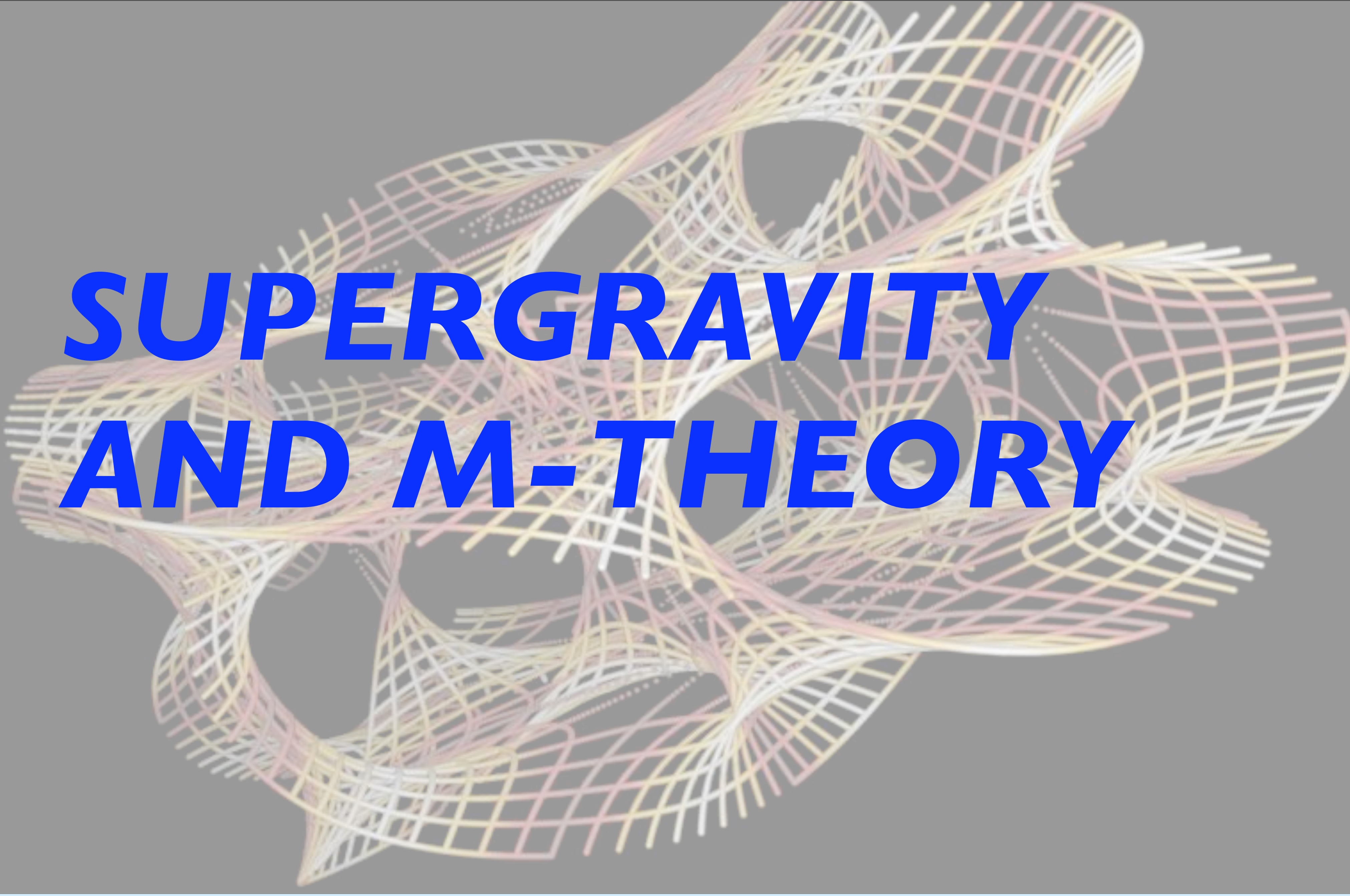

SUPERGRAVITY AND M-THEORY

Quantum Gravity: Challenges and Perspectives
Heraeus-Seminar, 14 - 16 April 2008

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Supergravity :

- provides the effective field theories for string compactifications
- provides the framework for many applications
- among those are topics that belong to the general relativity domain, such as black holes, cosmology, etcetera.

  There are many instances where supergravity can give the whole story, which is not surprising because a great many things depend strongly on the presence of symmetries, such as supersymmetry.

It seems, sometimes, that supergravity is already aware of the underlying string theory. In this talk we will see an example of this phenomenon.

Deforming supergravities by non-abelian gauge interactions reveals a restricted set of possible charges. Generically these gaugings involve a hierarchy of form fields of arbitrary rank.

This leads to intriguing connections with known results from M/String-Theory

Kaluza-Klein theory

toroidal compactification
pure gravity

$$\mathcal{M}^D \to \mathcal{M}^d \times T^n$$

$(D = d + n)$

$$g_{MN} \quad \to \quad g_{\mu\nu} + A_\mu{}^n + g_{mn}$$

→ massless states: graviton, n gauge fields (KK photons), $\frac{1}{2}n(n+1)$ scalar fields

infinite tower of massive graviton states

resulting theory is invariant under the group $\mathrm{GL}(n)$

non-linearly realized on the scalars: $\dfrac{\mathrm{GL}(n)}{\mathrm{SO}(n)}$

the massive states carry KK photon charges

charge lattice of KK tower: symmetry restricted to $\mathrm{GL}(n, \mathbb{Z})$

lower space-time dimensions do not follow the generic pattern:

three space-time dimensions: the vector fields can be dualized to scalars (Hodge duality)

massless: graviton (no states), $\frac{1}{2}n(n+3)$ scalars

symmetry non-linearly realized on the scalars $\dfrac{\mathrm{SL}(n+1)}{\mathrm{SO}(n+1)}$

systematic features of toroidal compactifications:

- ★ the rank of the invariance group increases with *n*
- ★ when starting with scalars that parametrize a homogeneous target space, the target space remains homogeneous
- ★ the presence of the massive states breaks the symmetry group to an arithmetic subgroup

Another example: graviton-tensor theory

the symmetry of the resulting compactified theory depends sensitively on the original theory

$$\mathcal{L}_D = -\tfrac{1}{2}\sqrt{g}\,R - \tfrac{3}{4}\sqrt{g}\big(\partial_{[M}B_{NP]}\big)^2$$

$$\begin{array}{ccc} g_{MN} & \rightarrow & g_{\mu\nu} + A_\mu{}^m + g_{mn} \\ B_{MN} & \rightarrow & B_{\mu\nu} + B_{m\mu} + B_{mn} \end{array}$$

$\Rightarrow$ $\mathrm{G} \subset \mathrm{SO}(n,n;\mathbb{Z})$

$\Rightarrow$ massless states: graviton, tensor, $2n$ spin-1 states, and n^2 spinless states

$\Rightarrow$ tower of massive graviton and tensor states not the generic pattern in five, four and three space-time dimensions !

e.g. upon including a dilaton in the original theory, one finds :

$$
\begin{array}{llll}
d > 5 & : & \mathrm{G} = \mathbb{R}^{+} \times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)\ \text{vectors} \\
d = 5 & : & \mathrm{G} = \mathbb{R}^{+} \times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)+1\ \text{vectors} \\
d = 4 & : & \mathrm{G} = \mathrm{SL}(2;\mathbb{Z}) \times \mathrm{SO}(n,n;\mathbb{Z}) & (n,n)+1\ \text{vectors} \\
d = 3 & : & \mathrm{G} = \mathrm{SO}(n+1,n+1;\mathbb{Z}) & 0\ \text{vectors}
\end{array}
$$

→ GOAL: study all possible deformations induced by gauging subgroups of G

The Hodge dilemma:

★ to increase the symmetry ⇒ dualize to lower-rank form fields

★ the presence of certain form fields may be an obstacle to certain gauge groups

★ what to do when the theory contains no (vector) gauge fields

Example: maximal supergravity in *3* space-time dimensions

Nicolai, Samtleben, 2000

gauging versus scalar-vector-tensor duality

128 scalars and *128* spinors, but no vectors !

obtained by dualizing vectors in order to realize the symmetry $E_{8(8)}(\mathbb{R})$

solution:

introduce *248* vector gauge fields with Chern-Simons terms

$$\mathcal{L}_{\rm CS} \propto g\,\varepsilon^{\mu\nu\rho}\,A_\mu{}^M\,\Theta_{MN}\left[\partial_\nu A_\rho{}^N - \frac{1}{3}g\,f_{PQ}{}^N A_\nu{}^P A_\rho{}^Q\right]$$

↑ EMBEDDING TENSOR

vectors 'invisible' at the level of the toroidal truncation

First the general analysis of gauge group embeddings :

Gauge group embeddings

gauge a subgroup of G, the symmetry group of the ungauged theory

with gauge fields $A_\mu{}^M$ transforming in some representation of G

gauge group encoded into the **EMBEDDING TENSOR** $\Theta_M{}^\alpha$

$$X_M = \Theta_M{}^\alpha \, t_\alpha$$

gauge group generators ↵ ↳ G generators

$\Theta_M{}^\alpha$ treated as a spurionic quantity,
transforming under the action of G
according to a product representation

dW, Nicolai, Samtleben, Trigiante, 2000-2008

This representation branches into irreducible representations.

Not all these representations are allowed !!

(for instance, because of supersymmetry)

Restricted deformations, for example:

EMBEDDING TENSORS FOR *D* = *3,4,5,6,7*

7	$\mathrm{SL}(5)$	$\mathbf{10} \times \mathbf{24} = \mathbf{10} + \boxed{\mathbf{15}} + \boxed{\overline{\mathbf{40}}} + \mathbf{175}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16} \times \mathbf{45} = \mathbf{16} + \boxed{\mathbf{144}} + \mathbf{560}$
5	$\mathrm{E}_{6(6)}$	$\mathbf{27} \times \mathbf{78} = \mathbf{27} + \boxed{\mathbf{351}} + \overline{\mathbf{1728}}$
4	$\mathrm{E}_{7(7)}$	$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \boxed{\mathbf{912}} + \mathbf{6480}$
3	$\mathrm{E}_{8(8)}$	$\mathbf{248} \times \mathbf{248} = \boxed{\mathbf{1}} + \mathbf{248} + \boxed{\mathbf{3875}} + \mathbf{27000} + \mathbf{30380}$

↑ D ↑ G ↑ M ↑ α

dW, Samtleben, Trigiante, 2002

- characterize all possible gaugings
- group-theoretical classification
- universal Lagrangians

one special (quadratic) constraint:

closure: $[X_M, X_N] = f_{MN}{}^P X_P$

$$\Theta_M{}^\beta \Theta_N{}^\gamma f_{\beta\gamma}{}^\alpha = f_{MN}{}^P \Theta_P{}^\alpha = -\Theta_M{}^\beta t_{\beta N}{}^P \Theta_P{}^\alpha$$

$f_{\beta\gamma}{}^\alpha \Theta_M{}^\beta \rightarrow -(X_M)_\gamma{}^\alpha$; $\Theta_M{}^\beta t_{\beta N}{}^P \rightarrow X_{MN}{}^P \in \mathrm{g}$

$\Leftrightarrow$ $\Theta_M{}^\alpha$ is invariant under the gauge group

$$\Leftrightarrow \quad [X_M, X_N] = X_{MN}{}^P X_P$$

$X_{MN}{}^P$ contains the gauge group structure constants, but is in general not symmetric in lower indices, unless contracted with the embedding tensor !!!!

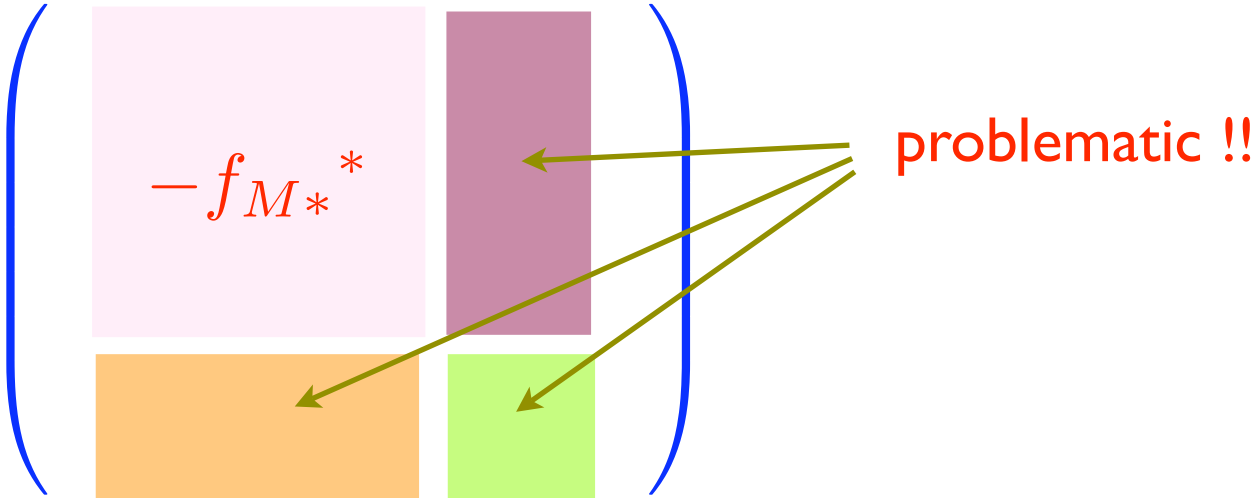
$$Z^M{}_{NP} \equiv X_{(NP)}{}^M \qquad\qquad Z^M{}_{NP}\,\Theta_M{}^\alpha = 0$$

Jacobi identity affected :

$$X_{[NP}{}^R X_{Q]R}{}^M = \tfrac{2}{3} Z^M{}_{R[N} X_{PQ]}{}^R$$

in special basis:

$$X_{MN}{}^{P} =$$



the gauge fields $A_\mu{}^M$ not involved in the gauging can still carry charges
this is known to be inconsistent ! To see this:

covariant derivative $\quad D_\mu = \partial_\mu - g\, A_\mu{}^M\, X_M$

Ricci identity $\quad [D_\mu, D_\nu] = -g\, \mathcal{F}_{\mu\nu}{}^M\, X_M$

field strength

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{NP}{}^M\, A_{[\mu}{}^N A_{\nu]}{}^P$$

anti-symmetric part

Palatini identity

$$\delta \mathcal{F}_{\mu\nu}{}^{M} = 2\, D_{[\mu} \delta A_{\nu]}{}^{M} - 2\, g\, Z^{M}{}_{PQ}\, \delta A_{[\mu}{}^{P} A_{\nu]}{}^{Q}$$

NOT covariant indeed !

options:

★ try to enlarge/change the gauge group

or .....

★ introduce an extra gauge transformation $\delta_{\Xi} A_{\mu}{}^{M} = -g\, Z^{M}{}_{NP} \Xi_{\mu}{}^{NP}$

and

introduce 2-form gauge fields $B_{\mu\nu}{}^{MN}$ whose variation cancels the undesirable terms:

$$\mathcal{F}_{\mu\nu}{}^{M} \rightarrow \mathcal{H}_{\mu\nu}{}^{M} = \mathcal{F}_{\mu\nu}{}^{M} + g\, Z^{M}{}_{NP} B_{\mu\nu}{}^{NP}$$

$Z^{M}{}_{NP}$ acts as an intertwining tensor between the gauge field representation and the 2-form field representation

subtle: regard (NP) as a single index, which does not map into the full symmetric tensor product !

This leads to, e.g.

$$
\begin{aligned}
\delta B_{\mu\nu}{}^{MN} &= 2\, D_{[\mu}\Xi_{\nu]}{}^{MN} - 2\,\Lambda^{\lceil M}\mathcal{H}_{\mu\nu}{}^{N\rfloor} \\
&\quad + 2\, A_{[\mu}{}^{\lceil M}\delta A_{\nu]}{}^{N\rfloor} \\
&\quad - g\, Y^{MN}{}_{P\lceil RS\rfloor}\, \Phi_{\mu\nu}{}^{P\lceil RS\rfloor}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{\mu\nu\rho}{}^{MN} &= 3\, D_{[\mu}B_{\nu\rho]}{}^{MN} \\
&\quad + 6\, A_{[\mu}{}^{\lceil M}\left(\partial_\nu A_{\rho]}{}^{N\rfloor} + \tfrac{1}{3} g X_{[PQ]}{}^{N\rfloor} A_\nu{}^P A_{\rho]}{}^Q\right) \\
&\quad + g\, Y^{MN}{}_{P\lceil RS\rfloor}\, C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}
\end{aligned}
$$

etcetera

where

	$\Phi_{\mu\nu}{}^{P\lceil RS\rfloor}$	new gauge parameter
	$C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}$	new tensor field
	$Y^{MN}{}_{P\lceil RS\rfloor}$	new covariant tensor proportional to the embedding tensor

Hierarchy of p-form fields

this structure continues indefinitely

$$A_\mu{}^{M} \longrightarrow B_{\mu\nu}{}^{MN} \longrightarrow C_{\mu\nu\rho}{}^{MNP} \longrightarrow \cdots$$

(*p*-form gauge fields)

$$\Lambda^{M} \longrightarrow \Xi_\mu{}^{MN} \longrightarrow \Phi_{\mu\nu}{}^{MNP} \longrightarrow \cdots$$

(transformation parameters)

$$Z^{M}{}_{NP} \longrightarrow Y^{MN}{}_{PQR} \longrightarrow Y^{MNP}{}_{QRST} \longrightarrow \cdots$$

(intertwining tensors)

the covariant intertwining tensors are all proportional to the embedding tensor and mutually orthogonal

dW, Samtleben, 2005
dW, Nicolai, Samtleben, 2008

Alternative deformations (digression)

An obvious question is whether the gaugings discussed so far are the only viable deformations. While it is true that other deformations are known in supergravity, there are indications that these deformations are already incorporated in the present approach.

$$
\begin{aligned}
\mathcal{H}_{\mu\nu}{}^{M} &= \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\, X_{NP}{}^M\, A_{[\mu}{}^N A_{\nu]}{}^P \\
&\quad + g\, Z^M{}_{NP} B_{\mu\nu}{}^{NP} \\
\mathcal{H}_{\mu\nu\rho}{}^{MN} &= 3\, D_{[\mu} B_{\nu\rho]}{}^{MN} \\
&\quad + 6\, A_{[\mu}{}^{\lceil M} \left(\partial_\nu A_{\rho]}{}^{N\rfloor} + \tfrac{1}{3} g X_{[PQ]}{}^{N\rfloor} A_\nu{}^P A_{\rho]}{}^Q \right) \\
&\quad + g\, Y^{MN}{}_{P\lceil RS\rfloor}\, C_{\mu\nu\rho}{}^{P\lceil RS\rfloor}
\end{aligned}
$$

✓ $\mathcal{O}(g^0)$: *survives* $g = 0$ *limit* *(known from Einstein-Maxwell SG)*

✓ $Z^M{}_{NP}\, \Theta_M{}^\alpha = 0 \Longrightarrow \Theta = 0\,,\ Z \neq 0$

(Romans massive deformation)

At this point there is no Lagrangian yet. (There exist universal Lagrangians!) In the context of a Lagrangian the transformations of the gauge hierarchy are subject to change.

Often the hierarchy breaks off at some point and higher rank forms do not appear in the Lagrangian (projection)

The physical degrees of freedom are shared between the various tensor fields in a way which depends on the embedding tensor.

studied in $D = 2,3,4,5,6,7$ space-time dimensions
in $D=4$, for $N = 2,4,8$ supergravities
in $D=3$, for $N = 1,\dots,6,8,9,10,12,16$ supergravities

Bergshoeff, de Vroome, dW, Herger, Nicolai, Samtleben, Schön, Sezgin, Trigiante, Weidner, etc

→ applications

Gauged supergravity, flux compactifications, etcetera, and.......

Maximal supergravities

Apply the embedding tensor formalism to the maximal supergravities, with the duality group, the representations of the vector gauge fields and the embedding tensor as input.

At this point, the number of space-time dimensions is not used.

This analysis yields the representations for the hierarchy of form fields.

Leads to :

	rank ⇨	1	2	3	4	5	6
7	SL(5)	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	SO(5, 5)	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$E_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Striking feature:

rank *D-2* : adjoint representation of the duality group

dW, Samtleben, Nicolai, 2008

note: restricted representation, not the full symmetric tensor product

	rank ⇨	1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
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3	$\mathrm{E}_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Striking feature:

rank *D-1* : embedding tensor !

	rank ⇨	1	2	3	4	5	6
7	$SL(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}} + \mathbf{40}$
6	$SO(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$E_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{133}+\mathbf{8165}$		
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Striking feature:

rank *D* : closure constraint on the embedding tensor !

	rank ⇨	1	2	3	4	5	6
7	SL(5)	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	SO(5, 5)	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
5	$E_{6(+6)}$	$\overline{\mathbf{27}}$	$\mathbf{27}$	$\mathbf{78}$	$\mathbf{351}$	$\mathbf{27}+\mathbf{1728}$	
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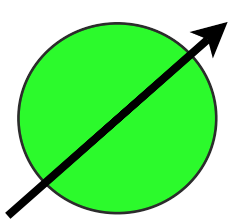
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$	$\mathbf{3875}+\mathbf{147250}$			

Perhaps most striking:

implicit connection between space-time electric/magnetic (Hodge) duality and the U-duality group



Θ dial

Probes new states in M-Theory!

Life at the end of the hierarchy:

		1	2	3	4	5	6
7	$\mathrm{SL}(5)$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
6	$\mathrm{SO}(5,5)$	$\mathbf{16}_c$	$\mathbf{10}$	$\overline{\mathbf{16}}_s$	$\mathbf{45}$	$\mathbf{144}_s$	$\mathbf{10}+\mathbf{126}_s+\mathbf{320}$
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It is possible to construct the hierarchy starting from the intermediate *(D-3)*-forms, assuming that they transform according to the conjugate of the representation associated with the vector fields. In this way one generates the *(D-2)*-, the *(D-1)*-, and the *D*-form fields, in accordance we the results found in the table. Note that the latter two forms are not related to any other forms by Hodge duality!

p-forms transforming in the conjugate of the representations of the *1*-forms, the adjoint representation, the embedding tensor and the constraints

$$\Delta \overset{[D-3]}{C}{}_{M} = D \overset{[D-4]}{\Phi}{}_{M} + \cdots - \boxed{Y_M{}^{\alpha}} \overset{[D-3]}{\Phi}{}_{\alpha}$$

$$\Delta \overset{[D-2]}{C}{}_{\alpha} = D \overset{[D-3]}{\Phi}{}_{\alpha} + \cdots - \boxed{Y_{\alpha,M}{}^{\beta}} \overset{[D-2]}{\Phi}{}^{M}{}_{\beta}$$

$$\Delta \overset{[D-1]}{C}{}^{M}{}_{\alpha} = D \overset{[D-2]}{\Phi}{}^{M}{}_{\alpha} + \cdots - \boxed{Y^{M}{}_{\alpha,PQ}{}^{\beta}} \overset{[D-1]}{\Phi}{}^{PQ}{}_{\beta}$$

$$\Delta \overset{[D]}{C}{}^{MN}{}_{\alpha} = D \overset{[D-1]}{\Phi}{}^{MN}{}_{\alpha} + \cdots - \boxed{Y^{MN}{}_{\alpha,PQR}{}^{\beta}} \boxed{\overset{[D]}{\Phi}{}^{PQR}{}_{\beta}}$$

$$\Delta \overset{[D+1]}{C}{}^{PQR}{}_{\alpha} = D \overset{[D]}{\Phi}{}^{PQR}{}_{\alpha} + \cdots - \cdots$$

intertwiners

closure constraint

$$\mathcal{Q}_{MN}{}^{\alpha} \equiv \delta_M \Theta_N{}^{\alpha} = \Theta_M{}^{\beta}\, \delta_\beta \Theta_N{}^{\alpha}$$

intertwiners

$$\begin{aligned} Y_M{}^{\alpha} &= \Theta_M{}^{\alpha} \\ Y_{\alpha,M}{}^{\beta} &= \delta_\alpha \Theta_M{}^{\beta} \\ Y^M{}_{\alpha,PQ}{}^{\beta} &= \frac{\delta}{\delta\, \Theta_M{}^{\alpha}}\, \mathcal{Q}_{PQ}{}^{\beta} \\ Y^{MN}{}_{\alpha,PQR}{}^{\beta} &= -\delta_P^M\, Y^N{}_{\alpha,QR}{}^{\beta} + X_{PQ}{}^{M}\, \delta_R^N \delta_\alpha^\beta + X_{PR}{}^{N}\, \delta_Q^M \delta_\alpha^\beta - X_{P\alpha}{}^{\beta}\, \delta_R^N \delta_Q^M \end{aligned}$$

orthogonality:

$$\begin{aligned} Y \times Y' &\propto \mathcal{Q}_{MN}{}^{\alpha} \\ Y^{MN}{}_{\alpha,PQR}{}^{\beta}\, \mathcal{Q}_{MN}{}^{\alpha} &= 0 \end{aligned}$$

What is the role of the higher form fields ?

This construction supports the following idea which has been worked out completely for three space-time dimensions:

Regard the embedding tensor as a space-time field transforming in the appropriate representation, but not satisfying the quadratic closure constraint. Add the gauge invariant Lagrangian with *(D-1)*- and *D*-form fields:

$$
\begin{aligned}
\mathcal{L} \;=\; & g\,\varepsilon^{\mu_1\mu_2\cdots\mu_D}\, C_{\mu_1\cdots\mu_{D-1}}{}^{M}{}_{\alpha}\, D_{\mu_D}\Theta_M{}^{\alpha} \\
& + g^2\,\varepsilon^{\mu_1\mu_2\cdots\mu_D}\, C_{\mu_1\cdots\mu_D}{}^{MN}{}_{\alpha}\, Q_{MN}{}^{\alpha}
\end{aligned}
$$

dW, Samtleben, Nicolai, 2008

M-theory implications:

		1	2	3	4	5	6
7	SL(5)	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{15}}+\mathbf{40}$
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The table coincides substantially with results based on several rather different conceptual starting points:

- M(atrix)-Theory compactified on a torus: duality representations of states
- Correspondence between toroidal compactifications of M-Theory and del Pezzo surfaces
- ***E11*** decompositions

• Algebraic Aspects of Matrix Theory on T^n

Elitzur, Giveon, Kutasov, Rabinovici, 1997

Based on the correspondence between super-Yang-Mills on T^n and M-Theory on $\tilde{T}^n$, a rectangular torus with radii $R_1, R_2, \ldots R_n$ in the infinite-momentum frame.

Invariance group consist of permutations of the R_i combined with the *T*-duality relations ($i \neq j \neq k$) :

$$R_i \to \frac{l_\mathrm{p}^3}{R_j R_k} \qquad R_j \to \frac{l_\mathrm{p}^3}{R_k R_i} \qquad R_k \to \frac{l_\mathrm{p}^3}{R_i R_j} \qquad l_\mathrm{p}^3 \to \frac{l_\mathrm{p}^6}{R_i R_j R_k}$$

generate a group isomorphic with the Weyl group of $\mathrm{E}_{n(n)}$

The explicit duality multiplets arise as representations of this group.

Example $n=4$ → $D=7$

4 KK states on T^n $\qquad M \sim \dfrac{1}{R_i}$

6 2-brane states wrapped on T^n $\qquad M \sim \dfrac{R_j R_k}{l_{\rm p}^3} \qquad j \neq k$

4 2-brane states wrapped on $T^n \times x^{11}$ $\qquad M \sim \dfrac{R_{11} R_i}{l_{\rm p}^3}$

1 5-brane state wrapped on $T^n \times x^{11}$ $\qquad M \sim \dfrac{R_{11} R_1 R_2 R_3 R_4}{l_{\rm p}^6}$

the dimensions of these two multiplets coincide with those of the multiplets presented previously for vectors and tensors

for higher n the multiplets are sometimes incomplete, because they are not generated as a single orbit by the Weyl group.

- **A Mysterious Duality**

Iqbal, Neitzke, Vafa, 2001

This cannot be a coincidence!

> It is important to uncover the physical interpretation of these duality relations. One possibility is that the del Pezzo surface is the moduli space of some probe in M-Theory. It must be a U-duality invariant probe ....

Such probe is the gauging encoded in the embedding tensor!

- ***E11* decomposition**

> Based on the conjecture that *E11* is the underlying symmetry of M-Theory. Decomposing the relevant *E11* representation to dimensions $D<11$ yields representations that substantially overlap with those generated for the gaugings.

West et. al., 2001-2007
Bergshoeff et. al.,2005-2007

Conclusions

- ✦ There are unexpected intriguing connections with other results derived on the basis of rather different concepts
- ✦ Gaugings probe new degrees of freedom of M-Theory
- ✦ Maximal supergravity theories contain subtle information about M-Theory. This may be interpreted as an indication that supergravity needs to be extended towards string/M-theory. This is also indicated by comparing degrees of freedom originating from the maximal theories in various dimensions.
- ✦ More work needs to be done on clarifying these connections