Quantum Geometrodynamics:

whence, whither?

Claus Kiefer

Institut für Theoretische Physik
Universität zu Köln

Contents

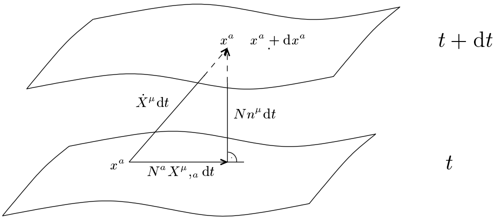
Main Approaches to Quantum Gravity

- Quantum general relativity
  - Covariant approaches (perturbation theory, path integrals, ...)
  - Canonical approaches (geometrodynamics, connection dynamics, loop dynamics, ...)
- String theory
- Other approaches
  (Quantization of topology, ...)

Topic here: canonical quantum geometrodynamics

Canonical Formalism

$$\dot{X}^\nu \equiv t^\nu = N n^\nu + N^a X^\nu_{,a}$$



$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N^2 \mathrm{d}t^2 + h_{ab}(\mathrm{d}x^a + N^a \mathrm{d}t)(\mathrm{d}x^b + N^b \mathrm{d}t) \\ &= (h_{ab}N^a N^b - N^2)\mathrm{d}t^2 + 2h_{ab}N^a \mathrm{d}x^b \mathrm{d}t + h_{ab}\mathrm{d}x^a \mathrm{d}x^b \ . \end{aligned}$$

configuration variable: three-metric h_{ab}

More fundamental viewpoint:

three-manifold Σ is given; only after solving the dynamical equations can we construct spacetime and interpret the time dependence of the metric h_{ab} of Σ as being brought about by 'wafting' Σ through a four-manifold via a one-parameter family of embeddings

- Six evolution equations for h_{ab} and its canonical momentum π^{ab}
- Four constraints

Spacetime can then be interpreted as a 'trajectory of spaces'

Constraints

Einstein's equations can be written as a dynamical system of evolution equations together with constraints:

$$
\begin{aligned}
H[h,\pi] &= 2\kappa\, G_{ab\,cd}\pi^{ab}\pi^{cd} - (2\kappa)^{-1}\sqrt{h}({}^{(3)}R - 2\Lambda) + \sqrt{h}\rho \approx 0\,, \\
D^a[h,\pi] &= -2\nabla_b\pi^{ab} + \sqrt{h}j^a \approx 0\,,
\end{aligned}
$$

with "DeWitt metric"

$$G_{ab\,cd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd})$$

$$\kappa = 8\pi G/c^4$$

Configuration space: Space of all three-geometries (=Superspace)

Constraints and evolution

I

Constraints are preserved in time $\Longleftrightarrow$ energy–momentum tensor of matter has vanishing covariant divergence

compare with electrodynamics: Gauss constraint preserved in time $\Longleftrightarrow$ charge conservation

II

Einstein's equations are the unique propagation law consistent with the constraints

compare with electrodynamics: Maxwell's equations are the unique propagation law consistent with the Gauss constraint

Constraints: "Laws of the Instant"'

Problem of time I

Restrict to *compact* three-spaces Σ

- The total Hamiltonian is a combination of pure constraints $\longrightarrow$ all of the evolution will be generated by constraints ('pure gauge')
- no external time parameter exists
- all physical time parameters are to be constructed from within our system, that is, as functional of the canonical variables;
  a priori there is no preferred choice of such an intrinsic time parameter

The absence of an extrinsic time and the non-preference of an intrinsic one is known as the (classical part of the) problem of time in canonical gravity (but: "trajectories" still present)

Quantum constraints

In the quantum theory, only the constraints remain; in the vacuum case, they read

$$\hat{H}\Psi \equiv \left(-2\kappa\hbar^2 G_{abcd}\frac{\delta^2}{\delta h_{ab}\delta h_{cd}} - (2\kappa)^{-1}\sqrt{h}\big({}^{(3)}R - 2\Lambda\big)\right)\Psi = 0$$

Wheeler–DeWitt equation

$$\hat{D}^a\Psi \equiv -2\nabla_b\frac{\hbar}{\mathrm{i}}\frac{\delta\Psi}{\delta h_{ab}} = 0$$

quantum diffeomorphism (momentum) constraint

Problem of time II

- In addition to classical ‘problem of time’: spacetime has *disappeared*!
- local intrinsic time can be defined through local hyperbolic structure of Wheeler–DeWitt equation (‘wave equation’)
- Related problem: *Hilbert-space problem* –
  which inner product, if any, to choose between wave functionals?
  - Schrödinger inner product?
  - Klein–Gordon inner product?

A Brief History of Geometrodynamics

- F. Klein, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse*, **1918**, 171–189:
  first four Einstein equations are “Hamiltonian”and “momentum density” equations
- L. Rosenfeld, *Annalen der Physik*, 5. Folge, **5**, 113–152 (1930):
  general constraint formalism; first four Einstein equations are constraints; consistency conditions in the quantum theory (“Dirac consistency”)

- P. Bergmann and collaborators (from 1949 on): general formalism (mostly classical); notion of observables
  Bergmann (1966): $H\psi = 0$, $\partial\psi/\partial t = 0$
  (“To this extent the Heisenberg and Schrödinger pictures are indistinguishable in any theory whose Hamiltonian is a constraint.”)
- P. Dirac (1951): general formalism; Dirac brackets
- P. Dirac (1958/59): application to the gravitational field; reduced quantization
  (“I am inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world.”)
- ADM (1959–1962): lapse and shift; rigorous definition of gravitational energy and radiation by canonical methods

- B. S. DeWitt, Quantum theory of gravity. I. The canonical theory. *Phys. Rev.*, **160**, 1113–48 (1967): general Wheeler–DeWitt equation; configuration space; quantum cosmology; semiclassical limit; conceptual issues, ...
- J. A. Wheeler, Superspace and the nature of quantum geometrodynamics. In *Battelle rencontres* (ed. C. M. DeWitt and J. A. Wheeler), pp. 242–307 (1968): general Wheeler–DeWitt equation; superspace; semiclassical limit; conceptual issues; ...

Erwin Schrödinger 1926:

We know today, in fact, that our classical mechanics fails for very small dimensions of the path and for very great curvatures. Perhaps this failure is in strict analogy with the failure of geometrical optics ... that becomes evident as soon as the obstacles or apertures are no longer great compared with the real, finite, wavelength. ... Then it becomes a question of searching for an undulatory mechanics, and the most obvious way is by an elaboration of the Hamiltonian analogy on the lines of undulatory optics.

Hamilton–Jacobi equation $\longrightarrow$ guess a wave equation

Hamilton–Jacobi equation

$$\begin{aligned} 16\pi G\, G_{abcd}\frac{\delta S}{\delta h_{ab}}\frac{\delta S}{\delta h_{cd}} - \frac{\sqrt{h}}{16\pi G}({}^{(3)}R - 2\Lambda) &= 0 \\ D_a\frac{\delta S}{\delta h_{ab}} &= 0 \end{aligned}$$

(Peres 1962)

Independent of its status on the most fundamental level, quantum geometrodynamics should be approximately valid away from the Planck scale

WKB approximation:

$$\Psi[h_{ab}] = C[h_{ab}] \exp\left(\frac{\mathrm{i}}{\hbar}S[h_{ab}]\right)$$

Born–Oppenheimer approximation

Ansatz:

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]\mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]}|\psi[h_{ab}]\rangle$$

This yields (up to higher-order terms)

$$\begin{aligned}\left(\hat{\mathcal{H}}_\perp^{\mathrm{m}} - \langle\psi|\hat{\mathcal{H}}_\perp^{\mathrm{m}}|\psi\rangle - \mathrm{i}G_{abcd}\frac{\delta S}{\delta h_{ab}}\frac{\delta}{\delta h_{cd}}\right)|\psi[h_{ab}]\rangle &= 0\\ \left(\hat{\mathcal{H}}_a^{\mathrm{m}} - \langle\psi|\hat{\mathcal{H}}_a^{\mathrm{m}}|\psi\rangle - \frac{2}{\mathrm{i}}h_{ab}D_c\frac{\delta}{\delta h_{bc}}\right)|\psi[h_{ab}]\rangle &= 0\end{aligned}$$

One now evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x},t)$, corresponding to a solution, $S[h_{ab}]$, of the Hamilton–Jacobi equations; this solution is obtained from

$$\dot{h}_{ab} = NG_{abcd}\frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}$$

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle = \int \mathrm{d}^3x\,\dot{h}_{ab}(\mathbf{x},t)\,\frac{\delta}{\delta h_{ab}(\mathbf{x})}|\psi[h_{ab}]\rangle$$

$\longrightarrow$ functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field:

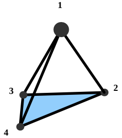
$$\begin{aligned}
\mathrm{i}\frac{\partial}{\partial t}\,|\psi(t)\rangle &= \hat{H}^{\mathrm{m}}|\psi(t)\rangle \\
\hat{H}^{\mathrm{m}} &\equiv \int \mathrm{d}^3x\left\{N(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{\perp}(\mathbf{x}) + N^a(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{a}(\mathbf{x})\right\}
\end{aligned}$$

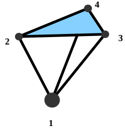
$\hat{H}^{\mathrm{m}}$: matter-field Hamiltonian in the Schrödinger picture, parametrically depending on (generally non-static) metric coefficients of the curved space–time background.

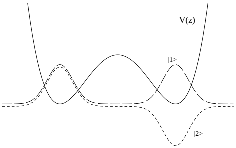
WKB time t controls the dynamics in this approximation

Time from Symmetry Breaking

Analogy from molecular physics: emergence of chirality







dynamical origin: decoherence due to scattering with light or air molecules

quantum cosmology: decoherence between $\exp(\mathrm{i}S_0/\hbar)$- and $\exp(-\mathrm{i}S_0/\hbar)$-part of wave function through interaction with multipoles

one example for decoherence factor:

$\exp\left(-\frac{\pi m H_0^2 a^3}{128\hbar}\right) \sim \exp\left(-10^{43}\right)$ (C. K. 1992)

Quantum gravitational corrections

Next order in the Born–Oppenheimer approximation gives

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2}\,(\text{various terms})$$

(C. K. and T. P. Singh (1991); A. O. Barvinsky and C. K. (1998))

Simple example: Quantum gravitational correction to trace anomaly in de Sitter space:

$$\delta\epsilon \approx -\frac{2G\hbar^2 H_0^6}{3(1440)^2\pi^3}$$

(C. K. 1996)

Quantum Black Holes

- wave functions for eternal spherically-symmetric black holes
- quantization of dust shells
- quantization of dust clouds (Lemaître–Tolman–Bondi model)
- entanglement entropy; e.g. from the no-boundary wave function of a black hole:

  $S = \frac{A}{360\pi l^2}$

  (Barvinsky *et al*. 1995)

Quantum Gravitational Collapse

Lemaître–Tolman–Bondi (LTB) model:
self-gravitating dust cloud with $T_{\mu\nu} = \epsilon(\tau,\rho) u_\mu u_\nu$

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \frac{(\partial_\rho R)^2}{1+2E(\rho)}\mathrm{d}\rho^2 + R^2(\rho)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

- exact quantum states of a particular type (cloud consists of decoupled shells)
- Hawking radiation and greybody factors
- BTZ black hole: Hawking radiation as well as microscopic derivation of black-hole entropy

(S. Gutti, C. K., J. Müller-Hill, T. P. Singh, C. Vaz, L. C. R. Wijewardhana, L. Witten in various combinations 2003–2008)

Entropy of the BTZ black hole

Jacob Bekenstein 1973

It is then natural to introduce the concept of black-hole entropy as the measure of the *inaccessibility* of information (to an exterior observer) as to which particular internal configuration of the black hole is actually realized in a given case

- discrete mass spectrum for the shells collapsing to the black hole;
- black-hole entropy is number of possible distributions of $\mathcal{N}$ identical shells between these levels;

$$S_{\text{can}} \approx 2\pi \sqrt{\left(1 - \frac{48lM_0}{\hbar}\right) \frac{lM}{6\hbar}}$$

with $l = |\Lambda|^{-1/2}$;

- is equal to Bekenstein–Hawking entropy for

$$M_0 = -\frac{1}{16G} + \frac{\hbar}{48l}$$

Quantum Cosmology

Closed Friedmann–Lemaître universe with scale factor a, containing a homogeneous massive scalar field ϕ (two-dimensional *minisuperspace*)

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega_3^2$$

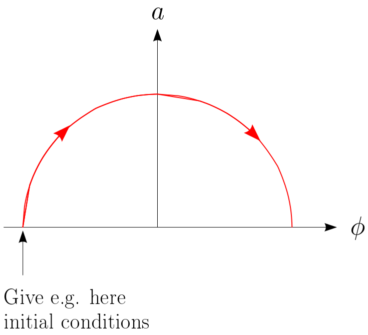
The Wheeler–DeWitt equation reads (with units $2G/3\pi = 1$)

$$\frac{1}{2}\left(\frac{\hbar^2}{a^2}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{\hbar^2}{a^3}\frac{\partial^2}{\partial \phi^2} - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2\right)\psi(a,\phi) = 0$$

Factor ordering chosen in order to achieve covariance in minisuperspace

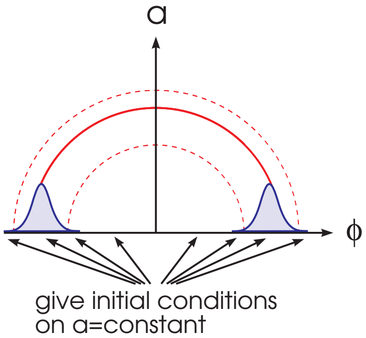
Determinism in classical and quantum theory

Classical theory



Recollapsing part is deterministic successor of expanding part

Quantum theory



‘Recollapsing’ wave packet must be present ‘initially’

Big-brake cosmology: Classical model

Equation of state $p = A/\rho$, $A > 0$, for a Friedmann universe with scale factor $a(t)$ and scalar field $\phi(t)$ with potential ($24\pi G = 1$)

$$V(\phi) = V_0 \left(\sinh(|\phi|) - \frac{1}{\sinh(|\phi|)} \right) \; ; \; V_0 = \sqrt{A/4}$$

develops pressure singularity (only $\ddot{a}(t)$ becomes singular)

- total lifetime: $t_0 \approx 7 \times 10^2 \frac{1}{\sqrt{V_0 \left[\frac{\mathrm{g}}{\mathrm{cm}^3} \right]}} \, \mathrm{s}$
- lifetime much bigger than current age of our Universe for

$$V_0 \ll 2.6 \times 10^{-30} \, \frac{\mathrm{g}}{\mathrm{cm}^3}$$

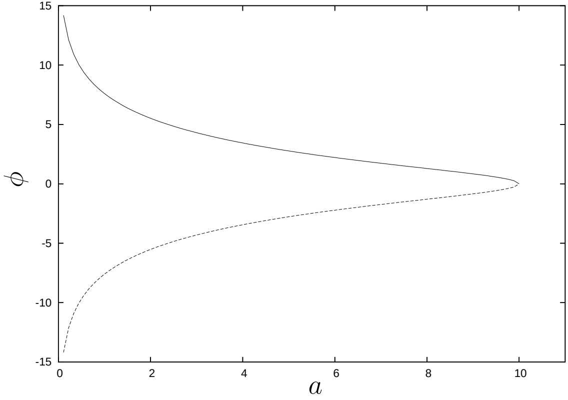


Figure: Classical trajectory in configuration space.

Big-brake cosmology: The quantum model

Wheeler–DeWitt equation:

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi)$$

$$+\quad V_0 e^{6\alpha}\left(\sinh\left(\sqrt{3\kappa^2}|\phi|\right)-\frac{1}{\sinh\left(\sqrt{3\kappa^2}|\phi|\right)}\right)\Psi(\alpha,\phi)=0$$

($\kappa^2 = 8\pi G$, $\alpha = \ln a$, Laplace–Beltrami factor ordering)

vicinity of big-brake singularity: region of small ϕ; therefore use

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi)-\frac{\tilde{V}_0}{|\phi|}e^{6\alpha}\Psi(\alpha,\phi)=0\ ,$$

where $\tilde{V}_0 = V_0/3\kappa^2$.

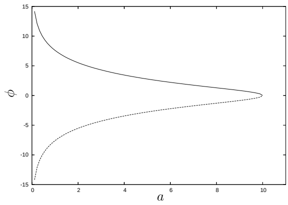
Normalizable solutions read

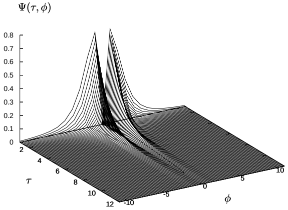
$$\Psi\left(\alpha,\phi\right)=\sum_{k=1}^{\infty}A(k)k^{-3/2}\mathrm{K}_0\left(\frac{1}{\sqrt{6}}\frac{V_\alpha}{\hbar^2 k\kappa}\right)$$
$$\times\left(2\frac{V_\alpha}{k}|\phi|\right)e^{-\frac{V_\alpha}{k|\phi|}}\mathrm{L}^1_{k-1}\left(2\frac{V_\alpha}{k}|\phi|\right)\ .$$

(K_0: Bessel function; L^1_{k-1}: Laguerre polynoms; $V_\alpha \equiv \tilde{V}_0 e^{6\alpha}$)

$\longrightarrow$ construction of wave packets

Normalizable solutions of the Wheeler–DeWitt equation vanish at the classical singularity





similar result for the corresponding loop quantum cosmology

(Kamenshchik, C. K., Sandhöfer 2007)

Quantum phantom cosmology

Classical model: Friedmann universe with scale factor $a(t)$ containing a scalar field with negative kinetic term ('phantom')
$\longrightarrow$ develops a big-rip singularity
(ρ and p diverge as a goes to infinity at a *finite time*)

Quantum model: Wave-packet solutions of the Wheeler–DeWitt equation disperse in the region of the classical big-rip singularity
$\longrightarrow$ time and the classical evolution come to an end;
only a stationary quantum state is left

Exhibition of quantum effects at large scales!

(Dąbrowski, C. K., Sandhöfer 2006)

Path Integral satisfies Constraints

- Quantum mechanics: path integral satisfies Schrödinger equation
- Quantum gravity: path integral satisfies Wheeler–DeWitt equation and diffeomorphism constraints

A. O. Barvinsky (1998): direct check in the one-loop approximation that the quantum-gravitational path integral satisfies the constraints
$\longrightarrow$ connection between covariant and canonical approach

application in quantum cosmology: no-boundary condition

Albert Einstein 1949:

Die Begriffe und Sätze erhalten “Sinn” bzw. “Inhalt” nur durch ihre Beziehung zu den Sinnenerlebnissen. Die Verbindung der letzteren mit den ersteren ist rein intuitiv, nicht selbst von logischer Natur. Der Grad der Sicherheit, mit der diese Beziehung bzw. intuitive Verknüpfung vorgenommen werden kann, und nichts anderes, unterscheidet die leere Phantasterei von der wissenschaftlichen “Wahrheit”.

Richard Feynman (as quoted by A. Ashtekar):

It is very important that we do not all follow the same fashion . . . It’s necessary to increase the amount of variety . . . the only way to do it is to implore you few guys to take a risk . . .

Ref.: C. K., *Quantum Gravity* (Second edition, Oxford 2007)