Bootstrapping the AdS Virasoro-Shapiro amplitude

Tobias Hansen, Durham University

Supergravity techniques and the CFT bootstrap
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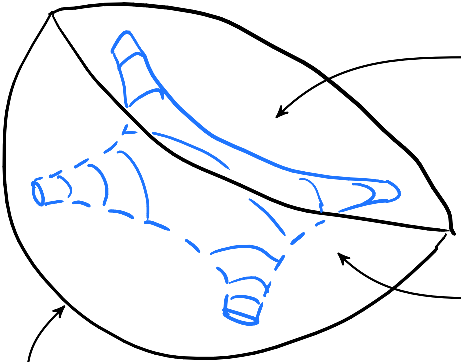
Based on:

2204.07542, 2209.06223, 2303.08834, 2305.03593 with Luis F. Alday, João Silva

2306.12786 with Luis F. Alday

2308.03683 with Giulia Fardelli, João Silva

1 process - 3 descriptions



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

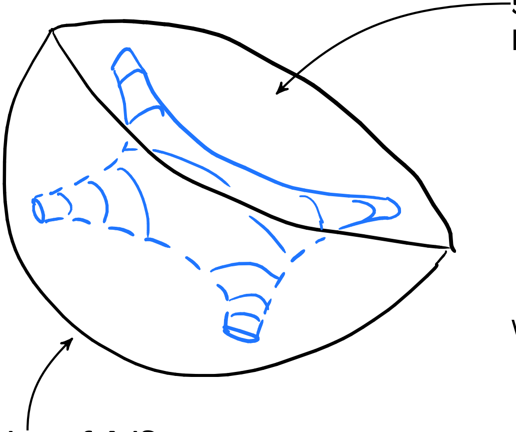
- strings on curved background

2d string world-sheet:
2d CFT???

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- non-abelian gauge theory
- conformal symmetry
- supersymmetry
- integrable

This talk:
Find the amplitude without quantizing the string.



5d bulk of AdS:
IIb string theory on $AdS_5 \times S^5$

- AdS radius R_{AdS}
- string length L_s
- string coupling g_s

4d boundary of AdS:
$\mathcal{N} = 4$ super Yang Mills theory

- $SU(N)$ gauge group
- coupling $\sqrt{\lambda} = \frac{R_{\text{AdS}}^2}{L_s^2}$

Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

Expansion around flat space:

$$\frac{R_{\text{AdS}}^2}{L_s^2} \gg 1 \quad \Leftrightarrow \quad \sqrt{\lambda} \gg 1$$

$$L_s^2 p_i \cdot p_j \text{ finite}$$

Plan for the talk

1. Review: String scattering in flat space
2. String scattering in AdS
    1. The CFT dispersion relation
    2. Single-valued functions for the world-sheet
    3. Checks: Integrability and Localization
    4. Including KK modes

1. Flat Space Review

The Virasoro-Shapiro amplitude (flat space)

In the beginning, there was the amplitude.
[Veneziano,1968;Virasoro,1969;Shapiro,1970]

Scattering of 4 gravitons in the type IIb superstring:

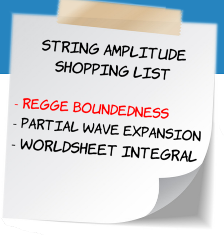
Virasoro-Shapiro amplitude

$$A^{(0)}(S,T) = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\,\Gamma(T+1)\,\Gamma(U+1)}$$

$$S = -\frac{L_s^2}{4}(p_1+p_2)^2\,,\ T = -\frac{L_s^2}{4}(p_1+p_3)^2\,,\ U = -\frac{L_s^2}{4}(p_1+p_4)^2$$

$$S+T+U=0$$

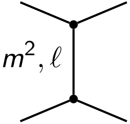
Regge boundedness (flat space)



String amplitudes have soft UV (Regge) bahaviour

$$\lim_{|S|\to\infty} A^{(0)}(S,T) \sim S^{T+\alpha_0}, \quad \mathrm{Re}(T) < 0$$

and higher spin resonances



$$= \frac{P_\ell(S)}{T - m^2} \qquad P_\ell(S) = S^\ell + O(S^{\ell-1})$$

Regge bahaviour places strong constraints on the coefficients $a_{\delta,\ell}$ in

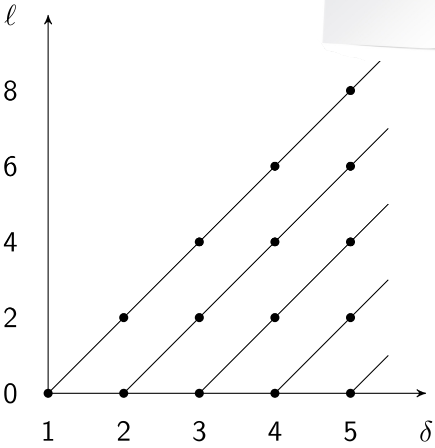
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T - \delta}$$

The spectrum (flat space)

The exchanged massive string spectrum is extracted via the partial wave expansion

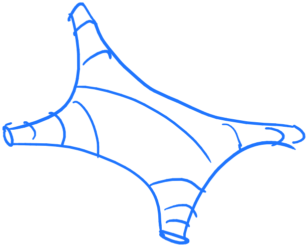
$$A^{(0)}(S,T) = \sum_{(\delta,\ell)} \frac{a_{\delta,\ell} P_\ell(S)}{T-\delta}$$

It forms linear Regge trajectories.



World-sheet integral (flat space)

The amplitude is also given by an integral over world-sheets:



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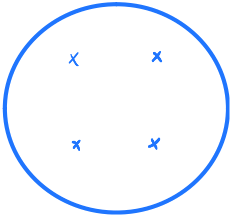


$$A^{(0)}(S,T)=\int d^2z\;|z|^{-2S-2}|1-z|^{-2T-2}G^{(0)}_{\text{tot}}(S,T,z)$$

$$G^{(0)}_{\text{tot}}(S,T,z)=\frac{1}{3}\left(\frac{1}{U^2}+\frac{|z|^2}{S^2}+\frac{|1-z|^2}{T^2}\right)$$

The integrand is a single-valued function of z!

Low energy expansion (flat space)

Low energy effective action (supergravity + derivative interactions)
→ Low energy expansion:

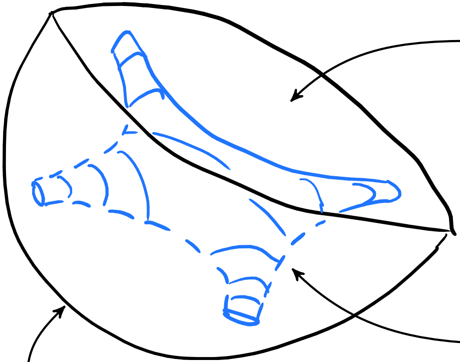
$$A^{(0)}(S,T) = \frac{1}{STU} + \sum_{a,b=0}^{\infty} (S^2+T^2+U^2)^a (STU)^b \alpha^{(0)}_{a,b}$$

$$= \underbrace{\frac{1}{STU}}_{\text{sugra}} + \underbrace{\alpha^{(0)}_{0,0}}_{R^4} + \underbrace{(S^2+T^2+U^2)\alpha^{(0)}_{1,0}}_{D^4 R^4} + \underbrace{(STU)\alpha^{(0)}_{0,1}}_{D^6 R^6} + \ldots$$

Wilson coefficients $\alpha^{(0)}_{a,b}$ are in the ring of single-valued multiple zeta values [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Example: $\alpha^{(0)}_{a,0} = \zeta(3+2a)\,, \qquad \alpha^{(0)}_{a,1} = \sum_{\substack{i_1,i_2=0 \\ i_1+i_2=a}}^{a} \zeta(3+2i_1)\zeta(3+2i_2)$

2. String scattering in AdS



5d bulk of AdS:
scattering amplitude of 4 gravitons:

$$A(S,T) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\text{AdS}}^2}\right)^k A^{(k)}(S,T)$$

2d string world-sheet:
world-sheet integrand:

$$G_{\text{tot}}(S,T,z) = \sum_{k=0}^{\infty} \left(\frac{L_s^2}{R_{\text{AdS}}^2}\right)^k G_{\text{tot}}^{(k)}(S,T,z)$$

4d boundary of AdS:
correlator of 4 stress-tensors:

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

Boundary correlator to bulk amplitude

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

↓ superconformal Ward identity

$$H(U,V) \qquad U = \frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2},\ V = \frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$

↓ Mellin transform

$$M(s,t)$$

↓ Borel transform (flat space limit [Penedones;2010])

$$A(S,T) = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{\lambda}}\right)^k A^{(k)}(S,T)$$

↓ world-sheet integral

$$A^{(k)}(S,T) = \int d^2z\ |z|^{-2S-2}|1-z|^{-2T-2} G^{(k)}_{\text{tot}}(S,T,z)$$

The Mellin transform

Mellin transform

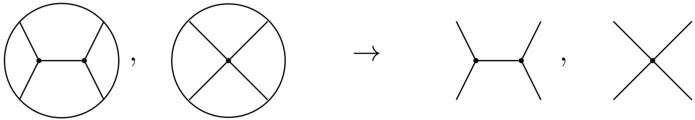
$$H(U,V)=\int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}}V^{\frac{t}{2}-2}\Gamma\left(2-\frac{s}{2}\right)^2\Gamma\left(2-\frac{t}{2}\right)^2\Gamma\left(2-\frac{u}{2}\right)^2 M(s,t)$$

The Borel transform

Borel transform

$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i}\, e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

1. Maps Witten diagrams to Feynman diagrams for $R_{\text{AdS}} \to \infty$ [Penedones;2010]



2. Borel summation of the low energy expansion:

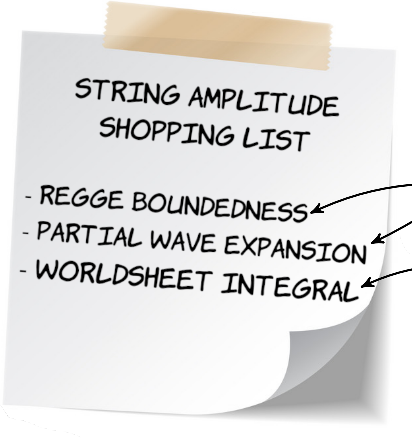
$$M(s,t) = \sum_{p,q} \frac{\Gamma(6+p+q)}{\lambda^{\frac{3}{2}}} \left(\frac{s}{2\sqrt{\lambda}}\right)^p \left(\frac{t}{2\sqrt{\lambda}}\right)^q \alpha_{p,q} \quad \Rightarrow \quad A(S,T) = \sum_{p,q} S^p T^q \alpha_{p,q}$$

3. Stringy flat space limit:

$$\sqrt{\lambda} = \frac{R_{\text{AdS}}^2}{L_s^2} >> 1\,, \qquad S \sim \frac{L_s^2}{R_{\text{AdS}}^2} s \sim L_s^2 (p_1+p_2)^2 \text{ finite}$$

Plan of attack

We attack the problem from 2 sides:



Dispersion relation for $M(s, t)$

Single-valued ansatz for $G(S, T, z)$

Both have unfixed data.
Equating the two expressions fixes the answer!

2.1. The CFT dispersion relation

Operator product expansion

We can expand $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ using:

Operator product expansion (OPE)

$$\mathcal{O}_2(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_{\Delta,\ell}\ \text{primaries}} C_{\Delta,\ell}\ c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\big|_{y=0}$$

OPE data

- $\Delta =$ dimension
- $\ell =$ spin
- $C_{\Delta,\ell} =$ OPE coefficients

$M(s,t)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Poles and residues of $M(s,t)$

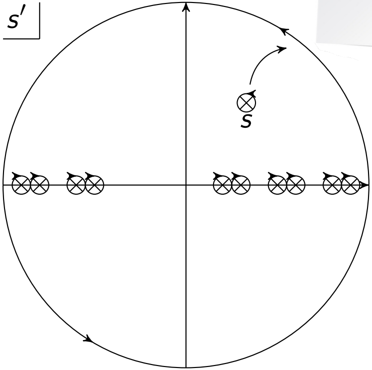
$$M(s,t) \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

$M(s,t)$ has only OPE poles:

$$\text{poles} \sim \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s' - (\Delta - \ell + 2m)}$$

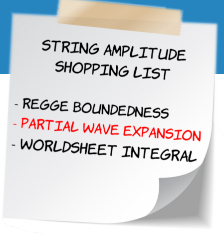
Regge bounded due to bound on chaos: [Maldacena,Shenker,Stanford;2015]

$$\lim_{|s|\to\infty} |M(s,t)| \lesssim |s|^{-2}, \ \mathrm{Re}(t) < 2$$



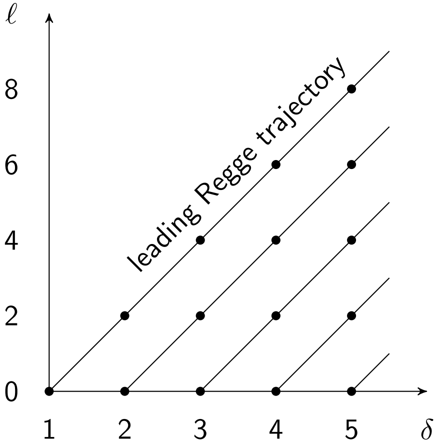
$$M(s,t) = \oint_s \frac{ds'}{2\pi i} \frac{M(s',t)}{(s'-s)} = -\sum_{\text{operators}} \left(\frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{s - (\Delta - \ell + 2m)} + \frac{C^2_{\Delta,\ell} Q_{\Delta,\ell,m}(t)}{u - (\Delta - \ell + 2m)} \right)$$

Spectrum of exchanged operators



Exchanged operators: massive string modes
$=$ unprotected single-trace operators of $\mathcal{N}=4$ SYM theory

$$\Delta(\Delta-d)=R^2m^2=R^2\frac{4\delta}{L_s^2}+O(\lambda^0)\quad\Rightarrow\quad\Delta=2\sqrt{\delta}\lambda^{\frac{1}{4}}+O(\lambda^0)$$



known from flat space ($A^{(0)}$ data)

	$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data
$\Delta_{\delta,\ell}=$	$\lambda^{\frac{1}{4}}\Delta^{(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$
$C^2_{\delta,\ell}=$	$C^{2(0)}_{\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$	$+$	$\lambda^{-1}C^{2(2)}_{\delta,\ell}$

$\Delta^{(1)}_{\delta,\ell}$, $\Delta^{(2)}_{\delta,\ell}$ on leading trajectory known from integrability

Degeneracies in the spectrum

The amplitude encodes OPE data of multiple degenerate superprimaries.

Find degeneracies starting from type IIb strings in flat 10d:
[Bianchi,Morales,Samtleben;2003],[Alday,TH,Silva;2023]

$$SO(9) \to SO(4)\times SO(5) \overset{KK}{\to} SO(4)\times SO(6)$$

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

Number of superconformal long multiplets with superprimary $\mathcal{O}_{\delta,\ell}$

- $SO(6)$ singlet
- $\Delta = 2\sqrt{\delta}\lambda^{\frac{1}{4}} + O(\lambda^0)$

Example: $\mathcal{O}_{1,0} = \text{Konishi} \sim \text{Tr}(\phi^I\phi_I)$

The counting was confirmed for $\delta \leq 3$ with quantum spectral curve.
[Gromov,Hegedus,Julius,Sokolova;2023]

Dispersion relation $\to$ Residues

Dispersion relation for $M(s,t) \rightsquigarrow A^{(k)}(S,T)$ expanded around $S=\delta=1,2,\ldots$:

$$A^{(k)}(S,T)=\frac{R^{(k)}_{3k+1}(T,\delta,C^{2(0)}_{\delta,\ell})}{(S-\delta)^{3k+1}}+\ldots+\frac{R^{(k)}_{1}(T,\delta,C^{2(0)}_{\delta,\ell},\ldots,\Delta^{(k)}_{\delta,\ell},C^{2(k)}_{\delta,\ell})}{S-\delta}+\text{reg.}$$

Two lessons:

1. (OPE data)$^{(k-1)}$ fixes most residues of $A^{(k)}(S,T)$!
2. $G^{(k)}_{\text{tot}}(S,T,z)$ should have transcendentality $3k$:

$$\int d^2z\,|z|^{-2S-2}|1-z|^{-2T-2}\log^{3k}|z|^2 \propto \frac{1}{(S-\delta)^{3k+1}}+O\left((S-\delta)^0\right)$$

Next steps (order by order):

- Write world-sheet ansatz for $A^{(k)}(S,T)$.
- Compute its residues and match with the above to fix ansatz.

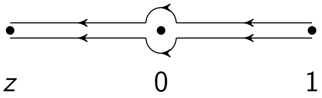
2.2. Single-valued functions for the world-sheet

Single-valued functions

Example for multivalued function:

$$\log z = \int_\sigma \frac{dt}{t}\,, \qquad \sigma = \text{path from 1 to z}$$

Depends on integration path:



$\Rightarrow \log z$ is well defined up to

$$n \oint_0 \frac{dt}{t} = 2\pi i n\,,\, n \in \mathbb{Z}$$

Single-valued version:

$$\mathcal{L}_0(z) \equiv 2\,\mathrm{Re}(\log z) = \log z + \log \bar{z} = \log |z|^2$$

Smaller function space $\rightarrow$ constraining power of imposing single-valuedness:

$$\text{multi-valued}: \ \log z, \log \bar{z}$$

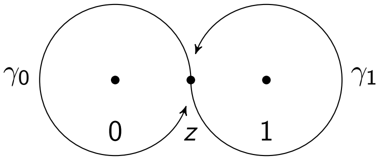
$$\text{single-valued}: \ \log |z|^2$$

More single-valued functions

Dilogarithm:

$$\mathrm{Li}_2(z) = \int\limits_{0\leq t_1\leq t_2\leq z} \frac{dt_1}{t_1-1}\frac{dt_2}{t_2}$$

M_x = analytic continuation along γ_x



$$M_1\mathrm{Li}_2(z) = \mathrm{Li}_2(z) + 2\pi i \log z$$
$$M_0\mathrm{Li}_2(z) = \mathrm{Li}_2(z)$$

Single-valued version:

$$\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1-\bar{z})\log|z|^2$$
$$M_1\mathcal{L}_{01}(z) = M_0\mathcal{L}_{01}(z) = \mathcal{L}_{01}(z)$$

For arbitrary iterated integrals [Brown;2004] :

$\mathcal{L}_{abc\ldots}(z)$	single-valued multiple polylogarithms (SVMPLs)
$\mathcal{L}_{abc\ldots}(1)$	single-valued multiple zeta values (SVMZVs)

e.g. $\mathrm{Li}_2(1) = \zeta(2)$, $\mathcal{L}_{01}(1) = 0$, $\mathcal{L}_{001}(1) = -\zeta^{\mathrm{sv}}(3) = -2\zeta(3)$

Toy model for strings in AdS

Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

$$= S_{\text{flat}} + \frac{1}{R_{\text{AdS}}^2} \underbrace{\lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} V^{\mu\nu}_{\text{graviton}}(q)}_{\equiv \tilde{V}} + \dots$$

AdS metric expanded around flat space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R_{\text{AdS}}^2} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} e^{iq\cdot X}$$

Amplitude:

$$A_4(p_i) \sim \int \mathcal{D}X \mathcal{D}g\, e^{-S_P} V^4_{\text{graviton}} = \int \mathcal{D}X \mathcal{D}g\, e^{-S_{\text{flat}}} \left(1 - \frac{\tilde{V}}{R_{\text{AdS}}^2} + \frac{1}{2}\frac{\tilde{V}^2}{R_{\text{AdS}}^4} + \dots\right) V^4_{\text{graviton}}$$

$$\Rightarrow \quad A_4^{(k)}(p_i) \sim \lim_{q_i\to 0} \left(\frac{\partial}{\partial q_i}\right)^{2k} A^{(0)}_{4+k}(p_i, q_i) + \dots$$

Soft gravitons in flat space

$$A_4^{(k)}(p_i) \sim \lim_{\epsilon \to 0} \left(\frac{\partial}{\epsilon\, \partial q_i} \right)^{2k} A_{4+k}^{(0)}(p_i, \epsilon q_i) + \ldots$$

Soft graviton theorem:

$$A_{n+1}(p_1, \ldots, p_n, \epsilon q) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon} \frac{\varepsilon_{\mu\nu} p_i^{\mu} p_i^{\nu}}{p_i \cdot q} + \frac{\varepsilon \cdot p_i \varepsilon_{\mu} q_{\nu} J_i^{\mu\nu}}{p_i \cdot q} + O(\epsilon) \right) A_n(p_1, \ldots, p_n)$$

Flat space amplitude with k soft gravitons:

$$A_{4+k}^{(0)}(p_i, \epsilon q_i) \sim \frac{1}{\epsilon^k} A_4^{(0)}(p_i) + \frac{1}{\epsilon^{k-1}} \text{"}\partial_{p_i}\text{"}\, A_4^{(0)}(p_i) + \ldots$$

$$\sim \int d^2 z |z|^{-2S-2} |1-z|^{-2T-2} \left(\frac{1}{\epsilon^k} + \frac{1}{\epsilon^{k-1}} \left(\# \log|z|^2 + \# \log|1-z|^2 \right) + \ldots + \epsilon^{2k} \mathcal{L}_{|w|=3k}(z) \right)$$

$$\Rightarrow \quad G_{\text{tot}}^{(k)}(S, T, z) \sim \text{single-valued multiple polylogs of weight } \leq 3k$$

World-sheet correlator (ansatz)

Ansatz:

$$A^{(k)}(S,T) = B^{(k)}(S,T) + B^{(k)}(U,T) + B^{(k)}(S,U)$$

$$B^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G^{(k)}(S,T,z)$$

Assumed properties of $G^{(k)}(S,T,z)$:

- uniform transcendentality $3k$ (SVMPLs(z), SVMZVs)
- rational function in S, T with homogeneity $2k-2$
- denominator $= U^n$, $n \leq 2$
- crossing symmetry: $G^{(k)}(S,T,z) = G^{(k)}(T,S,1-z)$

Recall (flat space):

$$G^{(0)}(S,T,z) = \frac{1}{3U^2}$$

World-sheet correlator (first correction)

Symmetrised single-valued multiple polylogs:

$$\mathcal{L}^{\pm}_w(z) = \mathcal{L}_w(z) \pm \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) \pm \mathcal{L}_w(1-\bar{z})$$

$$k = 1: \quad \text{weight 3 basis} = \text{4 symmetric} + \text{3 antisymmetric functions}$$

Solution:

$$\begin{aligned} G^{(1)}(S,T,z) = & -\frac{1}{6}\mathcal{L}^{+}_{000}(z) + 0\mathcal{L}^{+}_{001}(z) - \frac{1}{4}\mathcal{L}^{+}_{010}(z) + 2\zeta(3) \\ & + \frac{S-T}{S+T}\left(-\frac{1}{6}\mathcal{L}^{-}_{000}(z) + \frac{1}{3}\mathcal{L}^{-}_{001}(z) + \frac{1}{6}\mathcal{L}^{-}_{010}(z)\right) \end{aligned}$$

World-sheet correlator (second correction)

STRING AMPLITUDE SHOPPING LIST

- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- WORLDSHEET INTEGRAL

$k = 2:$ weight 6 basis = 25 symmetric + 20 antisymmetric functions:

$$\begin{aligned}\vec{L}^{+} = \Big(&\mathcal{L}^{+}_{000000}(z), \mathcal{L}^{+}_{000001}(z), \mathcal{L}^{+}_{000010}(z), \mathcal{L}^{+}_{000011}(z), \mathcal{L}^{+}_{000100}(z), \mathcal{L}^{+}_{000101}(z), \mathcal{L}^{+}_{000110}(z),\\ &\mathcal{L}^{+}_{000111}(z), \mathcal{L}^{+}_{001001}(z), \mathcal{L}^{+}_{001010}(z), \mathcal{L}^{+}_{001011}(z), \mathcal{L}^{+}_{001100}(z), \mathcal{L}^{+}_{001101}(z), \mathcal{L}^{+}_{001110}(z),\\ &\mathcal{L}^{+}_{010001}(z), \mathcal{L}^{+}_{010010}(z), \mathcal{L}^{+}_{010101}(z), \mathcal{L}^{+}_{010110}(z), \mathcal{L}^{+}_{011001}(z), \mathcal{L}^{+}_{011110}(z),\\ &\zeta(3)\mathcal{L}^{+}_{000}(z), \zeta(3)\mathcal{L}^{+}_{001}(z), \zeta(3)\mathcal{L}^{+}_{010}(z), \zeta(5)\mathcal{L}^{+}_{0}(z), \zeta(3)^2\Big)\\ \vec{L}^{-} = \Big(&\mathcal{L}^{-}_{000000}(z), \mathcal{L}^{-}_{000001}(z), \mathcal{L}^{-}_{000010}(z), \mathcal{L}^{-}_{000011}(z), \mathcal{L}^{-}_{000100}(z), \mathcal{L}^{-}_{000101}(z), \mathcal{L}^{-}_{000110}(z),\\ &\mathcal{L}^{-}_{001001}(z), \mathcal{L}^{-}_{001010}(z), \mathcal{L}^{-}_{001100}(z), \mathcal{L}^{-}_{001101}(z), \mathcal{L}^{-}_{001110}(z), \mathcal{L}^{-}_{010001}(z), \mathcal{L}^{-}_{010010}(z),\\ &\mathcal{L}^{-}_{010110}(z), \mathcal{L}^{-}_{011110}(z), \zeta(3)\mathcal{L}^{-}_{000}(z), \zeta(3)\mathcal{L}^{-}_{001}(z), \zeta(3)\mathcal{L}^{-}_{010}(z), \zeta(5)\mathcal{L}^{-}_{0}(z)\Big)\end{aligned}$$

Result:

$$G^{(2)}(S,T,z) = (S^2+T^2)\ \vec{r}_1 \cdot \vec{L}^{+} + ST\ \vec{r}_2 \cdot \vec{L}^{+} + \frac{(S^2+T^2)(S-T)}{S+T}\ \vec{r}_3 \cdot \vec{L}^{-} + \frac{ST(S-T)}{S+T}\ \vec{r}_4 \cdot \vec{L}^{-}$$

$$\vec{r}_1 = \left(-\tfrac{1}{18}, \tfrac{2971}{432}, \tfrac{13111}{3888}, -\tfrac{7271}{3888}, \ldots\right), \quad \vec{r}_2 = \ldots$$

We need to input the dimension of 1 operator ($\Delta^{(2)}_{1,0}$ = Konishi) to fix $A^{(2)}(S,T)$ completely.

2.3. Checks

OPE data

We compute $\forall \delta, \ell \qquad \# \in \mathbb{Q}$

$$k = 0: \qquad \langle C^{2(0)} \rangle_{\delta,\ell} = \#$$

$$k = 1: \qquad \sqrt{\delta} \langle C^{2(0)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\,, \quad \langle C^{2(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$k = 2: \qquad \langle C^{2(0)} (\Delta^{(1)})^2 \rangle_{\delta,\ell} = \#$$

$$\sqrt{\delta} \langle C^{2(0)} \Delta^{(2)} + C^{2(1)} \Delta^{(1)} \rangle_{\delta,\ell} = \#\zeta(3) + \#$$

$$\langle C^{2(2)} \rangle_{\delta,\ell} = \#\zeta(3)^2 + \#\zeta(5) + \#\zeta(3) + \#$$

Leading Regge trajectory:

$$\Delta\left(\tfrac{\ell}{2} + 1, \ell\right) = 2\sqrt{\tfrac{\ell}{2} + 1}\,\lambda^{\frac{1}{4}} - 2 + \frac{3\ell^2 + 10\ell + 16}{4\sqrt{2(\ell+2)}} \lambda^{-\frac{1}{4}} - \frac{21\ell^4 + 144\ell^3 + 292\ell^2 + 80\ell - 128 + 96(\ell+2)^3\zeta(3)}{32(2(\ell+2))^{\frac{3}{2}}} \lambda^{-\frac{3}{4}} + O(\lambda^{-\frac{5}{4}})\,,$$

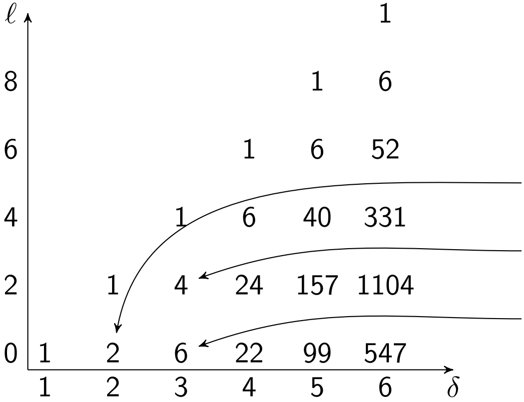
Agrees with integrability result!

[Gromov,Serban,Shenderovich,Volin;2011],[Basso;2011],[Gromov,Valatka;2011]

Averaged OPE data

$$\langle C^{2(0)}\rangle_{\delta,\ell} = \sum_I (C^{2(0)}_{\delta,\ell})_I\,, \quad \langle C^{2(0)}\Delta^{(1)}\rangle_{\delta,\ell} = \sum_I (C^{2(0)}_{\delta,\ell})_I(\Delta^{(1)}_{\delta,\ell})_I\,, \quad \ldots$$

+ results from the quantum spectral curve [Gromov,Hegedus,Julius,Sokolova;2023]
→ individual OPE coefficients



Many of them vanish!

$(C^{2(0)}_{2,0})_1 = (C^{2(1)}_{2,0})_1 = 0$

$(C^{2(0)}_{3,2})_{1,2} = 0$

$(C^{2(0)}_{3,0})_{1,2,3,4} = 0$

Explanation?

Wilson coefficients

$$A^{(k)}(S,T) = \mathrm{SUGRA}^{(k)} + \sum_{a,b=0}^{\infty} (S^2+T^2+U^2)^a (STU)^b \alpha^{(k)}_{a,b}$$

We compute $\forall a, b$ $\quad \# \in \mathbb{Q}$

$$\alpha^{(0)}_{a,b} = \sum_{k_i \text{ odd}} \# \zeta(k_1)\ldots\zeta(k_n)$$

$$\alpha^{(1)}_{a,b} = \sum_{k_i \text{ odd}} \# \zeta^{\mathrm{sv}}(k_1,k_2,k_3)\zeta(k_4)\ldots\zeta(k_n) + \ldots$$

$$\alpha^{(2)}_{a,b} = \sum_{k_i \text{ odd}} \# \zeta^{\mathrm{sv}}(k_1,k_2,k_3,k_4,k_5)\zeta(k_6)\ldots\zeta(k_n) + \ldots$$

In particular:

$$\alpha^{(1)}_{0,0} = 0\,, \quad \alpha^{(1)}_{1,0} = -\frac{22}{3}\zeta(3)^2\,, \quad \alpha^{(2)}_{0,0} = \frac{49}{4}\zeta(5)\,, \quad \alpha^{(2)}_{1,0} = \frac{4091}{16}\zeta(7)$$

Agrees with localisation result!
[Binder,Chester,Pufu,Wang;2019],[Chester,Pufu;2020],[Alday,TH,Silva;2022]

The low energy expansion ($S \sim T \sim 0$)
can be computed following [Vanhove,Zerbini;2018]

$$\int d^2z|z|^{-2S-2}|1-z|^{-2T-2}\mathcal{L}_w(z)$$

$$= \text{poles} + \sum_{p,q=0}^{\infty}(-S)^p(-T)^q\int\frac{d^2z}{|z|^2|1-z|^2}\underbrace{\mathcal{L}_{0^p}(z)\mathcal{L}_{1^q}(z)\mathcal{L}_w(z)}_{=\sum_{W\in 0^p ⧢ 1^q ⧢ w}\mathcal{L}_W(z)}$$

$$= \text{poles} + \sum_{p,q=0}^{\infty}(-S)^p(-T)^q\sum_{W\in 0^p ⧢ 1^q ⧢ w}\underbrace{\mathcal{L}_{0W}(1)-\mathcal{L}_{1W}(1)}$$

Single-valued multiple zeta values of weight $1+p+q+|w|$

As in flat space! [Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Alternative bootstrap method

Instead of making an ansatz for $G^{(k)}(S,T,z)$: combine the low energy expansion

$$M(s,t) = \text{SUGRA} + \sum_{a,b=0}^{\infty} \frac{\Gamma(6+2a+3b)}{\lambda^{\frac{3}{2}+a+\frac{3}{2}b}} (s^2+t^2+u^2)^a (stu)^b \left(\alpha_{a,b}^{(0)} + \frac{\alpha_{a,b}^{(1)}}{\sqrt{\lambda}} + \frac{\alpha_{a,b}^{(2)}}{\lambda} + \cdots \right)$$

with the dispersion relation

$$\Rightarrow \alpha_{a,b}^{(k)} = \sum_{\delta,\ell} \frac{f(\text{OPE data})}{\delta^{3+2a+3b}}$$

Ansatz:

$$\sum_{\ell} f(\text{OPE data}) = \sum \# \text{ Euler-Zagier sums}$$

$$Z_{s_1,\ldots,s_d}(N) = \sum_{\substack{n_1,\ldots,n_d \\ N\geq n_1>\ldots>n_d>0}} \frac{1}{n_1^{s_1}\cdots n_d^{s_d}}, \qquad \sum_{\delta=1}^{\infty} \frac{Z_{s_2,s_3,\ldots}(\delta-1)}{\delta^{s_1}} = \zeta(s_1,\ldots,s_d)$$

$\Rightarrow \alpha_{a,b}^{(k)} = \text{MZVs}$ Imposing $\alpha_{a,b}^{(k)} = \text{SVMZVs}$ fixes the #'s in the ansatz!

2.4. Including KK modes

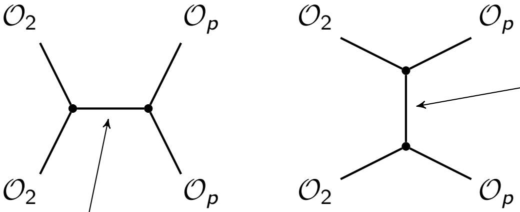
Correlators with Kaluza-Klein modes

We also computed the $O(1/\sqrt{\lambda})$ string amplitude for

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_p(x_3)\mathcal{O}_p(x_4)\rangle$$

$\mathcal{O}_p =$ KK mode
$\Delta = p = 3, 4, \ldots$
$[p, 0, 0]$ of $SO(6)$

- less crossing symmetry: $A(S,T) = A(S,U)$
- new operators:



new operators:
- odd spin
- non-zero R charge

same operators as in $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle$

World-sheet correlator for $\langle \mathcal{O}_2 \mathcal{O}_2 \mathcal{O}_p \mathcal{O}_p \rangle$

Ansatz:

$$A^{(1)}(S,T) = B_1^{(1)}(S,T) + B_1^{(1)}(S,U) + B_1^{(1)}(U,T) + B_2^{(1)}(S,T) + B_2^{(1)}(S,U)$$

$$B_i^{(1)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} G_i^{(1)}(S,T,z)\,, \quad i = 1,2$$

Result:

$$\begin{aligned} G_1^{(1)}(S,T,z) &= \frac{1}{24}\left(-p^2 \mathcal{L}^+_{000}(z) + 2(p-2)p\mathcal{L}^+_{001}(z) + (p^2-2p-6)\mathcal{L}^+_{010}(z) + 48\zeta(3)\right) \\ &\quad + \frac{p^2(S-T)}{24(S+T)}\left(-\mathcal{L}^-_{000}(z) + 2\mathcal{L}^-_{001}(z) + \mathcal{L}^-_{010}(z)\right) \\ G_2^{(1)}(S,T,z) &= \frac{p(p-2)}{24(S+T)}\left(3S\mathcal{L}^+_{000}(z) - 2(2S+T)\mathcal{L}^+_{001}(z) - (2S+T)\mathcal{L}^+_{010}(z)\right) \\ &\quad + \frac{p(p-2)}{24(S+T)}\left(3S\mathcal{L}^-_{000}(z) - 2(2S-T)\mathcal{L}^-_{001}(z) - (2S-T)\mathcal{L}^-_{010}(z)\right) \end{aligned}$$

Degeneracies of odd-spin operators

Even spin, $[0,0,0]$ of $SO(6)$:

ℓ \ δ	1	2	3	4	5	6
						1
8					1	6
6				1	6	52
4			1	6	40	331
2		1	4	24	157	1104
0	1	2	6	22	99	547

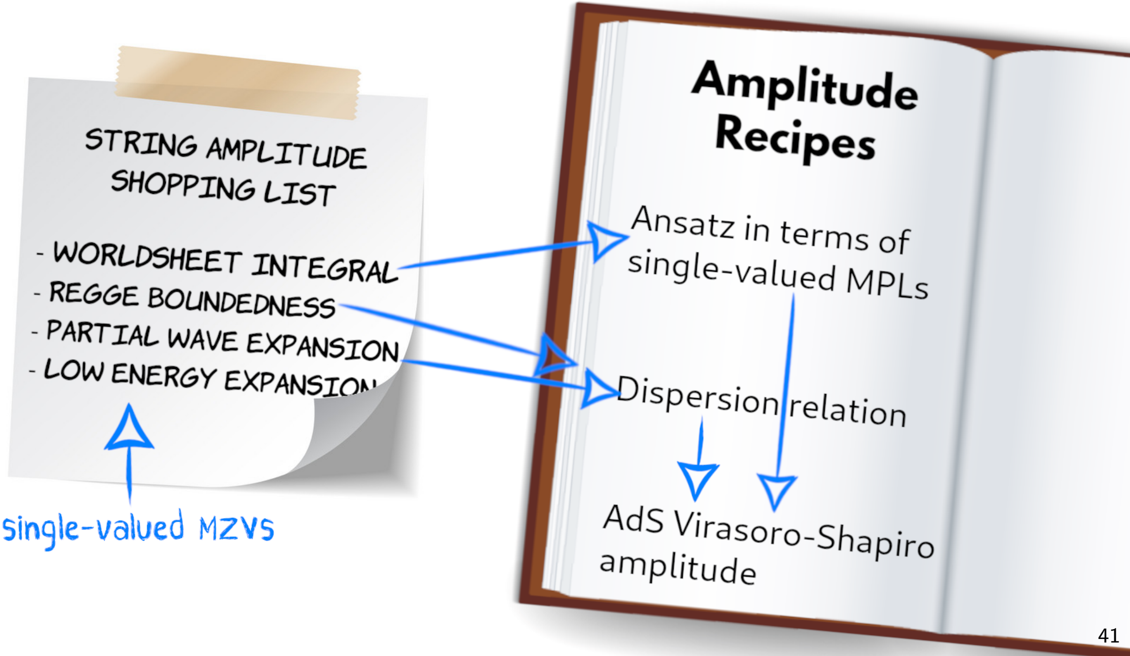
Odd spin, $[1,0,0]$ of $SO(6)$:

ℓ \ δ	1	2	3	4	5	6
9						2
7					2	32
5				2	28	316
3			2	22	206	1836
1		2	10	70	502	3536

The leading odd spin trajectory has very low degeneracies!

Good target for further study (our method, quantum spectral curve, ...).

Summary


STRING AMPLITUDE
SHOPPING LIST
- WORLDSHEET INTEGRAL
- REGGE BOUNDEDNESS
- PARTIAL WAVE EXPANSION
- LOW ENERGY EXPANSION
single-valued MZVs
Amplitude
Recipes
Ansatz in terms of
single-valued MPLs
Dispersion relation
AdS Virasoro-Shapiro
amplitude

Future directions

- Open strings / AdS Veneziano amplitude
  - Generalizations of KLT relations / single-valued map?
  - Gluon scattering on $AdS_5 \times S^3$ / $4d$ $\mathcal{N} = 2$ SCFT
  - Problem: no strong coupling OPE data known for consistency checks. Integrability?

- Other backgrounds
  - e.g. type IIA on $AdS_4 \times CP^3$ / ABJM

- Compute $A^{(k)}(S, T)$ directly from string theory?
  - Ramond-Ramond background flux. . .
  - String field theory?
  - Pure spinors?

Thank you!