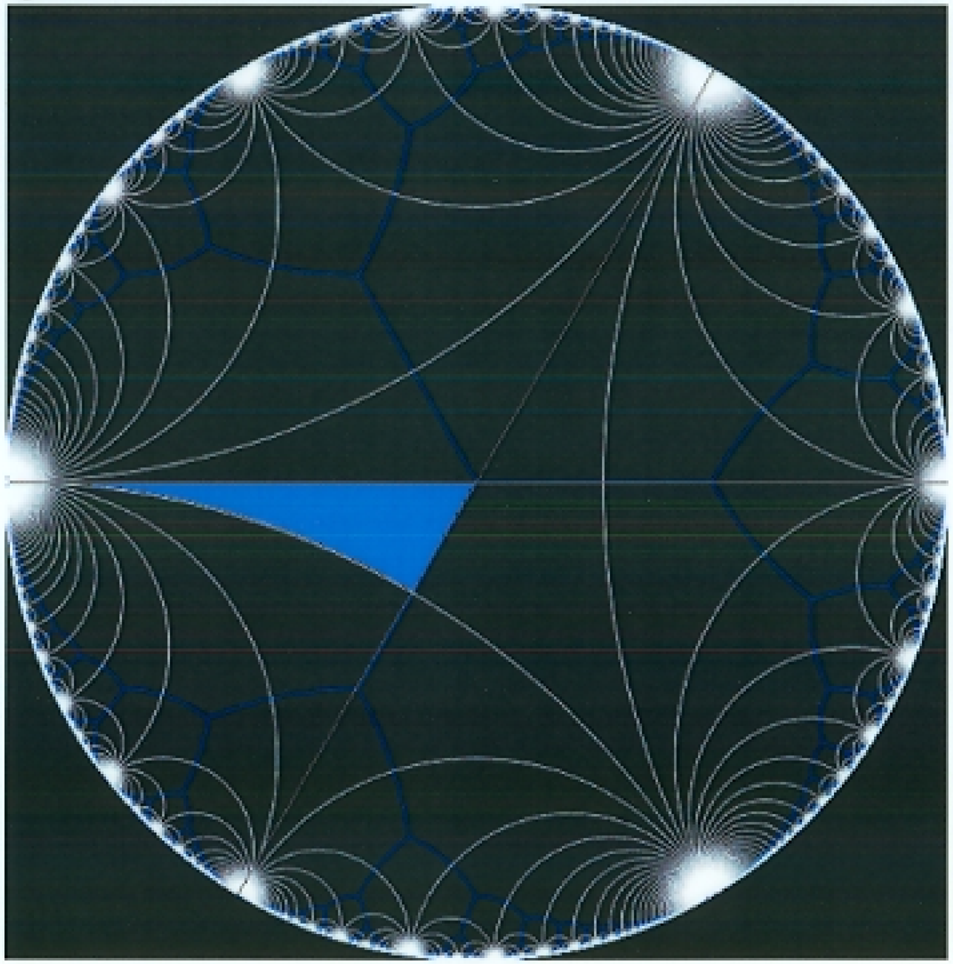
* dynamics decouple at each spatial point (infinite number of ordinary differential equations with respect to time – a finite # at each point)

True also in higher dimensions, with dilaton(s) and p-form(s).

Dynamics of scale factors (including dilatons) at a given point ~ dynamics of a billiard ball moving in a region of hyperbolic space.

Hamiltonian constraint: effectively removes one scale factor

Billiard description



Dynamics can be mapped on billiard dynamics in some region of hyperbolic space.

Free flight = Kasner behaviour

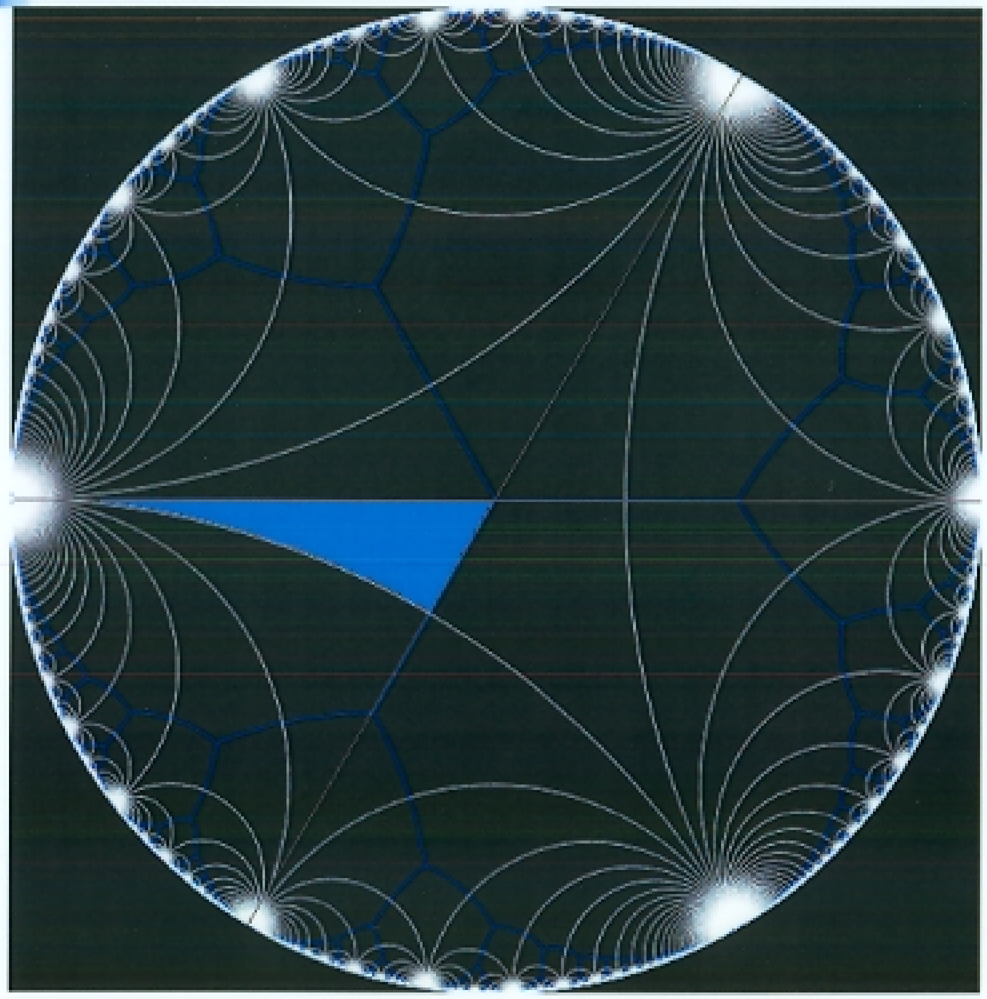
Collision against a wall = change from one Kasner regime to another

Furthermore, the billiard region is a "Coxeter polyhedron", i.e., a region bounded by hyperplanes ("billiard walls") with dihedral angles that are integer submultiples of π ($\pi/2$, $\pi/3$, $\pi/4$, ...)

It is a simplex

It is the fundamental domain of a discrete reflection group (in hyperbolic space) which is the Weyl group of a Lorentzian Kac-Moody algebra

Examples



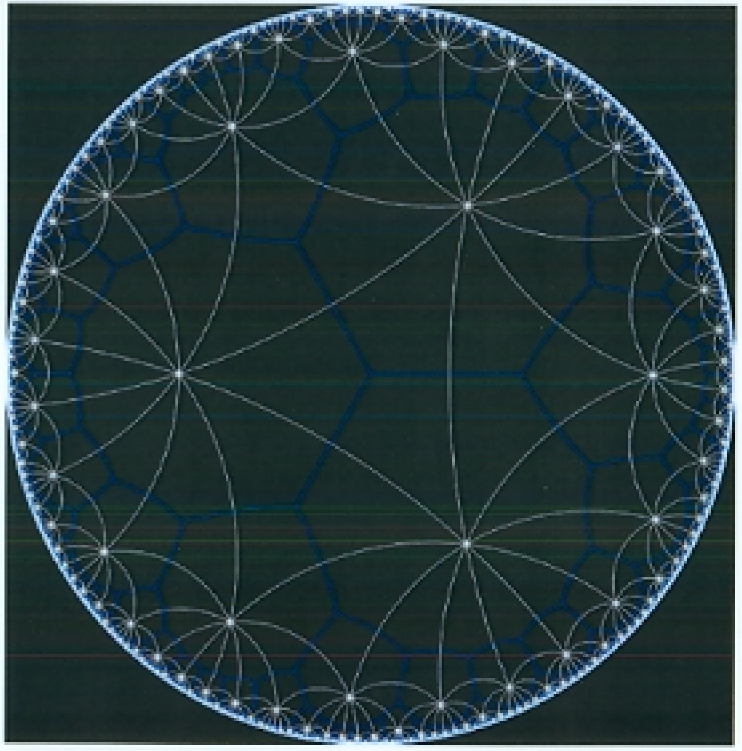
Pure gravity in 4 spacetime Dimensions.

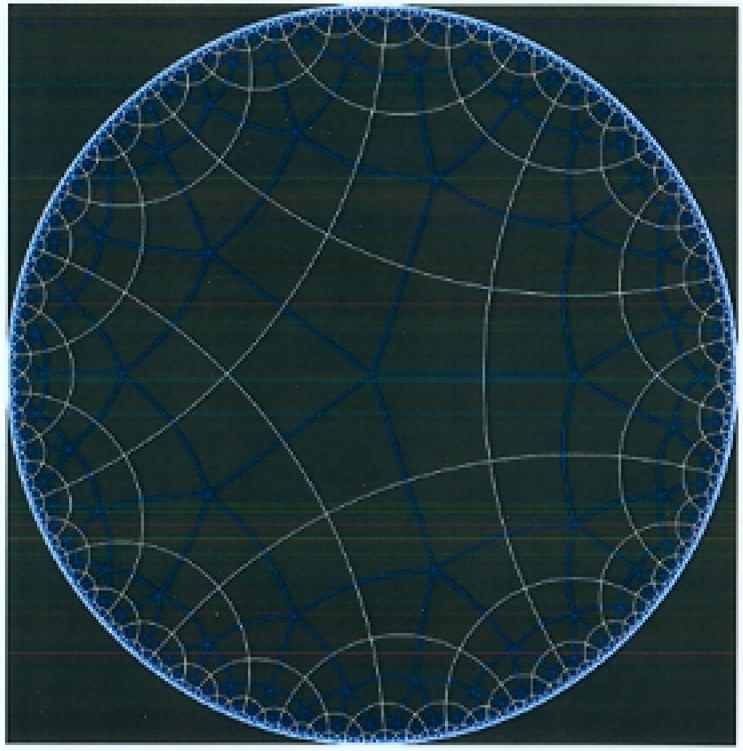
The billiard is a triangle with angles $\pi/2$, $\pi/3$ and 0, corresponding to the Coxeter group (2,3, infinity).

The triangle is the fundamental region of the group PGL(2,Z).

Arithmetical chaos

Tilings of the hyperbolic plane





http://www.hadron.org/~hatch/HyperbolicTesselations/

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Reflection groups in hyperbolic space

Simplex of finite volume: rank ≤ 10

(dimension ≤ 9)

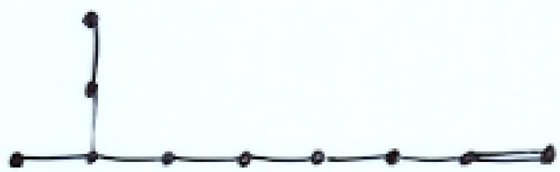
Finite volume

dimension < 996 / bound far from exact

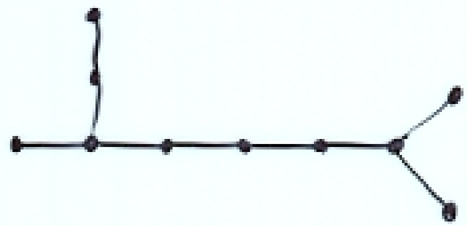
(known cases up to dim. 21)

$E_{10} = E_8^{++}$

$BE_{10} = B_8^{++}$



$DE_{10} = D_8^{++}$



- Furthermore, the matrix

$$A_{ij} = 2 \frac{(w_i | w_j)}{(w_i | w_i)}$$ is the Cartan

matrix of a Lorentzian Kac-Moody algebra

→ Conformal factor

(infinite-dimensional)

w_i : wall forms
$(\, | \,)$: scalar product in space of scale factors
(De Witt supermetric)

- Finite volume of billiard region

$\Longleftrightarrow$ KM algebra is hyperbolic

Billiard region =

fundamental Weyl chamber of KM algebra

Scale factors β^i $\longleftrightarrow$ Cartan degrees of freedom

Walls $\longleftrightarrow$ Hyperplanes orthogonal to (simple) roots

Reflection against a wall $\longleftrightarrow$ Weyl reflection

Kac-Moody algebra that appears:

Overextension of $\mathcal{U}_3$- duality group

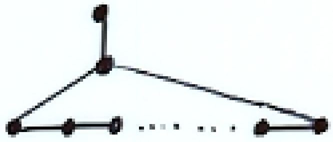
that appears in toroidal compactification

to 3 dimensions (Cremmer-Julia ...)

$\mathcal{U}_3 \longrightarrow \mathcal{U}_3^{+}$ (affine) $\longrightarrow \mathcal{U}_3^{++}$

(split case)

Pure gravity in dimension $D = d+1$

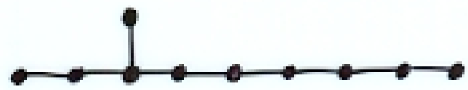
A_{d-2}^{++} 

A_7^{++} hyperbolic

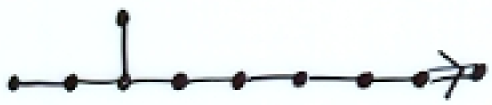
A_8^{++} non hyperbolic

A_1^{++} 

$D = 11$ sugra

$E_{10} \equiv E_8^{++}$ 

$D = 10$, $N = 1$ sugra + Maxwell multiplet

$B_8^{++} \equiv BE_{10}$ 

D = 10, pure $N = 1$ sugra

$D_8^{++} \equiv DE_{10}$ 

Does this signal a huge symmetry?

2. Trying to make the symmetry manifest

Idea: one sees only the Weyl group ... because the off-diagonal degrees of freedom have not been kept $\longrightarrow$ keep off-diagonal, "non Cartan" degrees of freedom

Analogy:

Theory of unitary matrices invariant under adjoint action

$$M \longrightarrow U M U^{+} \quad , \quad U, M \in SU(n)$$

$SU(n)$ is symmetry group

Can diagonalize M, $M = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

(can conjugate to Cartan subalgebra)

Invariance left: permutation of the α_i's,

i.e. Weyl group of $SU(n)$.

$\boxed{\text{Non-linear sigma-model on } G^{++}/K(G^{++}) \text{ in } 1+0 \text{ dimension}}$

$\mathcal{L}_{G^{++}/K(G^{++})}$

G^{++} admits a triangular decomposition

$$G^{++} = \mathcal{N}_- \oplus H \oplus \mathcal{N}_+$$

"lower triangular"	"diagonal"	"upper triangular"	
J_-	J_3	J_+	Roots
f_i	h_i	e_i	α_i
		$[e_i, e_j]$	$\alpha_i + \alpha_j$
		$[e_i, [e_j, e_k]]$	$\alpha_i + \alpha_j + \alpha_k$
		$\vdots$	

$$[h_i, h_j] = 0$$

$$[h_i, e_j] = A_{ij}\, e_j$$

Cartan matrix $\leftrightarrow$ Dynkin diagram

$$[h_i, f_j] = -A_{ij}\, f_j$$

$$[e_i, f_j] = \delta_{ij}\, h_i$$

$$\mathrm{ad}_{e_i}^{1-A_{ij}}\, e_j = 0 \qquad \mathrm{ad}_{f_i}^{1-A_{ij}}\, f_j = 0$$

Chevalley involution

$$\tau(h_i) = -h_i \qquad \tau(e_i) = -f_i \qquad \tau(f_i) = -e_i$$

$$\tau^2 = 1$$

$K(G^{++})$: subalgebra invariant under τ

$e_i - f_i$

α : root $\qquad e_{\alpha,s}$ $\qquad e_{\alpha,s} + \tau(e_{\alpha,s}) = k_{\alpha,s}$

degeneracy

Element in $SL(2,\mathbb{R})$

$$e^{H}\, e^{\mathcal{N}_-}\, e^{\mathcal{N}_+}$$

Element in $SL(2,\mathbb{R})$

$$g \sim h g \qquad h \in SO(2)$$

Triangular gauge

$$g \in \exp H \exp \mathcal{N}_+$$

$$\exp\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \exp\begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$$

Geodesic motion on $SL(2,\mathbb{R})/SO(2)$

$SO(3)/SO(2)$

$SL(2,\mathbb{R}) = \mathcal{N}_- \oplus H \oplus \mathcal{N}_+$

(triangular decomposition)

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$J_- \equiv f \qquad h \equiv J_3 \qquad e \equiv J_+$$

$$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h$$

Killing metric $+ \; + \; -$

$SO(2) \quad -$ $\qquad e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\operatorname{tr}(e-f)^2 = -2$$

$SL(2,\mathbb{R})/SO(2)$ has positive metric

$SO(2)$ is invariant under Chevalley involution

$$\tau(h) = -h \qquad \tau(e) = -f \qquad \tau(f) = -e$$

$G^{++}/K(G^{++})$ has same dimension as $H \oplus \mathcal{N}_+$

Lagrangian $\mathcal{L}_{G^{++}/K(G^{++})}$ for geodesic motion on $G^{++}/K(G^{++})$

(1+0 dimensional sigma model)

$$g \in G^{++}$$

$$g \sim hg \qquad (h \in K(G^{++}))$$

Triangular, or "Borel" gauge

$$g \in \exp(H \oplus \mathcal{N}_+)$$

$$v(t) = \dot{g} g^{-1} \in H \oplus \mathcal{N}_+ \quad (\subset G^{++})$$

$$\mathcal{P} = \frac{1}{2}\left(v - \tau(v)\right) \equiv v_{sym}$$

$$\mathcal{Q} = \frac{1}{2}\left(v + \tau(v)\right) \in K(G^{++})$$

$$\mathcal{L} = \langle \mathcal{P} | \mathcal{P} \rangle$$

invariant bilinear form on G^{++}

$\mathcal{L}$: invariant under right multiplication by an element of G^{++}

(+ reparametrization invariance)

What does this have to do with (super) gravity?

Gravity | σ-model

Finite number of fields $\phi^A(x^\mu)$ | ∞ number of variables $q^i(t)$

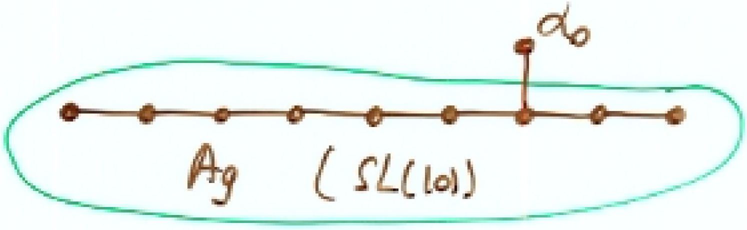
$\vec{x} = \text{const}$

$\phi^A(t), \partial_k \phi^A(t),$
$\partial_k \partial_l \phi^A(t), \ldots$ $\longleftrightarrow$ $q^i(t)$

$\vec{x}$ fixed

+New degrees of freedom (C)

Consider E_{10} for definitess (to be compared with 11D sugra)

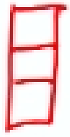


Level ℓ of a root : $\alpha = \ell \alpha_0 +$ others

Cartan of E_{10} — 10 diagonal elements

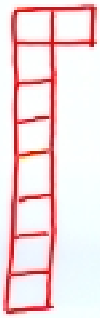
Level 0 — Positive root vectors of $SL(10)$

Level 1 — $E^{a_1 a_2 a_3}$



Level 2 — $E^{a_1 a_2 \cdots a_6}$

Level 3 — $E^{a_0 | a_1 - a_8}$



etc...

$$g(t) = \exp X_h(t) \cdot \exp X_A(t)$$

vielbein in triangular gauge

$$X_h(t) = h^a{}_b(t)\, K^b{}_a$$

generators of H and of $N_{SL(10)}$
$K^1{}_1, K^2{}_2, \ldots K^{10}{}_{10}$

(level 0)

$$X_A(t) = \frac{1}{3!} A_{abc}(t)\, E^{abc} \qquad \text{(level 1)}$$

$$+ \frac{1}{6!} A_{a_1 \ldots a_6}(t)\, E^{a_1 a_2 \ldots a_6} \qquad \text{(level 2)}$$

$$+ \frac{1}{9!} A_{a_0|a_1 \ldots a_8}(t)\, E^{a_0|a_1 \ldots a_8} \qquad \text{(level 3)}$$

$$+ \ldots$$

$$\mathcal{L} = \frac{1}{4}\left(g^{ac} g^{bd} - g^{ab} g^{cd}\right) \dot{g}_{ab}\, \dot{g}_{cd}$$ Level 0

$$+ \frac{1}{2}\ \frac{1}{3!}\ DA_{a_1 a_2 a_3}\ DA^{a_1 a_2 a_3}$$ Level 1

$$+ \frac{1}{2}\ \frac{1}{6!}\ DA_{a_1 \dots a_6}\ DA^{a_1 \dots a_6}$$ Level 2

$$+ \frac{1}{2}\ \frac{1}{8!}\ DA_{a_0 | a_1 \dots a_8}\ DA^{a_0 | a_1 \dots a_8}$$ Level 3

$+$ etc...

$$DA_{a_1 a_2 a_3} = \dot{A}_{a_1 a_2 a_3}$$

$$DA_{a_1 \dots a_6} = \dot{A}_{a_1 \dots a_6} + 10\, A_{[a_1 a_2 a_3}\, \dot{A}_{a_4 a_5 a_6]}$$

$$DA_{a_0 | a_1 \dots a_8} = \dot{A}_{a_0 | a_1 \dots a_8} + \text{"}\, A_3\, \dot{A}_6 + \dot{A}_3\, A_6$$
$$+ A_3\, A_3\, \dot{A}_3\, \text{"}$$

Coefficients fixed by group theory

Consistent truncation(s)

Set $DA_{(i)} = 0$ for $i > L$

Reduces the system to a system with a finite number of freedom.

Compare with truncated versions of supergravity in which some fields are set equal to zero and also some spatial gradients (in appropriate sense).

Works for truncations to low levels

$D = 11$ sugra

$$G_{ab}(t,\vec{x})$$

$$\mathcal{E}_{a_1a_2a_3}(t,\vec{x}) \sim F_{0a_1a_2a_3}(t,\vec{x})$$

$$F_{a_1a_2a_3a_4}(t,\vec{x})$$

$E_{10}/K(E_{10})$ $\qquad$ $D=11$ sugra

$L=0$

$g_{ab}(t)$

$DA_{(3)} = DA_{(6)} = \cdots = 0$

Assume metric homogeneous in space (spatial gradients $= 0$) ("reduce on a torus") and assume $G_{ab}(t,\cancel{\vec{x}})$

$\mathcal{E}_{a_1a_2a_3} = F_{a_1a_2a_3a_4} = 0$

$$g_{ab}(t) = G_{ab}(t)$$

$L=1$

$g_{ab}(t),\ A_{a_1a_2a_3}(t)$

$DA_{(6)} = DA_{(9)} = \cdots = 0$

Assume metric and electric field homogeneous in space

$G_{ab}(t,\cancel{\vec{x}})\ ,\ \mathcal{E}_{a_1a_2a_3}(t,\cancel{\vec{x}})$

and no magnetic field

$F_{a_1a_2a_3a_4} = 0$

$$g_{ab}(t) = G_{ab}(t)$$

$$DA_{a_1a_2a_3}(t) = \mathcal{E}_{a_1a_2a_3}(t)$$

$E_{10}/K(E_{10})$ $D=11$ sugra

$L=2$

$g_{ab}(t)$, $A_{a_1a_2a_3}(t)$,

$A_{a_1\cdots a_6}(t)$

$DA(g) = DA_{(12)} = \cdots = 0$

Assume

$G_{ab}(t, \cancel{\vec{x}})$

$\mathcal{E}_{a_1a_2a_3}(t, \cancel{\vec{x}})$

$F_{a_1a_2a_3a_4}(t, \cancel{\vec{x}})$

$$g_{ab}(t) = G_{ab}(t)$$

$$DA_{a_1a_2a_3}(t) = \mathcal{E}_{a_1a_2a_3}(t)$$

$$DA^{a_1\cdots a_6}(t) = -\frac{1}{4!}\,\varepsilon^{a_1-a_6 b_1-b_4} F_{b_1-b_4}(t)$$

Again, perfect matching (including CS term)

$E_{10}/K(E_{10})$ $\qquad$ $D = 11$ sugra

L=3 $\quad g_{ab}(t), A_{a_1a_2a_3}(t),$

$A_{a_1-a_6}(t), A_{a_0|a_1-a_8}(t)$ $\qquad$?

$DA_{(12)} = \cdots\cdots = 0$

Introduce spatial derivatives in a controlled way: reduce on a non abelian group manifold

$$d\sigma^2 = G_{ab}(t)\,\theta^a\theta^b$$

$$\theta^a = \theta^a_{\ i}(\vec{x})\,dx^i$$

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\,\theta^b\theta^b$$

↑ anholonomy coefficients

$C^a{}_{bc} \neq 0$, $\quad C^a{}_{ab} = 0$ (if "internal" manifold is compact)

$$g_{ab}(t) = G_{ab}(t)$$

$$DA_{a_1 a_2 a_3}(t) = \mathcal{E}_{a_1 a_2 a_3}(t)$$

$$DA^{a_1 - a_6}(t) = -\frac{1}{4}\,\varepsilon^{a_1 - a_6\, b_1 - b_4}\, F_{b_1 - b_4}(t)$$

"Dual to the graviton"

$$DA^{b|a_1 - a_8}(t) = \frac{3}{2}\,\varepsilon^{a_1 - a_8\, b_1 b_2}\, C^{b}{}_{b_1 b_2}$$

Dynamical Equations of motion:

* $A^{b|a_1 - a_8}$ $\qquad C^{b}{}_{b_1 b_2}$ $\qquad$ ok

* Maxwell $\qquad$ ok

* dynamical Einstein

Perfect match... up to height 30

$$\ddot{g}_{ab} = \underbrace{\cdots\cdots\cdots}_{\text{time derivative terms, ok}} + \underbrace{\text{“}\,{}^{(10)}R_{ab}\,\text{”}}_{\text{ok up to height 30}}$$

roots at level 3

$$ {}^{(10)}R_{ab} \sim \sum e^{-2\alpha(\beta)} \, C^2 $$

dep height $\geqslant 30$
then OK

(Real roots)

Duality

What about height $\gtrsim 30$?

What about dictionary at higher levels ?

No (good, solid) idea so far

(higher derivatives)

RECENT WORK on higher roots and cosmological deformations (Bergshoeff et al, West et al)

Higher order correction terms

$$e^{\lambda\varphi} R^N \sqrt{-g}$$

(ongoing work started by Damour and Nicolai)

Analysis à la BKL

$$e^{\lambda\varphi} R^N \sqrt{-g} \sim \sum c_i \, e^{-2w_i(\beta)}$$

$w_i(\beta)$: linear forms in the scale factors

These are <u>weights</u> of G^{++} (in fact, non spacelike weights)

The leading terms is a dominant weight

E_{10}

no φ

$N = 3k + 1$

$R^4 \sqrt{-g}\ ,\ R^7 \sqrt{-g}\ ,\ R^{10} \sqrt{-g} \ldots$

$R^4 \sqrt{-g}$ corresponds to the fundamental weight conjugate to the non gravitational root

Fermions can be included

Fit in representations of compact

subgroup ($K(E_{10})$)

E_{10} useful in understanding cosmological solutions of 11-D supergravity.

A. Kleinschmidt, H. Nicolai

M. H., M. Leston, D. Persson and P. Spindel, JHEP **10** (2006) 021

* Constraints (Damour, Kleinschmidt, Nicolai)

* Reduction and topology ("controlled chaos")

(Steinhardt, Turok, Wesley

M. H., Persson, Wesley)

CONCLUSIONS

* attempts to exhibit G^{++} symmetry yield encouraging results

* can go beyond height 1 seen by the billiard dynamics

* fermions can be included (rep. of $K(G^{++})$ up to same levels)

* intriguing match of quantum corrections with algebraic structure

* but approach probably too naive – need a good idea!

* Recent work on cosmological deformations

T. Damour and M.H., PRL 86 (2001) 4749

T. Damour, M.H. and H. Nicolai, PRL 89 (2002) 221601

T. Damour, M.H. and H. Nicolai, Class. Quant. Grav. 20 (2003) R145

M.H., M. Leston, D. Persson and P. Spindel, JHEP 10 (2006) 021