The Half-Maximal $D = 4$ Supergravities

Nikolaos Liatsos

School of Applied Mathematical and Physical Sciences,
National Technical University of Athens

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Outline

Introduction

The first instances of four-dimensional pure $\mathcal{N} = 4$ supergravities were constructed almost 50 years ago by [Das (1977), Cremmer and Scherk (1977), Cremmer, Scherk and Ferrara (1978), Freedman and Schwarz (1978)].

The coupling of $\mathcal{N} = 4$ supergravity to vector multiplets, as well as some of its gaugings, were analyzed a few years later, by [de Roo (1985), Bergshoeff, Koh and Sezgin (1985), de Roo and Wagemans (1985), Perret (1988)].

More recently, various gauged $\mathcal{N} = 4$ supergravity models originating from orientifold compactifications of type IIA or IIB supergravity were studied [D'Auria, Ferrara and Vaula (2002), D'Auria, Ferrara, Gargiulo, Trigiante and Vaula (2003), Berg, Haack and Kors (2003), Angelantonj, Ferrara and Trigiante (2003,2004), Villadoro and Zwirner (2004,2005), Derendinger, Kounnas, Petropoulos and Zwirner (2005), Dall'Agata, Villadoro and Zwirner (2009)].

The most general analysis of the structure of the gauged $D = 4$, $\mathcal{N} = 4$ supergravity is provided by [Schön and Weidner (2006)], where one can find a systematic discussion of the consistency constraints on the embedding tensor.

However, a specific symplectic frame is chosen, in which the rigid symmetry group of the ungauged Lagrangian is $G_{\mathcal{L}} = \mathrm{SO}(1,1) \times \mathrm{SO}(6,n)$ ($n =$ number of vector multiplets).

This choice is constraining, since for example the maximally supersymmetric anti-de Sitter vacuum cannot be obtained by a purely electric gauging in this frame [Louis and Triendl (2014)].

Our work provides the full Lagrangian and supersymmetry transformation rules for the gauged four-dimensional $\mathcal{N} = 4$ supergravity coupled to n vector multiplets in an arbitrary symplectic frame.

Any known (as well as yet unknown) vacuum of such a theory can be obtained from an electrically gauged theory, which is incorporated in our general Lagrangian.

The Ingredients of $\mathcal{N}=4$ Supergravity

$\mathcal{N}=4$ supergravity multiplet:

- graviton $g_{\mu\nu}$
- 4 gravitini ψ^i_μ, $i=1,\ldots,4$
- 6 vector fields $A^{ij}_\mu = -A^{ji}_\mu$
- 4 spin-1/2 fermions χ_i (dilatini)
- 1 complex scalar τ

n vector multiplets:

- *n* vector fields $A_\mu^{\underline{a}}$, $\underline{a} = 1, \ldots, n$
- $4n$ gaugini $\lambda^{\underline{a}i}$
- $6n$ real scalar fields

The scalar sector of the supergravity multiplet

The complex scalar of the $\mathcal{N}=4$ supergravity multiplet parametrizes the coset space SL(2,R)/SO(2).

Coset representative: complex SL(2,R) vector $\mathcal{V}_\alpha$, $\alpha=+,-$, which satisfies

$$\mathcal{V}_\alpha \mathcal{V}_\beta^* - \mathcal{V}_\alpha^* \mathcal{V}_\beta = -2i\epsilon_{\alpha\beta}, \qquad (1)$$

where $\epsilon_{\alpha\beta}=-\epsilon_{\beta\alpha}$ and $\epsilon_{+-}=1$.

$\mathcal{V}_\alpha$ carries SO(2) charge $+1$.

We also define

$$M_{\alpha\beta} = \mathrm{Re}(\mathcal{V}_\alpha \mathcal{V}_\beta^*)\,. \qquad (2)$$

$$\text{SL(2,R)/SO(2) zweibein}: P = \frac{i}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha d\mathcal{V}_\beta \tag{3}$$

$$\text{SO(2) connection}: \mathcal{A} = -\frac{1}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha d\mathcal{V}^*_\beta \tag{4}$$

Useful identities:

$$D\mathcal{V}_\alpha \equiv d\mathcal{V}_\alpha - i\mathcal{A}\mathcal{V}_\alpha = P\mathcal{V}^*_\alpha \tag{5}$$

$$DP \equiv dP - 2i\mathcal{A}\wedge P = 0 \tag{6}$$

$$F \equiv d\mathcal{A} = iP^*\wedge P\,. \tag{7}$$

The scalar sector of the vector multiplets

The $6n$ real scalars of the n vector multiplets parametrize the coset space SO(6,n)/(SO(6)$\times$SO(n)).
Coset representative: $(n+6)\times(n+6)$ matrix L with entries $L_M{}^{\underline{M}} = (L_M{}^{\underline{m}}, L_M{}^{\underline{a}})$, where $M=1,\dots,n+6$, $\underline{m}=1,\dots,6$, $\underline{a}=1,\dots,n$, which is an element of SO(6,n):

$$\eta_{MN} = \eta_{\underline{MN}} L_M{}^{\underline{M}} L_N{}^{\underline{N}} = L_M{}^{\underline{M}} L_{N\underline{M}} = L_M{}^{\underline{m}} L_{N\underline{m}} + L_M{}^{\underline{a}} L_{N\underline{a}}\,, \qquad (8)$$

where $\eta_{MN} = \eta_{\underline{MN}} = \mathrm{diag}(-1,-1,-1,-1,-1,-1,1,\dots,1)$.

We also introduce the positive definite symmetric matrix $M = LL^T$ with elements

$$M_{MN} = -L_M{}^{\underline{m}} L_{N\underline{m}} + L_M{}^{\underline{a}} L_{N\underline{a}} \,. \tag{9}$$

We can trade $L_M{}^{\underline{m}}$ for the antisymmetric SU(4) tensors $L_M{}^{ij} = -L_M{}^{ji}$, $i, j = 1, \ldots, 4$, defined by

$$L_M{}^{ij} = \Gamma_{\underline{m}}{}^{ij} L_M{}^{\underline{m}}, \tag{10}$$

where $\Gamma_{\underline{m}}{}^{ij}$ are six antisymmetric 4×4 matrices that realize the isomorphism between the fundamental representation of SO(6) and the twofold antisymmetric representation of SU(4).

$$\text{Pseudoreality}: L_{Mij} = (L_M{}^{ij})^* = \frac{1}{2}\epsilon_{ijkl} L_M{}^{kl} \tag{11}$$

$$\mathrm{SO}(6,n)/(\mathrm{SU}(4)\times \mathrm{SO}(n))\,\text{vielbein}:\; P_{\underline{a}}{}^{ij} = L^{M}{}_{\underline{a}}\, dL_{M}{}^{ij} \tag{12}$$

$$\mathrm{SU}(4)\,\text{connection}:\; \omega^{i}{}_{j} = L^{Mik}\, dL_{Mjk} \tag{13}$$

$$\mathrm{SO}(n)\,\text{connection}:\; \omega_{\underline{a}}{}^{\underline{b}} = L^{M}{}_{\underline{a}}\, dL_{M}{}^{\underline{b}} \tag{14}$$

Useful identities:

$$DL_{M}{}^{ij} \equiv dL_{M}{}^{ij} - \omega^{i}{}_{k} L_{M}{}^{kj} - \omega^{j}{}_{k} L_{M}{}^{ik} = L_{M}{}^{\underline{a}} P_{\underline{a}}{}^{ij} \tag{15}$$

$$DL_{M}{}^{\underline{a}} \equiv dL_{M}{}^{\underline{a}} + \omega^{\underline{a}}{}_{\underline{b}} L_{M}{}^{\underline{b}} = L_{M}{}^{ij} P^{\underline{a}}{}_{ij} \tag{16}$$

$$DP_{\underline{a}}{}^{ij} \equiv dP_{\underline{a}}{}^{ij} + \omega_{\underline{a}}{}^{\underline{b}} \wedge P_{\underline{b}}{}^{ij} - \omega^{i}{}_{k} \wedge P_{\underline{a}}{}^{kj} - \omega^{j}{}_{k} \wedge P_{\underline{a}}{}^{ik} = 0 \tag{17}$$

$$R^{i}{}_{j} \equiv d\omega^{i}{}_{j} - \omega^{i}{}_{k} \wedge \omega^{k}{}_{j} = P^{\underline{a}ik} \wedge P_{\underline{a}jk} \tag{18}$$

$$R_{\underline{a}}{}^{\underline{b}} \equiv d\omega_{\underline{a}}{}^{\underline{b}} + \omega_{\underline{a}}{}^{\underline{c}} \wedge \omega_{\underline{c}}{}^{\underline{b}} = -P_{\underline{a}ij} \wedge P^{\underline{b}ij} \tag{19}$$

The fermionic fields

Field	SO(2) charge
ψ_μ^i	$-\frac{1}{2}$
χ^i	$+\frac{3}{2}$
λ^{ai}	$+\frac{1}{2}$

$$\gamma_5\psi_\mu^i = \psi_\mu^i, \quad \gamma_5\chi^i = -\chi^i, \quad \gamma_5\lambda^{ai} = \lambda^{ai}. \tag{20}$$

$\psi_{i\mu} = (\psi_\mu^i)^c$, $\chi_i = (\chi^i)^c$ and $\lambda_i^a = (\lambda^{ai})^c$ have opposite SO(2) charges and chiralities.

Duality and Symplectic Frames

The ungauged theory for the four-dimensional $\mathcal{N} = 4$ Poincaré supergravity coupled to n vector multiplets contains $n + 6$ abelian vector fields A^{Λ}_{μ}, $\Lambda = 1, \ldots, n + 6$, and is described by a 2-derivative Lagrangian of the form

$$e^{-1}\mathcal{L} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} (*F^{\Sigma})^{\mu\nu} + \frac{1}{2} O^{\mu\nu}_{\Lambda} F^{\Lambda}_{\mu\nu} + e^{-1}\mathcal{L}_{\text{rest}}, \tag{21}$$

where $F^{\Lambda}_{\mu\nu} = 2\partial_{[\mu} A^{\Lambda}_{\nu]}$, $(*F^{\Lambda})_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma}$, $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are real symmetric matrices that depend on the scalar fields, with $\mathcal{I}_{\Lambda\Sigma}$ being negative definite, while $O^{\mu\nu}_{\Lambda}$ and $\mathcal{L}_{\text{rest}}$ do not depend on the vector fields.

We can associate with the field strengths $F^{\Lambda}_{\mu\nu}$ their magnetic duals $G_{\Lambda\mu\nu}$ defined by

$$G_{\Lambda\mu\nu} \equiv -e^{-1}\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial F^{\Lambda}_{\rho\sigma}} = \mathcal{R}_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu} - \mathcal{I}_{\Lambda\Sigma}(*F^{\Sigma})_{\mu\nu} - (*O_{\Lambda})_{\mu\nu}\,. \tag{22}$$

The equations of motion for the vector fields read

$$\partial_{[\mu|}G_{\Lambda|\nu\rho]} = 0 \tag{23}$$

and imply the local existence of $n+6$ dual magnetic vector fields $A_{\Lambda\mu}$ such that

$$G_{\Lambda\mu\nu} = 2\partial_{[\mu|}A_{\Lambda|\nu]}. \tag{24}$$

The group of global transformations that leave the full set of Bianchi identities and equations of motion of the ungauged $D = 4$, $\mathcal{N} = 4$ matter-coupled supergravity invariant is

$$G = \mathrm{SL}(2,\mathrm{R}) \times \mathrm{SO}(6,n) \subset \mathrm{Sp}(2(n+6),\mathrm{R})\,. \qquad (25)$$

The vector fields A_μ^Λ, which are those appearing in the ungauged Lagrangian and will be referred to as electric vectors, together with their magnetic duals $A_{\Lambda\mu}$ form an SL(2,R)$\times$SO(6,n) vector $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha} = (A_\mu^\Lambda, A_{\Lambda\mu})$, which is also a symplectic vector of Sp(2(6 + n),R).

Every electric/magnetic split $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha} = (A_\mu^\Lambda, A_{\Lambda\mu})$ such that the symplectic form

$$\mathbb{C}^{\mathcal{M}\mathcal{N}} = \mathbb{C}^{M\alpha N\beta} \equiv \eta^{MN}\epsilon^{\alpha\beta} \qquad (26)$$

decomposes as

$$\mathbb{C}^{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \mathbb{C}^{\Lambda\Sigma} & \mathbb{C}^{\Lambda}{}_{\Sigma} \\ \mathbb{C}_{\Lambda}{}^{\Sigma} & \mathbb{C}_{\Lambda\Sigma} \end{pmatrix} = \begin{pmatrix} 0 & \delta_\Sigma^\Lambda \\ -\delta_\Lambda^\Sigma & 0 \end{pmatrix}, \qquad (27)$$

defines a symplectic frame and any two symplectic frames are related by a symplectic rotation.

It is convenient to parametrize the choice of the symplectic frame by means of projectors $\Pi^{\Lambda}{}_{\mathcal{M}}$ and $\Pi_{\Lambda\mathcal{M}}$ that extract the electric and magnetic components of a symplectic vector $V^{\mathcal{M}}=(V^{\Lambda},V_{\Lambda})$ respectively, according to

$$V^{\Lambda}=\Pi^{\Lambda}{}_{\mathcal{M}}V^{\mathcal{M}}, \qquad V_{\Lambda}=\Pi_{\Lambda\mathcal{M}}V^{\mathcal{M}}. \tag{28}$$

These projectors must satisfy the properties

$$\Pi^{\Lambda}{}_{\mathcal{M}}\Pi^{\Sigma}{}_{\mathcal{N}}\,\mathbb{C}^{\mathcal{M}\mathcal{N}}=0\,, \tag{29}$$

$$\Pi^{\Lambda}{}_{\mathcal{M}}\Pi_{\Sigma\mathcal{N}}\,\mathbb{C}^{\mathcal{M}\mathcal{N}}=\delta^{\Lambda}_{\Sigma}\,, \tag{30}$$

$$\Pi_{\Lambda\mathcal{M}}\Pi_{\Sigma\mathcal{N}}\,\mathbb{C}^{\mathcal{M}\mathcal{N}}=0\,, \tag{31}$$

$$\Pi^{\Lambda}{}_{\mathcal{M}}\Pi_{\Lambda\mathcal{N}}-\Pi_{\Lambda\mathcal{M}}\Pi^{\Lambda}{}_{\mathcal{N}}=\mathbb{C}_{\mathcal{M}\mathcal{N}}\,, \tag{32}$$

where $\mathbb{C}_{\mathcal{M}\mathcal{N}}=\mathbb{C}_{M\alpha N\beta}\equiv\eta_{MN}\epsilon_{\alpha\beta}$

Solution of the Bianchi Identities of the Ungauged Theory

Geometric or rheonomic approach [Castellani, D' Auria and Fré (1991)]:

1. Promotion of the spacetime one-forms

$$e^a = e^a_\mu dx^\mu,\ \psi^i = \psi^i_\mu dx^\mu,\ \psi_i = \psi_{i\mu} dx^\mu,$$
$$A^{M\alpha} = A^{M\alpha}_\mu dx^\mu,\ \omega_{ab} = \omega_{\mu ab} dx^\mu,$$

and the spacetime zero-forms

$$\chi^i,\ \chi_i,\ \lambda^{\underline{a}i},\ \lambda_i^{\underline{a}},\ \mathcal{V}_\alpha,\ \mathcal{V}^*_\alpha,\ L_M{}^{ij},\ L_M{}^{\underline{a}}$$

to super-one-forms and super-zero-forms in $\mathcal{N}=4$ superspace respectively.

These superforms depend on the supespace coordinates $(x^\mu, \theta^i, \theta_i)$ in such a way that their projections on the spacetime submanifold, i.e. the $\theta^i = d\theta^i = 0$ hypersurface, are equal to the corresponding spacetime quantities.

A basis of one-forms in $\mathcal{N} = 4$ superspace is given by the supervielbein (e^a, ψ^i, ψ_i).

2. Supercurvatures:

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \tag{33}$$

$$\begin{aligned} T^a &= de^a + \omega^a{}_b \wedge e^b - \bar{\psi}^i \wedge \gamma^a \psi_i \\ &= De^a - \bar{\psi}^i \wedge \gamma^a \psi_i \end{aligned} \tag{34}$$

$$\rho_i = D\psi_i = d\psi_i + \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi_i - \frac{i}{2}\mathcal{A} \wedge \psi_i - \omega_i{}^j \wedge \psi_j \tag{35}$$

$$V_i = D\chi_i = d\chi_i + \frac{1}{4}\omega^{ab}\gamma_{ab}\chi_i + \frac{3i}{2}\mathcal{A}\chi_i - \omega_i{}^j\chi_j \tag{36}$$

$$\begin{aligned} \Lambda_{\underline{a}i} = D\lambda_{\underline{a}i} &= d\lambda_{\underline{a}i} + \frac{1}{4}\omega^{ab}\gamma_{ab}\lambda_{\underline{a}i} + \frac{i}{2}\mathcal{A}\lambda_{\underline{a}i} - \omega_i{}^j\lambda_{\underline{a}j} \\ &+ \omega_{\underline{a}}{}^{\underline{b}}\lambda_{\underline{b}i} \end{aligned} \tag{37}$$

$$\mathcal{F}^{M\alpha} = dA^{M\alpha} - (\mathcal{V}^{\alpha})^* L^{Mij} \bar{\psi}_i \wedge \psi_j - \mathcal{V}^{\alpha} L^M{}_{ij} \bar{\psi}^i \wedge \psi^j \tag{38}$$

$$P = \frac{i}{2} \epsilon^{\alpha\beta} \mathcal{V}_{\alpha} d\mathcal{V}_{\beta} \tag{39}$$

$$P_{\underline{a}ij} = L^M{}_{\underline{a}} dL_{Mij} \tag{40}$$

3. Bianchi identities:

$$DR^{ab} = 0 \tag{41}$$

$$DT^a = R^a{}_b \wedge e^b + \bar{\psi}_i \wedge \gamma^a \rho^i + \bar{\psi}^i \wedge \gamma^a \rho_i \tag{42}$$

$$D\rho_i = \frac{1}{4} R^{ab} \wedge \gamma_{ab} \psi_i - \frac{i}{2} F \wedge \psi_i - R_i{}^j \wedge \psi_j \tag{43}$$

$$DV_i = \frac{1}{4} R^{ab} \gamma_{ab} \chi_i + \frac{3i}{2} F \chi_i - R_i{}^j \chi_j \tag{44}$$

$$D\Lambda_{\underline{a}i} = \frac{1}{4} R^{ab} \gamma_{ab} \lambda_{\underline{a}i} + \frac{i}{2} F \lambda_{\underline{a}i} - R_i{}^j \lambda_{\underline{a}j} + R_{\underline{a}}{}^{\underline{b}} \lambda_{\underline{b}i} \tag{45}$$

$$\begin{aligned} D\mathcal{F}^{M\alpha} = & - \mathcal{V}^\alpha L^{Mij} P^* \wedge \bar{\psi}_i \wedge \psi_j - (\mathcal{V}^\alpha)^* L^{M\underline{a}} P_{\underline{a}}{}^{ij} \wedge \bar{\psi}_i \wedge \psi_j \\ & + 2(\mathcal{V}^\alpha)^* L^{Mij} \bar{\psi}_i \wedge \rho_j - (\mathcal{V}^\alpha)^* L^M{}_{ij} P \wedge \bar{\psi}^i \wedge \psi^j \\ & - \mathcal{V}^\alpha L^{M\underline{a}} P_{\underline{a}ij} \wedge \bar{\psi}^i \wedge \psi^j + 2\mathcal{V}^\alpha L^M{}_{ij} \bar{\psi}^i \wedge \rho^j \end{aligned} \tag{46}$$

$$DP = 0 \quad (47)$$

$$DP_{\underline{a}ij} = 0 \quad (48)$$

4. Solution of the Bianchi identities:

- The supercurvatures can be expanded along the intrinsic bases of one- and two-forms in $\mathcal{N}=4$ superspace built out of the supervielbein (e^a, ψ^i, ψ_i).
- **Rheonomy principle**: The components of the supercurvatures along the basis elements that involve at least one of ψ^i, ψ_i (outer components) must be expressed in terms of the supercurvature components along the basis elements e^a and $e^a \wedge e^b$ (inner components) and the physical superfields $\Rightarrow$ No new d.o.f. are introduced in the theory.

Then, one writes down the expansions of the supercurvatures in a form that is compatible with all the symmetries of the theory, i.e. **covariance under local SO(2), SU(4), SO(n) and Lorentz transformations**.

The unknown coefficients in these expansions are determined by requiring closure of the Bianchi identities.

We also impose the kinematic constraint

$$T^a = 0\,. \tag{49}$$

Rheonomic parametrizations of the supercurvatures:

$$P = P_a e^a + \bar{\psi}_i \chi^i \tag{50}$$

$$P_{\underline{a}ij} = P_{\underline{a}ija} e^a + 2\bar{\psi}_{[i|}\lambda_{\underline{a}|j]} + \epsilon_{ijkl}\bar{\psi}^k \lambda^l_{\underline{a}} \tag{51}$$

$$V_i = V_{ia} e^a - \frac{i}{4} L_{Mij} \mathcal{V}^*_\alpha \mathcal{F}^{M\alpha}_{ab} \gamma^{ab}\psi^j - (\bar{\lambda}_{\underline{a}i}\lambda^{\underline{a}}_j)\psi^j + \gamma^a P^*_a \psi_i \tag{52}$$

$$\Lambda_{\underline{a}i} = \Lambda_{\underline{a}ia} e^a - P_{\underline{a}ija}\gamma^a\psi^j + \frac{i}{8} L_{M\underline{a}} \mathcal{V}^*_\alpha \mathcal{F}^{M\alpha}_{ab}\gamma^{ab}\psi_i + (\bar{\chi}_i \lambda^j_{\underline{a}})\psi_j - \frac{1}{2}(\bar{\chi}_j \lambda^j_{\underline{a}})\psi_i \tag{53}$$

$$\begin{aligned}\mathcal{F}^{M\alpha} = &\frac{1}{2}\mathcal{F}^{M\alpha}_{ab} e^a \wedge e^b + \Big(-\frac{1}{4}\mathcal{V}^\alpha L^{Mij} \bar{\lambda}_{\underline{a}i}\gamma_{ab}\lambda^{\underline{a}}_j \, e^a \wedge e^b \\ &+ \frac{1}{4}\mathcal{V}^\alpha L^{M\underline{a}} \bar{\chi}_i \gamma_{ab}\lambda^i_{\underline{a}} \, e^a \wedge e^b + (\mathcal{V}^\alpha)^* L^M{}_{ij} \bar{\chi}^i \gamma_a \psi^j \wedge e^a \\ &+ (\mathcal{V}^\alpha)^* L^{M\underline{a}} \bar{\lambda}^i_{\underline{a}} \gamma_a \psi_i \wedge e^a + c.c. \Big)\end{aligned} \quad (54)$$

$$
\begin{aligned}
\rho_i =& \frac{1}{2}\rho_{iab} e^a \wedge e^b - \frac{i}{8} L_{Mij} \mathcal{V}_\alpha \mathcal{F}^{M\alpha}_{bc} \gamma^{bc} \gamma_a \psi^j \wedge e^a \\
&+ \frac{1}{4}\epsilon_{ijkl} (\bar{\lambda}^j_{\underline{a}} \gamma_{ab} \lambda^{\underline{a}k}) \gamma^a \psi^l \wedge e^b + \frac{1}{4} (\bar{\chi}_i \gamma_a \chi^j) \psi_j \wedge e^a \\
&- \frac{1}{4} (\bar{\chi}_j \gamma_a \chi^j) \psi_i \wedge e^a + \frac{1}{4} (\bar{\chi}_i \gamma^a \chi^j) \gamma_{ab} \psi_j \wedge e^b \\
&- \frac{1}{8} (\bar{\chi}_j \gamma^a \chi^j) \gamma_{ab} \psi_i \wedge e^b + \frac{1}{2} (\bar{\lambda}^{\underline{a}}_i \gamma_a \lambda^j_{\underline{a}}) \psi_j \wedge e^a \\
&+ \frac{1}{2} (\bar{\lambda}^{\underline{a}}_i \gamma^a \lambda^j_{\underline{a}}) \gamma_{ab} \psi_j \wedge e^b - \frac{1}{4} (\bar{\lambda}^{\underline{a}}_j \gamma^a \lambda^j_{\underline{a}}) \gamma_{ab} \psi_i \wedge e^b \\
&- \frac{1}{2}\epsilon_{ijkl} \chi^j (\bar{\psi}^k \wedge \psi^l),
\end{aligned}
\tag{55}
$$

where $\mathcal{F}^{M\alpha}_{ab}$ satisfy

$$\epsilon_{abcd}\mathcal{F}^{M\alpha cd} = -2\, M^M{}_N M^\alpha{}_\beta \mathcal{F}^{N\beta}_{ab}\,. \tag{56}$$

We define the $2(6+n)\times 2(6+n)$ matrix

$$\mathcal{M}_{\mathcal{M}\mathcal{N}} = \mathcal{M}_{M\alpha N\beta} = M_{\alpha\beta}M_{MN} \tag{57}$$

The restriction of the superspace equation (56) to spacetime reads

$$\begin{aligned}(*F^{M\alpha})_{\mu\nu} = &\mathbb{C}^{M\alpha\mathcal{N}}\mathcal{M}_{\mathcal{N}\mathcal{P}}F^{\mathcal{P}}_{\mu\nu} + \big(-2i(\mathcal{V}^\alpha)^* L^{Mij}\bar\psi_{i\mu}\psi_{j\nu} + \epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^\alpha)^* L^{Mij}\bar\psi^\rho_i\psi^\sigma_j \\ &- i\mathcal{V}^\alpha L^{Mij}\bar\lambda_{\underline{a}i}\gamma_{\mu\nu}\lambda^{\underline{a}}_j - i\mathcal{V}^\alpha L^{M\underline{a}}\bar\chi_i\gamma_{\mu\nu}\lambda^i_{\underline{a}} + 2i(\mathcal{V}^\alpha)^* L^M{}_{ij}\bar\chi^i\gamma_{[\mu}\psi^j_{\nu]} \\ &- \epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^\alpha)^* L^M{}_{ij}\bar\chi^i\gamma^\rho\psi^{j\sigma} + 2i\mathcal{V}^\alpha L^{M\underline{a}}\bar\lambda_{\underline{a}i}\gamma_{[\mu}\psi^i_{\nu]} \\ &- \epsilon_{\mu\nu\rho\sigma}\mathcal{V}^\alpha L^{M\underline{a}}\bar\lambda_{\underline{a}i}\gamma^\rho\psi^{i\sigma} + \text{c.c.}\big),\end{aligned} \tag{58}$$

where $F^{M\alpha}_{\mu\nu} = 2\partial_{[\mu}A^{M\alpha}_{\nu]}$.

Comparing (58) with the matrix equation

$$\begin{aligned}\begin{pmatrix}(*F^\Lambda)_{\mu\nu}\\(*G_\Lambda)_{\mu\nu}\end{pmatrix}=&\begin{pmatrix}(\mathcal{I}^{-1}\mathcal{R})^\Lambda{}_\Sigma & -(\mathcal{I}^{-1})^{\Lambda\Sigma}\\(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & -(\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma\end{pmatrix}\begin{pmatrix}F^\Sigma_{\mu\nu}\\G_{\Sigma\mu\nu}\end{pmatrix}\\&+\begin{pmatrix}-(\mathcal{I}^{-1})^{\Lambda\Sigma}(*O_\Sigma)_{\mu\nu}\\O_{\Lambda\mu\nu}-(\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma(*O_\Sigma)_{\mu\nu}\end{pmatrix}\end{aligned}\tag{59}$$

and identifying $G_{\Lambda\mu\nu}$ with $F_{\Lambda\mu\nu}$, we find that the matrix $\mathcal{M}_{\mathcal{M}\mathcal{N}}$ decomposes as

$$\mathcal{M}_{\mathcal{M}\mathcal{N}}=\begin{pmatrix}\mathcal{M}_{\Lambda\Sigma} & \mathcal{M}_\Lambda{}^\Sigma\\\mathcal{M}^\Lambda{}_\Sigma & \mathcal{M}^{\Lambda\Sigma}\end{pmatrix}=\begin{pmatrix}-(\mathcal{I}+\mathcal{R}\mathcal{I}^{-1}\mathcal{R})_{\Lambda\Sigma} & (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma\\(\mathcal{I}^{-1}\mathcal{R})^\Lambda{}_\Sigma & -(\mathcal{I}^{-1})^{\Lambda\Sigma}\end{pmatrix},\tag{60}$$

and that $O_{\Lambda\mu\nu}$ is given by

$$\begin{aligned} O_{\Lambda\mu\nu} =& \mathcal{I}_{\Lambda\Sigma}\Pi^{\Sigma}{}_{M\alpha}\big(-2(\mathcal{V}^\alpha)^* L^{Mij}\bar\psi_{i\mu}\psi_{j\nu} - i\epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^\alpha)^* L^{Mij}\bar\psi_i^\rho\psi_j^\sigma \\ &+ \mathcal{V}^\alpha L^{Mij}\bar\lambda_{\underline{a}i}\gamma_{\mu\nu}\lambda_j^{\underline{a}} - \mathcal{V}^\alpha L^{M\underline{a}}\bar\chi_i\gamma_{\mu\nu}\lambda_{\underline{a}}^i + 2(\mathcal{V}^\alpha)^* L^M{}_{ij}\bar\chi^i\gamma_{[\mu}\psi_{\nu]}^j \\ &+ i\epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^\alpha)^* L^M{}_{ij}\bar\chi^i\gamma^\rho\psi^{j\sigma} + 2\mathcal{V}^\alpha L^{M\underline{a}}\bar\lambda_{\underline{a}i}\gamma_{[\mu}\psi_{\nu]}^i \\ &+ i\epsilon_{\mu\nu\rho\sigma}\mathcal{V}^\alpha L^{M\underline{a}}\bar\lambda_{\underline{a}i}\gamma^\rho\psi^{i\sigma} + \text{c.c.}\big)\,. \end{aligned} \tag{61}$$

We also derive an expression for $O_{\Lambda\mu\nu} - (\mathcal{R}\mathcal{I}^{-1})_\Lambda{}^\Sigma(*O_\Sigma)_{\mu\nu}$, which is consistent with (61), if $\mathcal{N}_{\Lambda\Sigma} \equiv \mathcal{R}_{\Lambda\Sigma} + i\mathcal{I}_{\Lambda\Sigma}$ satisfies

$$\mathcal{N}_{\Lambda\Sigma}\Pi^{\Sigma}{}_{M\alpha}\mathcal{V}^\alpha L^{Mij} = \Pi_{\Lambda M\alpha}\mathcal{V}^\alpha L^{Mij}, \tag{62}$$

$$\mathcal{N}_{\Lambda\Sigma}\Pi^{\Sigma}{}_{M\alpha}(\mathcal{V}^\alpha)^* L^{M\underline{a}} = \Pi_{\Lambda M\alpha}(\mathcal{V}^\alpha)^* L^{M\underline{a}}. \tag{63}$$

The local supersymmetry transformation δ_ϵ of each spacetime field is equal to the projection on spacetime of the Lie derivative $\ell_\epsilon = di_\epsilon + i_\epsilon d$ of the corresponding superform along the tangent vector

$$\epsilon = \bar{\epsilon}^i D_i + \bar{\epsilon}_i D^i, \tag{64}$$

where the basis tangent vectors D_i, D^i are dual to the gravitino super-one-forms so that

$$i_\epsilon \psi^i = \epsilon^i, \; i_\epsilon \psi_i = \epsilon_i \,. \tag{65}$$

For the super-one-forms e^a, ψ_i and $A^{M\alpha}$ we have

$$\ell_\epsilon e^a = i_\epsilon T^a + \bar{\epsilon}^i \gamma^a \psi_i + \bar{\epsilon}_i \gamma^a \psi^i, \tag{66}$$

$$\ell_\epsilon \psi_i = D\epsilon_i + i_\epsilon \rho_i, \tag{67}$$

$$\ell_\epsilon A^{M\alpha} = i_\epsilon \mathcal{F}^{M\alpha} + 2(\mathcal{V}^\alpha)^* L^{Mij} \bar{\epsilon}_i \psi_j + 2\mathcal{V}^\alpha L^M{}_{ij} \bar{\epsilon}^i \psi^j, \tag{68}$$

while for the super-zero-forms

$$\nu^I \equiv (\mathcal{V}_\alpha, \mathcal{V}^*_\alpha, L_{Mij}, L_{M\underline{a}}, \chi^i, \chi_i, \lambda^i_{\underline{a}}, \lambda_{\underline{a}i}), \tag{69}$$

we have the simpler result

$$\ell_\epsilon \nu^I = i_\epsilon D\nu^I. \tag{70}$$

Duality Covariant Gauging

Embedding tensor formalism [Nicolai and Samtleben (2001), de Wit, Samtleben and Trigiante (2003,2005,2007)]:

- gauge fields $A_\mu^{\mathcal{M}} = A_\mu^{M\alpha}$ that decompose into electric gauge fields A_μ^Λ and magnetic gauge fields $A_{\Lambda\mu}$
- gauge group generators $X_{\mathcal{M}} = (X_\Lambda, X^\Lambda)$ expressed as linear combinations of the generators t_A of SL(2,R)$\times$SO(6,n)

$$X_{\mathcal{M}} = \Theta_{\mathcal{M}}{}^A t_A \,, \tag{71}$$

where $A = ([MN], (\alpha\beta))$ is an index labeling the adjoint representation of SL(2,R)$\times$SO(6,n) and $\Theta_{\mathcal{M}}{}^A = (\Theta_\Lambda{}^A, \Theta^{\Lambda A})$ is a constant tensor, called the *embedding tensor*.

The components of the embedding tensor are given by [Schön and Weidner (2006)]

$$\Theta_{\alpha M}{}^{NP} = f_{\alpha M}{}^{NP} - \xi_\alpha^{[N} \delta_M^{P]}, \qquad \Theta_{\alpha M}{}^{\beta\gamma} = \delta_\alpha^{(\beta} \xi_M^{\gamma)}, \qquad (72)$$

where $\xi_{\alpha M}$ and $f_{\alpha MNP} = f_{\alpha[MNP]}$ are two real constant SL(2,R)$\times$SO(6,n) tensors, so that

$$X_{(\mathcal{MNP})} = X_{(\mathcal{MN}}{}^{\mathcal{Q}} \mathbb{C}_{\mathcal{P})\mathcal{Q}} = 0, \qquad (73)$$

where $X_{\mathcal{MN}}{}^{\mathcal{P}} \equiv \Theta_{\mathcal{M}}{}^{A} (t_A)_{\mathcal{N}}{}^{\mathcal{P}}$ are the matrix elements of the gauge generators $X_{\mathcal{M}}$ in the fundamental representation of SL(2,R)$\times$SO(6,n).

Furthermore, the embedding tensor must be invariant under the action of the gauge group G_g that it defines, which is equivalent to the following quadratic constraints on the tensors $\xi_{\alpha M}$ and $f_{\alpha MNP}$ [Schön and Weidner (2006)]

$$\xi_\alpha^M \xi_{\beta M} = 0\,, \tag{74}$$

$$\xi_{(\alpha}^P f_{\beta)PMN} = 0\,, \tag{75}$$

$$3 f_{\alpha R[MN|} f_{\beta|PQ]}{}^R + 2\xi_{(\alpha|[M|} f_{|\beta)|NPQ]} = 0\,, \tag{76}$$

$$\epsilon^{\alpha\beta}(\xi_\alpha^P f_{\beta PMN} + \xi_{\alpha M}\xi_{\beta N}) = 0\,, \tag{77}$$

$$\epsilon^{\alpha\beta}(f_{\alpha MNR} f_{\beta PQ}{}^R - \xi_\alpha^R f_{\beta R[M[P}\eta_{Q]N]} - \xi_{\alpha[M|} f_{\beta|N]PQ} + \xi_{\alpha[P|} f_{\beta|Q]MN}) = 0\,. \tag{78}$$

These quadratic constraints guarantee the closure of the gauge algebra:

$$[X_{\mathcal{M}}, X_{\mathcal{N}}] = -X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} X_{\mathcal{P}}\,, \tag{79}$$

and also imply the locality constraint

$$\mathbb{C}^{\mathcal{M}\mathcal{N}} \Theta_{\mathcal{M}}{}^{A} \Theta_{\mathcal{N}}{}^{B} = 0\,. \tag{80}$$

The latter ensures that for any gauging there exists a symplectic frame in which the gauging is purely electric and guarantees that

$$\dim G_g \leq n + 6 \tag{81}$$

In the gauged theory, the ordinary exterior derivative d is replaced by a gauge-covariant one

$$\begin{aligned}\hat{d} &= d - gA^{\mathcal{M}}X_{\mathcal{M}} \\ &= d - gA^{M\alpha}\Theta_{\alpha M}{}^{NP}t_{NP} + gA^{M(\alpha}\epsilon^{\beta)\gamma}\xi_{\gamma M}t_{\alpha\beta}\,, \end{aligned} \qquad (82)$$

where we have introduced the one-forms $A^{\mathcal{M}} = A^{M\alpha} = A_\mu^{M\alpha}dx^\mu$.

The gauge-covariant 2-form field strengths of the vector gauge fields are defined by [Schön and Weidner (2006)]

$$H^{M\alpha} = dA^{M\alpha} - \frac{g}{2}\hat{f}_{\beta NP}{}^{M} A^{N\beta} \wedge A^{P\alpha} - \frac{g}{2}\Theta^{\alpha M}{}_{NP} B^{NP} + \frac{g}{2}\xi^{M}_{\beta} B^{\alpha\beta}, \qquad (83)$$

where

$$\hat{f}_{\alpha MNP} = f_{\alpha MNP} - \xi_{\alpha[M}\eta_{P]N} - \frac{3}{2}\xi_{\alpha N}\eta_{MP} \qquad (84)$$

and $B^{NP} = B^{[NP]}$, $B^{\alpha\beta} = B^{(\alpha\beta)}$ are 2-form gauge fields in the adjoint representations of SO(6,n) and SL(2,R) respectively.

The field strengths of the two-form gauge fields are defined by [de Wit and Samtleben (2005)]

$$\mathcal{H}^{(3)MN} \equiv dB^{MN} + 2g\Theta_{\alpha PQ}{}^{[M|}A^{P\alpha} \wedge B^{|N]Q} + \epsilon_{\alpha\beta}A^{[M|\alpha} \wedge \left(dA^{|N]\beta} + \frac{g}{3}X_{P\gamma Q\delta}{}^{|N]\beta}A^{P\gamma} \wedge A^{Q\delta}\right), \tag{85}$$

$$\mathcal{H}^{(3)\alpha\beta} \equiv dB^{\alpha\beta} - g\xi^{(\alpha|M}A_{M\gamma} \wedge B^{|\beta)\gamma} - g\xi_{\gamma M}A^{M(\alpha} \wedge B^{\beta)\gamma} - \eta_{MN}A^{M(\alpha|} \wedge \left(dA^{N|\beta)} + \frac{g}{3}X_{P\gamma Q\delta}{}^{N|\beta)}A^{P\gamma} \wedge A^{Q\delta}\right) \tag{86}$$

$$\text{gauged SL(2,R)/SO(2) zweibein} : \hat{P} = \frac{i}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha \hat{d}\mathcal{V}_\beta \quad (87)$$

$$\text{gauged SO(2) connection} : \hat{\mathcal{A}} = -\frac{1}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha \hat{d}\mathcal{V}^*_\beta\,, \quad (88)$$

where

$$\hat{d}\mathcal{V}_\alpha \equiv d\mathcal{V}_\alpha + \frac{1}{2}g\xi_{\alpha M}A^{M\beta}\mathcal{V}_\beta + \frac{1}{2}g\xi^{M\beta}A_{M\alpha}\mathcal{V}_\beta\,. \quad (89)$$

Useful relations:

$$\hat{D}\mathcal{V}_\alpha \equiv \hat{d}\mathcal{V}_\alpha - i\hat{\mathcal{A}}\mathcal{V}_\alpha = \hat{P}\mathcal{V}^*_\alpha \quad (90)$$

$$\hat{D}\hat{P} \equiv d\hat{P} - 2i\hat{\mathcal{A}}\wedge\hat{P} = \frac{i}{2}g\xi_{\alpha M}\mathcal{V}^\alpha\mathcal{V}_\beta H^{M\beta} \quad (91)$$

$$\hat{F} \equiv d\hat{\mathcal{A}} = i\hat{P}^*\wedge\hat{P} + \frac{g}{2}\xi^\alpha_M M_{\alpha\beta}H^{M\beta} \quad (92)$$

$$\text{gauged } \mathrm{SO}(6,n)/(\mathrm{SU}(4)\times\mathrm{SO}(n))\,\text{vielbein} : \hat{P}_{\underline{a}}{}^{ij} = L^{M}{}_{\underline{a}}\hat{d}L_{M}{}^{ij} \tag{93}$$

$$\text{gauged } \mathrm{SU}(4)\,\text{connection} : \hat{\omega}^{i}{}_{j} = L^{Mik}\hat{d}L_{Mjk} \tag{94}$$

$$\text{gauged } \mathrm{SO}(n)\,\text{connection} : \hat{\omega}_{\underline{a}}{}^{\underline{b}} = L^{M}{}_{\underline{a}}\hat{d}L_{M}{}^{\underline{b}} \tag{95}$$

where

$$\hat{d}L_{M}{}^{\underline{M}} \equiv dL_{M}{}^{\underline{M}} + gA^{N\alpha}\Theta_{\alpha NM}{}^{P}L_{P}{}^{\underline{M}} \tag{96}$$

Useful relations:

$$\hat{D} L_M{}^{ij} \equiv \hat{d} L_M{}^{ij} - \hat{\omega}^i{}_k L_M{}^{kj} - \hat{\omega}^j{}_k L_M{}^{ik} = L_M{}^{\underline{a}} \hat{P}_{\underline{a}}{}^{ij} \tag{97}$$

$$\hat{D} L_M{}^{\underline{a}} \equiv \hat{d} L_M{}^{\underline{a}} + \hat{\omega}^{\underline{a}}{}_{\underline{b}} L_M{}^{\underline{b}} = L_M{}^{ij} \hat{P}^{\underline{a}}{}_{ij} \tag{98}$$

$$\begin{aligned} \hat{D} \hat{P}_{\underline{a}}{}^{ij} &\equiv d \hat{P}_{\underline{a}}{}^{ij} + \hat{\omega}_{\underline{a}}{}^{\underline{b}} \wedge \hat{P}_{\underline{b}}{}^{ij} - \hat{\omega}^i{}_k \wedge \hat{P}_{\underline{a}}{}^{kj} - \hat{\omega}^j{}_k \wedge \hat{P}_{\underline{a}}{}^{ik} \\ &= g \Theta_{\alpha M}{}^{NP} L_{N\underline{a}} L_P{}^{ij} H^{M\alpha} \end{aligned} \tag{99}$$

$$\begin{aligned} \hat{R}^i{}_j \equiv d\hat{\omega}^i{}_j - \hat{\omega}^i{}_k \wedge \hat{\omega}^k{}_j &= \hat{P}^{\underline{a}ik} \wedge \hat{P}_{\underline{a}jk} \\ &+ g \Theta_{\alpha M}{}^{NP} L_N{}^{ik} L_{Pjk} H^{M\alpha} \end{aligned} \tag{100}$$

$$\begin{aligned} \hat{R}_{\underline{a}}{}^{\underline{b}} \equiv d\hat{\omega}_{\underline{a}}{}^{\underline{b}} + \hat{\omega}_{\underline{a}}{}^{\underline{c}} \wedge \hat{\omega}_{\underline{c}}{}^{\underline{b}} &= -\hat{P}_{\underline{a}ij} \wedge \hat{P}^{\underline{b}ij} \\ &+ g \Theta_{\alpha M}{}^{NP} L_{N\underline{a}} L_P{}^{\underline{b}} H^{M\alpha} \end{aligned} \tag{101}$$

Solution of the Gauged Bianchi Identities

Gauged supercurvatures:

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \tag{102}$$

$$\begin{aligned} T^a &= de^a + \omega^a{}_b \wedge e^b - \bar{\psi}^i \wedge \gamma^a \psi_i \\ &= \hat{D}e^a - \bar{\psi}^i \wedge \gamma^a \psi_i \end{aligned} \tag{103}$$

$$\hat{\rho}_i = \hat{D}\psi_i = d\psi_i + \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi_i - \frac{i}{2}\hat{\mathcal{A}} \wedge \psi_i - \hat{\omega}_i{}^j \wedge \psi_j \tag{104}$$

$$\hat{V}_i = \hat{D}\chi_i = d\chi_i + \frac{1}{4}\omega^{ab}\gamma_{ab}\chi_i + \frac{3i}{2}\hat{\mathcal{A}}\chi_i - \hat{\omega}_i{}^j \chi_j \tag{105}$$

$$\begin{aligned} \hat{\Lambda}_{\underline{a}i} = \hat{D}\lambda_{\underline{a}i} &= d\lambda_{\underline{a}i} + \frac{1}{4}\omega^{ab}\gamma_{ab}\lambda_{\underline{a}i} + \frac{i}{2}\hat{\mathcal{A}}\lambda_{\underline{a}i} - \hat{\omega}_i{}^j \lambda_{\underline{a}j} \\ &+ \hat{\omega}_{\underline{a}}{}^{\underline{b}}\lambda_{\underline{b}i} \end{aligned} \tag{106}$$

$$\mathcal{H}^{M\alpha} = dA^{M\alpha} - \frac{g}{2}\hat{f}_{\beta NP}{}^{M} A^{N\beta} \wedge A^{P\alpha} - \frac{g}{2}\Theta^{\alpha M}{}_{NP} B^{NP} + \frac{g}{2}\xi^{M}_{\beta} B^{\alpha\beta} - (\mathcal{V}^{\alpha})^{*} L^{Mij} \bar{\psi}_i \wedge \psi_j - \mathcal{V}^{\alpha} L^{M}{}_{ij} \bar{\psi}^i \wedge \psi^j \tag{107}$$

$$\mathcal{H}^{(3)MN} = \hat{d}B^{MN} + \epsilon_{\alpha\beta} A^{[M|\alpha} \wedge \left(dA^{|N]\beta} + \frac{g}{3} X_{P\gamma Q\delta}{}^{|N]\beta} A^{P\gamma} \wedge A^{Q\delta} \right) \tag{108}$$

$$\mathcal{H}^{(3)\alpha\beta} = \hat{d}B^{\alpha\beta} - \eta_{MN} A^{M(\alpha|} \wedge \left(dA^{N|\beta)} + \frac{g}{3} X_{P\gamma Q\delta}{}^{N|\beta)} A^{P\gamma} \wedge A^{Q\delta} \right) \tag{109}$$

$$\hat{P} = \frac{i}{2}\epsilon^{\alpha\beta}\mathcal{V}_\alpha \hat{d}\mathcal{V}_\beta \tag{110}$$

$$\hat{P}_{\underline{a}ij} = L^M{}_{\underline{a}}\hat{d}L_{Mij} \tag{111}$$

Bianchi identities:

$$\hat{D}R^{ab} = 0 \tag{112}$$

$$\hat{D}T^a = R^a{}_b \wedge e^b + \bar{\psi}_i \wedge \gamma^a \hat{\rho}^i + \bar{\psi}^i \wedge \gamma^a \hat{\rho}_i \tag{113}$$

$$\hat{D}\hat{\rho}_i = \frac{1}{4}R^{ab} \wedge \gamma_{ab}\psi_i - \frac{i}{2}\hat{F} \wedge \psi_i - \hat{R}_i{}^j \wedge \psi_j \tag{114}$$

$$\hat{D}\hat{V}_i = \frac{1}{4}R^{ab}\gamma_{ab}\chi_i + \frac{3i}{2}\hat{F}\chi_i - \hat{R}_i{}^j\chi_j \tag{115}$$

$$\hat{D}\hat{\Lambda}_{\underline{a}i} = \frac{1}{4}R^{ab}\gamma_{ab}\lambda_{\underline{a}i} + \frac{i}{2}\hat{F}\lambda_{\underline{a}i} - \hat{R}_i{}^j\lambda_{\underline{a}j} + \hat{R}_{\underline{a}}{}^{\underline{b}}\lambda_{\underline{b}i} \tag{116}$$

$$\begin{aligned}\hat{D}\mathcal{H}^{M\alpha} =& -\mathcal{V}^{\alpha}L^{Mij}\hat{P}^{*}\wedge\bar{\psi}_i\wedge\psi_j-(\mathcal{V}^{\alpha})^{*}L^{M\underline{a}}\hat{P}_{\underline{a}}{}^{ij}\wedge\bar{\psi}_i\wedge\psi_j\\ &+2(\mathcal{V}^{\alpha})^{*}L^{Mij}\bar{\psi}_i\wedge\hat{\rho}_j-(\mathcal{V}^{\alpha})^{*}L^{M}{}_{ij}\hat{P}\wedge\bar{\psi}^i\wedge\psi^j\\ &-\mathcal{V}^{\alpha}L^{M\underline{a}}\hat{P}_{\underline{a}ij}\wedge\bar{\psi}^i\wedge\psi^j+2\mathcal{V}^{\alpha}L^{M}{}_{ij}\bar{\psi}^i\wedge\hat{\rho}^j\\ &-\frac{g}{2}\Theta^{\alpha M}{}_{NP}\mathcal{H}^{(3)NP}+\frac{g}{2}\xi^{M}_{\beta}\mathcal{H}^{(3)\alpha\beta}\end{aligned}\tag{117}$$

$$\begin{aligned}-\frac{1}{2}\Theta^{\alpha M}{}_{NP}\hat{D}\mathcal{H}^{(3)NP}+\frac{1}{2}\xi^{M}_{\beta}\hat{D}\mathcal{H}^{(3)\alpha\beta} =& X_{N\beta P\gamma}{}^{M\alpha}\Big[\mathcal{H}^{N\beta}\\ &+(\mathcal{V}^{\beta})^{*}L^{Nij}\bar{\psi}_i\wedge\psi_j+\mathcal{V}^{\beta}L^{N}{}_{ij}\bar{\psi}^i\wedge\psi^j\Big]\wedge\Big[\mathcal{H}^{P\gamma}\\ &+(\mathcal{V}^{\gamma})^{*}L^{Pkl}\bar{\psi}_k\wedge\psi_l+\mathcal{V}^{\gamma}L^{P}{}_{kl}\bar{\psi}^k\wedge\psi^l\Big]\end{aligned}\tag{118}$$

$$\hat{D}\hat{P} = \frac{i}{2} g \xi_{\alpha M} \mathcal{V}^{\alpha} \mathcal{V}_{\beta} \mathcal{H}^{M\beta} - g \xi_{\alpha M} \mathcal{V}^{\alpha} L^{Mij} \bar{\psi}_i \wedge \psi_j \tag{119}$$

$$\hat{D}\hat{P}_{\underline{a}ij} = g \Theta_{\alpha M}{}^{NP} L_{N\underline{a}} L_{Pij} \Big[\mathcal{H}^{M\alpha} + (\mathcal{V}^{\alpha})^* L^{Mkl} \bar{\psi}_k \wedge \psi_l + \mathcal{V}^{\alpha} L^{M}{}_{kl} \bar{\psi}^k \wedge \psi^l \Big] \tag{120}$$

The above Bianchi identities are solved by appropriate rheonomic parametrizations of the supercurvatures:

- Constraint of vanishing supertorsion:

$$T^a = 0 \tag{121}$$

- $\hat{P}$, $\hat{P}_{\underline{a}ij}$ and $\mathcal{H}^{M\alpha}$ have the same outer components as their ungauged counterparts, while their inner components are denoted by $\hat{P}_a$, $\hat{P}_{\underline{a}ija}$ and $\mathcal{H}^{M\alpha}_{ab}$ respectively.

  Furthermore, $\mathcal{H}^{M\alpha}_{ab}$ must satisfy

$$\epsilon_{abcd}\mathcal{H}^{M\alpha cd} = -2M^M{}_N M^\alpha{}_\beta \mathcal{H}^{N\beta}_{ab} \tag{122}$$

- For the super-field strengths of the super-two-forms $B^{M\alpha} \equiv -\frac{1}{2}\Theta^{\alpha M}{}_{NP}B^{NP} + \frac{1}{2}\xi^M_\beta B^{\alpha\beta}$,

$$
\begin{aligned}
\mathcal{H}^{(3)M\alpha} \equiv & -\frac{1}{2}\Theta^{\alpha M}{}_{NP}\mathcal{H}^{(3)NP} + \frac{1}{2}\xi^M_\beta \mathcal{H}^{(3)\alpha\beta} = \frac{1}{6}\mathcal{H}^{(3)M\alpha}_{abc} e^a \wedge e^b \wedge e^c \\
& + i\Theta^{\alpha MNP} L_N{}^{\underline{a}} L_P{}^{ij} \bar{\lambda}_{\underline{a}i}\gamma_{ab}\psi_j \wedge e^a \wedge e^b \\
& - \frac{1}{4}\xi^M_\beta (\mathcal{V}^\alpha)^* (\mathcal{V}^\beta)^* \bar{\chi}^i \gamma_{ab}\psi_i \wedge e^a \wedge e^b \\
& - i\Theta^{\alpha MNP} L_N{}^{\underline{a}} L_{Pij} \bar{\lambda}^i_{\underline{a}}\gamma_{ab}\psi^j \wedge e^a \wedge e^b \\
& - \frac{1}{4}\xi^M_\beta \mathcal{V}^\alpha \mathcal{V}^\beta \bar{\chi}_i \gamma_{ab}\psi^i \wedge e^a \wedge e^b \\
& + 2i\Theta^{\alpha MNP} L_N{}^{ik} L_{Pjk} \bar{\psi}^j \wedge \gamma_a \psi_i \wedge e^a \\
& - \frac{1}{2}\xi^M_\beta M^{\alpha\beta} \bar{\psi}^i \wedge \gamma_a \psi_i \wedge e^a
\end{aligned}
\qquad (123)
$$

- The rheonomic parametrizations of the femionic supercurvatures in the gauged theory,

$$\hat{V}_i \equiv \hat{D}\chi_i,\ \hat{\Lambda}_{\underline{a}i} \equiv \hat{D}\lambda_{\underline{a}i},\ \hat{\rho}_i \equiv \hat{D}\psi_i \tag{124}$$

are obtained from their ungauged counterparts with the replacements

$$P_a \to \hat{P}_a,\ P_{\underline{a}ija} \to \hat{P}_{\underline{a}ija},\ \mathcal{F}^{M\alpha}_{ab} \to \mathcal{H}^{M\alpha}_{ab} \tag{125}$$

and the addition of suitable fermion shift terms proportional to g:

$$\hat{D}\chi_i \supset \frac{2}{3} g \bar{A}_{2ij}\psi^j \tag{126}$$

$$\hat{D}\lambda_{\underline{a}i} \supset g \bar{A}_{2\underline{a}}{}^{j}{}_{i}\psi_j \tag{127}$$

$$\hat{D}\psi_i \supset -\frac{1}{3} g \bar{A}_{1ij}\gamma_a\psi^j \wedge e^a \,, \tag{128}$$

where [Schön and Weidner (2006)]

$$A_2^{ij} = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{kl} L^{Nik} L^{Pjl} + \frac{3}{2}\xi_{\alpha M}\mathcal{V}^\alpha L^{Mij} \tag{129}$$

$$A_{2\underline{a}i}{}^{j} = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{\underline{a}} L^N{}_{ik} L^{Pjk} - \frac{1}{4}\delta^j_i \xi_{\alpha M}\mathcal{V}^\alpha L^M{}_{\underline{a}} \tag{130}$$

$$A_1^{ij} = f_{\alpha MNP}(\mathcal{V}^\alpha)^* L^M{}_{kl} L^{Nik} L^{Pjl} \tag{131}$$

Constraints on the inner components of the fermionic supercurvatures:

$$\gamma^a \hat{V}_{ia} = \frac{i}{4}\mathcal{V}^*_\alpha L_{M\underline{a}}\mathcal{H}^{M\alpha}_{ab}\gamma^{ab}\lambda^{\underline{a}}_i - 2g\bar{A}_2{}^{\underline{a}j}{}_i\lambda_{\underline{a}j} + 2g\bar{A}_2{}^{\underline{a}j}{}_j\lambda_{\underline{a}i} + \dots \tag{132}$$

$$\gamma^a \hat{\Lambda}_{\underline{a}ia} = \frac{i}{4}\mathcal{V}^*_\alpha L_{Mij}\mathcal{H}^{M\alpha}_{ab}\gamma^{ab}\lambda^j_{\underline{a}} + \frac{i}{8}\mathcal{V}_\alpha L_{M\underline{a}}\mathcal{H}^{M\alpha}_{ab}\gamma^{ab}\chi_i \tag{133}$$

$$- gA_{2\underline{a}i}{}^j\chi_j + gA_{2\underline{a}j}{}^j\chi_i + 2g\bar{A}_{\underline{ab}ij}\lambda^{\underline{b}j} + \frac{2}{3}g\bar{A}_{2(ij)}\lambda^j_{\underline{a}} + \dots ,$$

where

$$A_{\underline{ab}}{}^{ij} \equiv f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{\underline{a}}L^N{}_{\underline{b}}L^{Pij}. \tag{134}$$

$$\gamma^b \hat{\rho}_{iba} = \frac{i}{2}\mathcal{V}_\alpha L_{M\underline{a}}\mathcal{H}^{M\alpha}_{ab}\gamma^b\lambda_i^{\underline{a}} - \frac{i}{2}\mathcal{V}^*_\alpha L_{Mij}\mathcal{H}^{M\alpha}_{ab}\gamma^b\chi^j$$
$$+ \hat{P}_a\chi_i + 2\hat{P}_{\underline{a}ija}\lambda^{\underline{a}j} \tag{135}$$
$$+ \frac{1}{3}g\bar{A}_{2ji}\gamma_a\chi^j + gA_{2\underline{a}i}{}^j\gamma_a\lambda_j^{\underline{a}} + \dots$$

The restrictions of these constraints to spacetime are identified with the equations of motions for the fermionic spacetime fields.

In the rheonomic approach the action is written as

$$S = \int_{\mathcal{M}^4} \mathcal{L}\,, \tag{136}$$

where

- $\mathcal{M}^4$ is a four-dimensional bosonic hypersurface embedded in $\mathcal{N} = 4$ superspace
- $\mathcal{L}$ is a super-four-form Lagrangian

Provided $\mathcal{L}$ does not contain the Hodge duality operator, the equations of motion implied by the variational principle $\delta S = 0$ are independent from the choice of $\mathcal{M}^4$ and are thus valid in the whole superspace.

In order to construct $\mathcal{L}$,

1. write down the most general super-four-form Lagrangian invariant under local Lorentz, SO(2), SU(4) and SO(n) transformations
   - in particular, introduce auxiliary super-0-forms providing a first-order description of the kinetic terms of the bosonic superfields which avoids the introduction of the Hodge duality operator
2. require the equations of motion arising from the variational principle $\delta S = 0$ be solved by the constraint $T^a = 0$, the rheonomic parametrizations of the supercurvatures and the superspace equations of motion obtained by demanding closure of the Bianchi identities

The Lagrangian

The spacetime Lagrangian for the gauged $D = 4$, $\mathcal{N} = 4$ supergravity in an arbitrary symplectic frame follows from the restriction of the corresponding superspace four-form Lagrangian to spacetime and can be split in 6 terms as follows

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{fermion mass}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{top}} + \mathcal{L}_{\text{4fermi}} \qquad (137)$$

where

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{kin}} = & \frac{1}{2}R + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\left(\bar{\psi}^i_\mu\gamma_\nu\hat{\rho}_{i\rho\sigma} - \bar{\psi}_{i\mu}\gamma_\nu\hat{\rho}^i_{\rho\sigma}\right) \\
& - \frac{1}{2}\left(\bar{\chi}^i\gamma^\mu\hat{D}_\mu\chi_i + \bar{\chi}_i\gamma^\mu\hat{D}_\mu\chi^i\right) \\
& - \left(\bar{\lambda}^{\underline{a}}_i\gamma^\mu\hat{D}_\mu\lambda^i_{\underline{a}} + \bar{\lambda}^i_{\underline{a}}\gamma^\mu\hat{D}_\mu\lambda^{\underline{a}}_i\right) \\
& - \hat{P}^*_\mu\hat{P}^\mu - \frac{1}{2}\hat{P}_{\underline{a}ij\mu}\hat{P}^{\underline{a}ij\mu} + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}H^\Lambda_{\mu\nu}H^{\Sigma\mu\nu} \\
& + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}\mathcal{R}_{\Lambda\Sigma}H^\Lambda_{\mu\nu}H^\Sigma_{\rho\sigma}\,,
\end{aligned}
\tag{138}
$$

where the field strengths of the fermionic fields have the following expressions

$$\hat{\rho}_{i\mu\nu} \equiv 2\partial_{[\mu|}\psi_{i|\nu]} + \frac{1}{2}\omega_{[\mu|}{}^{ab}(e,\psi)\gamma_{ab}\psi_{i|\nu]} - i\hat{\mathcal{A}}_{[\mu|}\psi_{i|\nu]} - 2\hat{\omega}_i{}^j{}_{[\mu|}\psi_{j|\nu]}, \tag{139}$$

$$\hat{D}_\mu\chi_i \equiv \partial_\mu\chi_i + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab}\chi_i + \frac{3i}{2}\hat{\mathcal{A}}_\mu\chi_i - \hat{\omega}_i{}^j{}_\mu\chi_j\,, \tag{140}$$

$$\hat{D}_\mu\lambda_{\underline{a}i} \equiv \partial_\mu\lambda_{\underline{a}i} + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab}\lambda_{\underline{a}i} + \frac{i}{2}\hat{\mathcal{A}}_\mu\lambda_{\underline{a}i} - \hat{\omega}_i{}^j{}_\mu\lambda_{\underline{a}j} + \hat{\omega}_{\underline{a}}{}^{\underline{b}}{}_\mu\lambda_{\underline{b}i}\,, \tag{141}$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{Pauli}} = & \hat{P}^{*}_{\mu}\left(\bar{\chi}^{i}\psi^{\mu}_{i} - \bar{\chi}^{i}\gamma^{\mu\nu}\psi_{i\nu}\right) + \hat{P}_{\mu}\left(\bar{\chi}_{i}\psi^{i\mu} - \bar{\chi}_{i}\gamma^{\mu\nu}\psi^{i}_{\nu}\right) \\
& - 2\hat{P}_{\underline{a}ij\mu}\left(\bar{\lambda}^{\underline{a}i}\psi^{j\mu} - \bar{\lambda}^{\underline{a}i}\gamma^{\mu\nu}\psi^{j}_{\nu}\right) \qquad (142) \\
& - 2\hat{P}^{\underline{a}ij\mu}\left(\bar{\lambda}_{\underline{a}i}\psi_{j\mu} - \bar{\lambda}_{\underline{a}i}\gamma_{\mu\nu}\psi^{\nu}_{j}\right) + \frac{1}{2}H^{\Lambda}_{\mu\nu}O^{\mu\nu}_{\Lambda}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
O_{\Lambda\mu\nu} = & \mathcal{I}_{\Lambda\Sigma}\Pi^{\Sigma}{}_{M\alpha}\Big(-2(\mathcal{V}^{\alpha})^{*}L^{Mij}\bar{\psi}_{i\mu}\psi_{j\nu} - i\epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^{\alpha})^{*}L^{Mij}\bar{\psi}^{\rho}_{i}\psi^{\sigma}_{j} \\
& + \mathcal{V}^{\alpha}L^{Mij}\bar{\lambda}_{\underline{a}i}\gamma_{\mu\nu}\lambda^{\underline{a}}_{j} - \mathcal{V}^{\alpha}L^{M\underline{a}}\bar{\chi}_{i}\gamma_{\mu\nu}\lambda^{i}_{\underline{a}} + 2(\mathcal{V}^{\alpha})^{*}L^{M}{}_{ij}\bar{\chi}^{i}\gamma_{[\mu}\psi^{j}_{\nu]} \\
& + i\epsilon_{\mu\nu\rho\sigma}(\mathcal{V}^{\alpha})^{*}L^{M}{}_{ij}\bar{\chi}^{i}\gamma^{\rho}\psi^{j\sigma} + 2\mathcal{V}^{\alpha}L^{M\underline{a}}\bar{\lambda}_{\underline{a}i}\gamma_{[\mu}\psi^{i}_{\nu]} \qquad (143) \\
& + i\epsilon_{\mu\nu\rho\sigma}\mathcal{V}^{\alpha}L^{M\underline{a}}\bar{\lambda}_{\underline{a}i}\gamma^{\rho}\psi^{i\sigma} + \text{c.c.}\Big)\,.
\end{aligned}
$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{fermion mass}} = & -2g\bar{A}_2{}^{aj}{}_i\bar{\chi}^i\lambda_{\underline{a}j} + 2g\bar{A}_2{}^{ai}{}_i\bar{\chi}^j\lambda_{\underline{a}j} + 2gA_{\underline{ab}}{}^{ij}\bar{\lambda}_i^{\underline{a}}\lambda_j^{\underline{b}} \\
& + \frac{2}{3}gA_2^{ij}\bar{\lambda}_i^{\underline{a}}\lambda_{\underline{a}j} + \frac{2}{3}g\bar{A}_{2ij}\bar{\chi}^i\gamma^\mu\psi_\mu^j \qquad (144) \\
& + 2gA_{2\underline{a}j}{}^i\bar{\lambda}_i^{\underline{a}}\gamma^\mu\psi_\mu^j - \frac{2}{3}g\bar{A}_{1ij}\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^j + c.c.\,,
\end{aligned}
$$

$$
e^{-1}\mathcal{L}_{\text{pot}} = g^2\left(\frac{1}{3}A_1^{ij}\bar{A}_{1ij} - \frac{1}{9}A_2^{ij}\bar{A}_{2ij} - \frac{1}{2}A_{2\underline{a}i}{}^j\bar{A}_2{}^{\underline{a}i}{}_j\right), \qquad (145)
$$

where the A tensors are given by [Schön and Weidner (2006)]

$$A_1^{ij} = f_{\alpha MNP}(\mathcal{V}^\alpha)^* L^M{}_{kl} L^{Nik} L^{Pjl}, \qquad (146)$$

$$A_{2\underline{a}i}{}^j = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{\underline{a}} L^N{}_{ik} L^{Pjk} - \frac{1}{4}\delta_i^j \xi_{\alpha M}\mathcal{V}^\alpha L^M{}_{\underline{a}}, \qquad (147)$$

$$A_2^{ij} = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{kl} L^{Nik} L^{Pjl} + \frac{3}{2}\xi_{\alpha M}\mathcal{V}^\alpha L^{Mij}, \qquad (148)$$

$$A_{\underline{ab}}{}^{ij} = f_{\alpha MNP}\mathcal{V}^\alpha L^M{}_{\underline{a}} L^N{}_{\underline{b}} L^{Pij} \qquad (149)$$

and satisfy the Ward identity

$$\frac{2}{3}A_1^{jk}\bar{A}_{1ik} - \frac{2}{9}A_2^{kj}\bar{A}_{2ki} - A_{2\underline{a}i}{}^k \bar{A}_2{}^{\underline{a}j}{}_k = \frac{1}{4}\delta_i^j\left(\frac{2}{3}A_1^{kl}\bar{A}_{1kl} - \frac{2}{9}A_2^{kl}\bar{A}_{2kl} - A_{2\underline{a}k}{}^l \bar{A}_2{}^{\underline{a}k}{}_l\right). \qquad (150)$$

The topological term $\mathcal{L}_{\text{top}}$ reads [de Wit, Samtleben and Trigiante (2005)]

$$e^{-1}\mathcal{L}_{\text{top}} = \frac{1}{8} g \epsilon^{\mu\nu\rho\sigma} \Pi^{\Lambda}{}_{M\alpha} \Pi_{\Lambda N\beta} \left(\Theta^{\alpha M}{}_{PQ} B^{PQ}_{\mu\nu} - \xi^{M}_{\gamma} B^{\alpha\gamma}_{\mu\nu}\right) \times$$

$$\left(2\partial_\rho A^{N\beta}_{\sigma} - g \hat{f}_{\delta RS}{}^{N} A^{R\delta}_{\rho} A^{S\beta}_{\sigma} - \frac{1}{4} g \Theta^{\beta N}{}_{RS} B^{RS}_{\rho\sigma} + \frac{1}{4} g \xi^{N}_{\delta} B^{\beta\delta}_{\rho\sigma}\right)$$

$$- \frac{1}{6} g \epsilon^{\mu\nu\rho\sigma} \left(\Pi^{\Lambda}{}_{R\epsilon} \Pi_{\Lambda S\zeta} + 2\Pi_{\Lambda R\epsilon} \Pi^{\Lambda}{}_{S\zeta}\right) X_{M\alpha N\beta}{}^{R\epsilon} A^{M\alpha}_{\mu} A^{N\beta}_{\nu} \times$$

$$\left(\partial_\rho A^{S\zeta}_{\sigma} + \frac{1}{4} g X_{P\gamma Q\delta}{}^{S\zeta} A^{P\gamma}_{\rho} A^{Q\delta}_{\sigma}\right) \qquad (151)$$

$$e^{-1}\mathcal{L}_{\text{4fermi}} = \frac{1}{8} (\mathcal{I}^{-1})^{\Lambda\Sigma} O_{\Lambda\mu\nu} O^{\mu\nu}_{\Sigma} \qquad (152)$$

$+$ terms independent of the choice of frame

The supersymmetry transformation rules

The Lagrangian (137) is invariant up to a total derivative under the local supersymmetry transformations

$$\delta_\epsilon e_\mu^a = \bar{\epsilon}^i \gamma^a \psi_{i\mu} + \bar{\epsilon}_i \gamma^a \psi_\mu^i \,, \tag{153}$$

$$\delta_\epsilon \mathcal{V}_\alpha = \mathcal{V}_\alpha^* \bar{\epsilon}_i \chi^i , \tag{154}$$

$$\delta_\epsilon L_{Mij} = L_{M\underline{a}} (2\bar{\epsilon}_{[i} \lambda_{j]}^{\underline{a}} + \epsilon_{ijkl} \bar{\epsilon}^k \lambda^{\underline{a}l}) \,, \tag{155}$$

$$\delta_\epsilon L_M{}^{\underline{a}} = 2 L_M{}^{ij} \bar{\epsilon}_i \lambda_j^{\underline{a}} + c.c. \,, \tag{156}$$

$$\begin{aligned} \delta_\epsilon A_\mu^{M\alpha} = {} & (\mathcal{V}^\alpha)^* L^M{}_{ij} \bar{\epsilon}^i \gamma_\mu \chi^j - \mathcal{V}^\alpha L^{M\underline{a}} \bar{\epsilon}^i \gamma_\mu \lambda_{\underline{a}i} \\ & + 2\mathcal{V}^\alpha L^M{}_{ij} \bar{\epsilon}^i \psi_\mu^j + c.c. \,, \end{aligned} \tag{157}$$

$$\begin{aligned}\delta_\epsilon B^{M\alpha}_{\mu\nu} =& 2i\Theta^{\alpha MNP} L_N{}^{\underline{a}} L_P{}^{ij} \bar{\epsilon}_i \gamma_{\mu\nu} \lambda_{\underline{a}j} + \frac{1}{2}\xi^M_\beta (\mathcal{V}^\alpha)^* (\mathcal{V}^\beta)^* \bar{\epsilon}_i \gamma_{\mu\nu} \chi^i \\ & - 2i\Theta^{\alpha MNP} L_N{}^{\underline{a}} L_{Pij} \bar{\epsilon}^i \gamma_{\mu\nu} \lambda^j_{\underline{a}} + \frac{1}{2}\xi^M_\beta \mathcal{V}^\alpha \mathcal{V}^\beta \bar{\epsilon}^i \gamma_{\mu\nu} \chi_i \\ & - 4i\Theta^{\alpha MNP} L_N{}^{ik} L_{Pjk} \left(\bar{\epsilon}^j \gamma_{[\mu|} \psi_{i|\nu]} + \bar{\epsilon}_i \gamma_{[\mu} \psi^j_{\nu]} \right) \qquad (158) \\ & + \xi^M_\beta M^{\alpha\beta} \left(\bar{\epsilon}^i \gamma_{[\mu|} \psi_{i|\nu]} + \bar{\epsilon}_i \gamma_{[\mu} \psi^i_{\nu]} \right) \\ & - \Theta^{\alpha M}{}_{NP} \epsilon_{\beta\gamma} A^{N\beta}_{[\mu|} \delta_\epsilon A^{P\gamma}_{|\nu]} - \xi^M_\beta \eta_{NP} A^{N(\alpha|}_{[\mu|} \delta_\epsilon A^{P|\beta)}_{|\nu]},\end{aligned}$$

where $B^{M\alpha}_{\mu\nu} \equiv -\frac{1}{2}\Theta^{\alpha M}{}_{NP} B^{NP}_{\mu\nu} + \frac{1}{2}\xi^M_\beta B^{\alpha\beta}_{\mu\nu}$,

$$\delta_\epsilon \psi_{i\mu} = \hat{D}_\mu \epsilon_i - \frac{i}{8} \mathcal{V}_\alpha L_{Mij} \mathcal{G}^{M\alpha}_{\nu\rho} \gamma^{\nu\rho} \gamma_\mu \epsilon^j - \frac{1}{3} g \bar{A}_{1ij} \gamma_\mu \epsilon^j + \dots \,, \quad (159)$$

$$\delta_\epsilon \lambda_{\underline{a}i} = \frac{i}{8} \mathcal{V}^*_\alpha L_{M\underline{a}} \mathcal{G}^{M\alpha}_{\mu\nu} \gamma^{\mu\nu} \epsilon_i - \hat{P}_{\underline{a}ij\mu} \gamma^\mu \epsilon^j + g \bar{A}_{2\underline{a}}{}^j{}_i \epsilon_j + \dots \,, \quad (160)$$

$$\delta_\epsilon \chi_i = - \frac{i}{4} \mathcal{V}^*_\alpha L_{Mij} \mathcal{G}^{M\alpha}_{\mu\nu} \gamma^{\mu\nu} \epsilon^j + \gamma^\mu \epsilon_i \hat{P}^*_\mu + \frac{2}{3} g \bar{A}_{2ij} \epsilon^j + \dots \,, \quad (161)$$

where

$$\hat{D}_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4} \omega_{\mu ab}(e, \psi) \gamma^{ab} \epsilon_i - \frac{i}{2} \hat{\mathcal{A}}_\mu \epsilon_i - \hat{\omega}_i{}^j{}_\mu \epsilon_j \,, \quad (162)$$

and we have introduced the symplectic vector $\mathcal{G}^{M\alpha}_{\mu\nu} = (H^{\Lambda}_{\mu\nu}, \mathcal{G}_{\Lambda\mu\nu})$, where

$$\mathcal{G}_{\Lambda\mu\nu} \equiv -e^{-1}\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial H^{\Lambda}_{\rho\sigma}} = \mathcal{R}_{\Lambda\Sigma}H^{\Sigma}_{\mu\nu} - \mathcal{I}_{\Lambda\Sigma}(*H^{\Sigma})_{\mu\nu} - (*O_{\Lambda})_{\mu\nu}\,. \qquad (163)$$

Vacua

In order to derive the conditions satisfied by the critical points of the scalar potential

$$V = -e^{-1}\mathcal{L}_{\text{pot}} = g^2 \left(-\frac{1}{3} A_1^{ij} \bar{A}_{1ij} + \frac{1}{9} A_2^{ij} \bar{A}_{2ij} + \frac{1}{2} A_{2\underline{a}i}{}^{j} \bar{A}_2{}^{\underline{a}i}{}_{j} \right), \tag{164}$$

we compute its variation induced by the action of an infinitesimal rigid SL(2,R) $\times$ SO(6,n) transformation that is orthogonal to the isotropy group SO(2) $\times$ SU(4) $\times$ SO(n) of the scalar manifold on the coset representatives $\mathcal{V}_\alpha$ and $L_M{}^{\underline{M}}$ [de Wit and Nicolai (1984)].

Such a transformation can be written as

$$\delta\mathcal{V}_\alpha = \Sigma\mathcal{V}^*_\alpha,\ \delta L_M{}^{ij} = \Sigma_{\underline{a}}{}^{ij} L_M{}^{\underline{a}},\ \delta L_M{}^{\underline{a}} = \Sigma^{\underline{a}}{}_{ij} L_M{}^{ij}, \qquad (165)$$

where Σ denotes the complex SL(2,R)/SO(2) scalar fluctuation and $\Sigma_{\underline{a}ij} = (\Sigma_{\underline{a}}{}^{ij})^* = \frac{1}{2}\epsilon_{ijkl}\Sigma_{\underline{a}}{}^{kl}$ are the SO(6,n)/(SO(6) $\times$ SO(n)) scalar fluctuations.
The variation of the scalar potential is given by

$$\delta V = g^2\ \left(X\Sigma + X^*\Sigma^* + X^{\underline{a}ij}\Sigma_{\underline{a}ij} \right), \qquad (166)$$

where

$$X = -\frac{2}{9}A_1^{ij}\bar{A}_{2ij} + \frac{1}{18}\epsilon^{ijkl}\bar{A}_{2ij}\bar{A}_{2kl} - \frac{1}{2}\bar{A}_{2\underline{a}}{}^i{}_j\bar{A}_2{}^{\underline{a}j}{}_i + \frac{1}{4}\bar{A}_{2\underline{a}}{}^i{}_i\bar{A}_2{}^{\underline{a}j}{}_j, \tag{167}$$

$$X^{\underline{a}ij} = -\frac{2}{3}A_1^{[i|k}A_2{}^{\underline{a}}{}_k{}^{|j]} - \frac{1}{3}A_2^{[i|k}\bar{A}_2{}^{\underline{a}|j]}{}_k - \frac{1}{3}A_2^{k[i|}\bar{A}_2{}^{\underline{a}|j]}{}_k - \frac{1}{4}A_2^{[ij]}\bar{A}_2{}^{\underline{a}k}{}_k - A^{\underline{ab}[i|k}\bar{A}_{2\underline{b}}{}^{|j]}{}_k + \frac{1}{4}A^{\underline{ab}ij}\bar{A}_{2\underline{b}}{}^k{}_k + \epsilon^{ijlm}\Big(-\frac{1}{3}\bar{A}_{1kl}\bar{A}_2{}^{\underline{a}k}{}_m - \frac{1}{3}\bar{A}_{2(kl)}A_2{}^{\underline{a}}{}_m{}^k - \frac{1}{8}\bar{A}_{2lm}A_2{}^{\underline{a}}{}_k{}^k + \frac{1}{2}\bar{A}^{\underline{ab}}{}_{kl}A_{2\underline{b}m}{}^k + \frac{1}{8}\bar{A}^{\underline{ab}}{}_{lm}A_{2\underline{b}k}{}^k\Big). \tag{168}$$

The stationary points of the scalar potential correspond to solutions of the following system of $6n + 2$ real equations

$$X = 0\,, \qquad X^{\underline{a}ij} = 0\,. \tag{169}$$

Scalar masses

We can specify the mass spectrum of the scalar fields by computing the second variation of the scalar potential under (165). Mass terms for the scalar fluctuations:

$$e^{-1}\mathcal{L}_{\text{scalar mass}} = -\frac{1}{2}\delta^2 V. \tag{170}$$

We then introduce the real scalar fluctuations

$$\Sigma_1 = \sqrt{2}\,\text{Re}\Sigma, \quad \Sigma_2 = \sqrt{2}\,\text{Im}\Sigma, \quad \Sigma_{\underline{am}} = -\Gamma_{\underline{m}ij}\Sigma_{\underline{a}}{}^{ij}, \tag{171}$$

and substitute the expansions of the coset representatives around their vacuum expectation values into the kinetic terms for the scalars.

We find that the kinetic and mass terms for the scalar fluctuations read

$$\begin{aligned} e^{-1}\mathcal{L} \supset & -\frac{1}{2}(\partial_\mu \Sigma_1)(\partial^\mu \Sigma_1) - \frac{1}{2}(\partial_\mu \Sigma_2)(\partial^\mu \Sigma_2) \\ & -\frac{1}{2}\delta^{\underline{ab}}\delta^{\underline{mn}}(\partial_\mu \Sigma_{\underline{am}})(\partial^\mu \Sigma_{\underline{bn}}) \\ & -\frac{1}{2}(\mathcal{M}_0^2)^{1,1}\Sigma_1^2 - \frac{1}{2}(\mathcal{M}_0^2)^{2,2}\Sigma_2^2 \\ & -(\mathcal{M}_0^2)^{1,\underline{am}}\Sigma_1\Sigma_{\underline{am}} - (\mathcal{M}_0^2)^{2,\underline{am}}\Sigma_2\Sigma_{\underline{am}} \\ & -\frac{1}{2}(\mathcal{M}_0^2)^{\underline{am},\underline{bn}}\Sigma_{\underline{am}}\Sigma_{\underline{bn}}\,, \end{aligned} \tag{172}$$

where the elements of the squared mass matrix for the scalars $\mathcal{M}_0^2$ are given by

$$(\mathcal{M}_0^2)^{1,1} = (\mathcal{M}_0^2)^{2,2} = g^2\Big(-\frac{2}{9}A_1^{ij}\bar{A}_{1ij} - \frac{2}{9}A_2^{(ij)}\bar{A}_{2ij} + \frac{2}{9}A_2^{[ij]}\bar{A}_{2ij} + A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_j\Big), \tag{173}$$

$$(\mathcal{M}_0^2)^{1,\underline{am}} = (\mathcal{M}_0^2)^{\underline{am},1} = \frac{\sqrt{2}}{4}g^2\big(-\bar{A}_{2ij}\bar{A}_2{}^{\underline{a}k}{}_k + 4\bar{A}^{\underline{ab}}{}_{ik}\bar{A}_{2\underline{b}}{}^k{}_j - \bar{A}^{\underline{ab}}{}_{ij}\bar{A}_{2\underline{b}}{}^k{}_k\big)\Gamma^{\underline{m}ij} + c.c.\,, \tag{174}$$

$$(\mathcal{M}_0^2)^{2,\underline{am}} = (\mathcal{M}_0^2)^{\underline{am},2} = \frac{i\sqrt{2}}{4}g^2\big(-\bar{A}_{2ij}\bar{A}_2{}^{\underline{a}k}{}_k + 4\bar{A}^{\underline{ab}}{}_{ik}\bar{A}_{2\underline{b}}{}^k{}_j - \bar{A}^{\underline{ab}}{}_{ij}\bar{A}_{2\underline{b}}{}^k{}_k\big)\Gamma^{\underline{m}ij} + c.c.\,, \tag{175}$$

$$
\begin{aligned}
(\mathcal{M}_0^2)^{\underline{am},\underline{bn}} = {} & \frac{1}{2} g^2 \left(2\bar{A}_2{}^{\underline{a}j}{}_k A_2{}^{\underline{b}}{}_l{}^i - A^{\underline{a}cij} \bar{A}^{\underline{b}}{}_{\underline{c}kl} \right) \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}kl} \\
& + \frac{1}{2} g^2 \Big(- 2A_2{}^{\underline{a}}{}_k{}^j \bar{A}_2{}^{\underline{b}k}{}_l + 2\bar{A}_2{}^{\underline{a}j}{}_k A_2{}^{\underline{b}}{}_l{}^k - 2A_2{}^{\underline{a}}{}_l{}^k \bar{A}_2{}^{\underline{b}j}{}_k + A_2{}^{\underline{a}}{}_k{}^k \bar{A}_2{}^{\underline{b}j}{}_l \\
& + A_2{}^{\underline{a}}{}_l{}^j \bar{A}_2{}^{\underline{b}k}{}_k - \frac{1}{3} \epsilon_{klmn} A_1^{jk} A^{\underline{ab}mn} - \frac{1}{3} \epsilon^{jkmn} \bar{A}_{1kl} \bar{A}^{\underline{ab}}{}_{mn} + 2A_2^{(jk)} \bar{A}^{\underline{ab}}{}_{kl} \\
& + 2\bar{A}_{2(kl)} A^{\underline{ab}jk} + A^{\underline{abc}} \bar{A}_{2\underline{c}}{}^j{}_l - \bar{A}^{\underline{abc}} A_{2\underline{c}l}{}^j - 4A^{\underline{a}cjk} \bar{A}^{\underline{b}}{}_{\underline{c}kl} \Big) \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}il} \\
& + \frac{1}{4} g^2 A_2{}^{\underline{b}}{}_k{}^k \bar{A}_2{}^{\underline{a}l}{}_l \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}ij} \\
& + \frac{1}{2} g^2 \left(\frac{1}{3} A_2^{ij} \bar{A}_{2kl} - 2A_{2\underline{c}l}{}^i \bar{A}_2{}^{\underline{c}j}{}_k \right) \delta^{\underline{ab}} \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}kl} \\
& + \frac{1}{2} g^2 \Big(- \frac{8}{9} A_1^{jk} \bar{A}_{1kl} + 2A_{2\underline{c}l}{}^k \bar{A}_2{}^{\underline{c}j}{}_k - A_{2\underline{c}k}{}^k \bar{A}_2{}^{\underline{c}j}{}_l - A_{2\underline{c}l}{}^j \bar{A}_2{}^{\underline{c}k}{}_k \\
& + \frac{8}{9} A_2^{(jk)} \bar{A}_{2(kl)} \Big) \delta^{\underline{ab}} \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}il} + \frac{1}{8} g^2 A_{2\underline{c}k}{}^k \bar{A}_2{}^{\underline{c}l}{}_l \delta^{\underline{ab}} \Gamma^{\underline{m}}{}_{ij} \Gamma^{\underline{n}ij} \\
& + (\underline{a} \leftrightarrow \underline{b}, \underline{m} \leftrightarrow \underline{n}),
\end{aligned}
\tag{176}
$$

where

$$A_{\underline{abc}} \equiv f_{\alpha MNP} \mathcal{V}^{\alpha} L^{M}{}_{\underline{a}} L^{N}{}_{\underline{b}} L^{P}{}_{\underline{c}}. \tag{177}$$

Vector masses

Equations of motion for the vector gauge fields:

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu \mathcal{G}^{M\alpha}_{\rho\sigma} = ig\xi^M_\beta \left(\mathcal{V}^\alpha \mathcal{V}^\beta (\hat{P}^\mu)^* - (\mathcal{V}^\alpha)^* (\mathcal{V}^\beta)^* \hat{P}^\mu\right) + 2g\,\Theta^{\alpha M}{}_{NP} L^N{}_{\underline{a}} L^P{}_{ij} \hat{P}^{\underline{a}ij\mu} + \dots , \tag{178}$$

where the ellipses represent terms of higher order in the fields.

Using the twisted self-duality condition

$$\epsilon_{\mu\nu\rho\sigma}\mathcal{G}^{M\alpha\rho\sigma} = 2\eta^{MN}\epsilon^{\alpha\beta}M_{NP}M_{\beta\gamma}\mathcal{G}^{P\gamma}_{\mu\nu} + \text{(2-fermion terms)} \tag{179}$$

and that $\mathcal{G}^{M\alpha}_{\mu\nu}$ is on-shell identified with $H^{M\alpha}_{\mu\nu}$, we can write (178) as

$$e^{-1}\partial_\nu(eH^{M\alpha\nu\mu}) = (\mathcal{M}_1^2)^{M\alpha}{}_{N\beta}A^{N\beta\mu} + \dots , \tag{180}$$

where

$$(\mathcal{M}_1^2)^{M\alpha}{}_{N\beta} = \frac{i}{4}g^2 M^{MP}\xi_{\gamma P}\xi^{\delta}_{N}\left((\mathcal{V}^\alpha)^*(\mathcal{V}^\gamma)^*\mathcal{V}_\beta\mathcal{V}_\delta - \mathcal{V}^\alpha\mathcal{V}^\gamma\mathcal{V}^*_\beta\mathcal{V}^*_\delta\right) + g^2\Theta_{\gamma PQR}\Theta_{\beta NST}M^{MP}M^{\alpha\gamma}L^Q{}_{\underline{a}}L^{S\underline{a}}L^R{}_{ij}L^{Tij} \tag{181}$$

is the squared mass matrix of the vector fields.

The matrix $\mathcal{M}_1^2$ is a $(12 + 2n) \times (12 + 2n)$ matrix. However, the locality constraint on the embedding tensor implies that $6 + n$ vector fields are not physical. Therefore, at least half of the eigenvalues of this matrix are zero at any vacuum.

Fermion masses

After eliminating the mass mixing terms between the gravitini and the spin-1/2 fermions,

$$e^{-1}\mathcal{L}_{\text{mix}} = -g\bar{\psi}^{i}_{\mu}\gamma^{\mu}G_i + c.c.\,, \tag{182}$$

where

$$G_i \equiv \frac{2}{3}\bar{A}_{2ji}\chi^j + 2A_{2\underline{a}i}{}^{j}\lambda^{\underline{a}}_{j}, \tag{183}$$

the mass matrix of the spin-1/2 fermions for Minkowski vacua that completely break $\mathcal{N}=4$ supersymmetry is given by

$$\mathcal{M}_{\frac{1}{2}} = \begin{pmatrix} (\mathcal{M}_{\frac{1}{2}})_{ij} & (\mathcal{M}_{\frac{1}{2}})_i{}^{\underline{b}j} \\ (\mathcal{M}_{\frac{1}{2}})^{\underline{a}i}{}_j & (\mathcal{M}_{\frac{1}{2}})^{\underline{a}i,\underline{b}j} \end{pmatrix}$$

$$\equiv g \begin{pmatrix} 0 & -\sqrt{2}\bar{A}_2{}^{\underline{b}j}{}_i + \sqrt{2}\delta^j_i \bar{A}_2{}^{\underline{b}k}{}_k \\ -\sqrt{2}\bar{A}_2{}^{\underline{a}i}{}_j + \sqrt{2}\delta^i_j \bar{A}_2{}^{\underline{a}k}{}_k & 2A^{\underline{ab}ij} + \frac{2}{3}\delta^{\underline{ab}} A_2^{(ij)} \end{pmatrix} \quad (184)$$

$$+ g \begin{pmatrix} -\frac{4}{9}(\bar{A}_1^{-1})^{kl}\bar{A}_{2ik}\bar{A}_{2jl} & -\frac{2\sqrt{2}}{3}(\bar{A}_1^{-1})^{kl}\bar{A}_{2ik}A_2{}^{\underline{b}}{}_l{}^j \\ -\frac{2\sqrt{2}}{3}(\bar{A}_1^{-1})^{kl}\bar{A}_{2jk}A_2{}^{\underline{a}}{}_l{}^i & -2(\bar{A}_1^{-1})^{kl}A_2{}^{\underline{a}}{}_k{}^i A_2{}^{\underline{b}}{}_l{}^j \end{pmatrix}$$

The equations of motion for the gravitini read

$$\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_{i\rho} = -\frac{2}{3}g\bar{A}_{1ij}\gamma^{\mu\nu}\psi_\nu^j + \dots\,, \tag{185}$$

so the mass matrix of the gravitini is given by

$$(\mathcal{M}_{\frac{3}{2}})_{ij} = -\frac{2}{3}g\bar{A}_{1ij}. \tag{186}$$

Supertrace relations

Supertrace of the squared mass matrices:

$$\mathrm{STr}(\mathcal{M}^2) \equiv \sum_{\text{spins } J} (-1)^{2J}(2J+1)\mathrm{Tr}(\mathcal{M}_J^2)$$

$$= \mathrm{Tr}\left(\mathcal{M}_0^2\right) - 2\mathrm{Tr}\left(\mathcal{M}_{\frac{1}{2}}^\dagger \mathcal{M}_{\frac{1}{2}}\right) + 3\mathrm{Tr}\left(\mathcal{M}_1^2\right)$$

$$- 4\mathrm{Tr}\left(\mathcal{M}_{\frac{3}{2}}^\dagger \mathcal{M}_{\frac{3}{2}}\right) . \qquad (187)$$

This supertrace controls the quadratic divergences of the 1-loop effective potential [Coleman and Weinberg (1973), Weinberg (1973)].

Using the critical point conditions, the vanishing of the cosmological constant and the quadratic constraints on the embedding tensor, we find

$$\mathrm{Tr}\left(\mathcal{M}^{\dagger}_{\frac{3}{2}}\mathcal{M}_{\frac{3}{2}}\right) = \left(\bar{\mathcal{M}}_{\frac{3}{2}}\right)^{ij}\left(\mathcal{M}_{\frac{3}{2}}\right)_{ij} = \frac{4}{9}g^2 A_1^{ij}\bar{A}_{1ij}\,. \tag{188}$$

$$\mathrm{Tr}(\mathcal{M}_1^2) = (\mathcal{M}_1^2)^{M\alpha}{}_{M\alpha} = \left(\frac{4}{3}+\frac{1}{9}n\right) g^2 A_2^{[ij]}\bar{A}_{2ij} + 2g^2 A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_j + g^2 A^{\underline{ab}ij}\bar{A}_{\underline{ab}ij}\,, \tag{189}$$

$$\begin{aligned}\mathrm{Tr}\left(\mathcal{M}_{\frac{1}{2}}^{\dagger}\mathcal{M}_{\frac{1}{2}}\right) &= \left(\bar{\mathcal{M}}_{\frac{1}{2}}\right)^{ij}\left(\mathcal{M}_{\frac{1}{2}}\right)_{ij} + 2\left(\bar{\mathcal{M}}_{\frac{1}{2}}\right)_{\underline{a}i}{}^{j}\left(\mathcal{M}_{\frac{1}{2}}\right)_{j}{}^{\underline{a}i} \\ &\quad + \left(\bar{\mathcal{M}}_{\frac{1}{2}}\right)_{\underline{a}i,\underline{b}j}\left(\mathcal{M}_{\frac{1}{2}}\right)^{\underline{a}i,\underline{b}j} \\ &= -\frac{16}{9}g^2 A_1^{ij}\bar{A}_{1ij} + 4g^2 A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_{j} + \frac{4}{9}ng^2 A_2^{(ij)}\bar{A}_{2ij} \\ &\quad + 4g^2 A^{\underline{ab}ij}\bar{A}_{\underline{ab}ij} + \frac{32}{9}g^2 A_2^{[ij]}\bar{A}_{2ij}, \end{aligned} \tag{190}$$

$$\begin{aligned}
\mathrm{Tr}(\mathcal{M}_0^2) &= (\mathcal{M}_0^2)^{1,1} + (\mathcal{M}_0^2)^{2,2} + \delta_{\underline{ab}}\delta_{\underline{mn}}(\mathcal{M}_0^2)^{\underline{am},\underline{bn}} \\
&= -\frac{4}{9}(3n+1)g^2 A_1^{ij}\bar{A}_{1ij} + \frac{4}{9}(3n-1)g^2 A_2^{(ij)}\bar{A}_{2ij} \\
&\quad + \frac{1}{9}(n+24)\, g^2 A_2^{[ij]}\bar{A}_{2ij} \\
&\quad + 2ng^2 A_{2\underline{a}i}{}^{j}\bar{A}_2{}^{\underline{a}i}{}_j + 5g^2 A^{\underline{ab}ij}\bar{A}_{\underline{ab}ij}\,.
\end{aligned} \tag{191}$$

Altogether, the supertrace of the squared mass eigenvalues equals

$$\mathrm{STr}(\mathcal{M}^2) = 4(n-1)V = 0 \qquad (192)$$

for any Minkowski vacuum of $D = 4$, $\mathcal{N} = 4$ supergravity that completely breaks $\mathcal{N} = 4$ supersymmetry irrespective of the number of vector multiplets and the choice of the gauge group.

Conclusion

- Construction of the complete Lagrangian that incorporates all gauged $\mathcal{N}=4$ matter-coupled supergravities in four spacetime dimensions.
- $\mathrm{STr}(\mathcal{M}^2)=0$ for all Minkowski vacua that completely break $\mathcal{N}=4$ supersymmetry $\Rightarrow$ the one-loop effective potential at such vacua has no quadratic divergence

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