Wilson loop correlators at strong coupling in $4d$ $\mathcal{N} = 2$ SCFTs

Based on: arXiv:2303.08210 and arXiv:2308.03848

In collaboration with: Paolo Vallarino

Alessandro Pini

6 November, 2023

HUMBOLDT-UNIVERSITÄT ZU BERLIN

Introduction and motivations

4d SCFTs $\mathcal{N} = 4$, $\mathcal{N} = 2$ $\Rightarrow$ **Holographic** and **localization** techniques.

Mostly focus on SQCD or quiver gauge theories in the large N-limit

- Partition function, Correlators among Chiral/Anti-Chiral. [F. Passerini and K.Zarembo (2011)], [J.G. Russo and K.Zarembo (2012)], [E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)], [D. Rodríguez-Gómez and J.G. Russo (2016)],[B. Fiol and A.R. Fukelman (2019), (2020)],[M. Beccaria, G.V.Dunne and A.A. Tseytlin (2021)],[M. Beccaria, G.P. Korchemsky and A.A. Tseytlin (2022),(2023)],[M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (2022)]

For $\mathcal{N} = 2$ finding exact results at **strong coupling** is **difficult**

- BPS Wilson loops, Chiral and Wilson loops $\Rightarrow$ many properties **still** deserve to be analysed [S. -J. Rey and T. Suyama (2011)], [H. Ouyang (2021)], [K.Zarembo (2020)],[F. Galvagno and M.Preti (2021)]

Introduction and motivations

Specific
$4d\ \ \mathcal{N}=2$ SCFTs:
("close" to $\mathcal{N}=4$)

1. Type-E theory
   [I.G.Koh and S. Rajpoot (1984)]
2. $\mathbb{Z}_M$ orbifold of $\mathcal{N}=4$

Powerful tool: the **X-matrix**
[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

Sum up the ⇓ perturbative series

Exact results in the 't Hooft limit

Contents of the talk

1. $\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$ in the type-E theory

2. $\langle \underbrace{W \cdots W}_{n} \rangle$ in the $4d$ $\mathcal{N} = 2$ circular quiver

3. Conclusions

Contents of the talk

1. $\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$ in the type-E theory
2. $\langle \underbrace{W \cdots W}_{n} \rangle$ in the $4d$ $\mathcal{N} = 2$ circular quiver
3. Conclusions

Type-E theory

[I.G.Koh and S. Rajpoot (1984)]

Matter content

$G = SU(N)$, $\mathcal{V}_{\mathcal{N}=2}$ Adj, $\mathcal{H}_{\mathcal{N}=2}$ Sym, $\mathcal{H}_{\mathcal{N}=2}$ Anti-Sym,

$4d\ \mathcal{N} = 2$ $\quad SU(2)_R \times U(1)_r \quad$ $\beta = 0 \Rightarrow$ **CFT**

Chiral Operator

$$\varphi(x) \in \mathcal{V}_{\mathcal{N}=2} \quad \Rightarrow \quad \mathcal{O}_{\mathbf{n}}(x) = \mathrm{tr}\varphi^{n_1}(x) \cdots \mathrm{tr}\varphi^{n_\ell}(x) \quad \mathbf{n} = \{n_1, \cdots, n_\ell\}$$

chiral C.P.O.

$$\overline{\mathcal{Q}}^a_{\dot{\alpha}} \mathcal{O}_{\mathbf{n}}(x) = 0,$$

Conformal dimension

$$\Delta_{\mathbf{n}} = n_1 + n_2 + \cdots + n_\ell$$

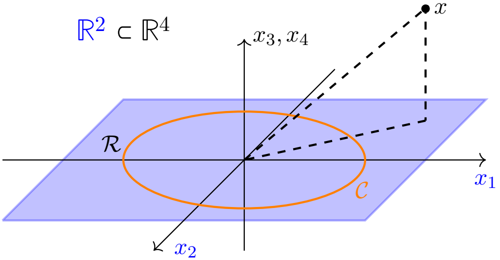
Defect correlator in the Type-E theory

Half-BPS Circular Wilson loop

$$W_{\mathcal{C}} = \frac{1}{N} \operatorname{tr} \mathcal{P} \exp \left\{ g \oint_{\mathcal{C}} d\tau \left(i\, A_\mu(x) \dot{x}(\tau) + \frac{\mathcal{R}}{2} (\varphi(x) + \overline{\varphi}(x)) \right) \right\}$$

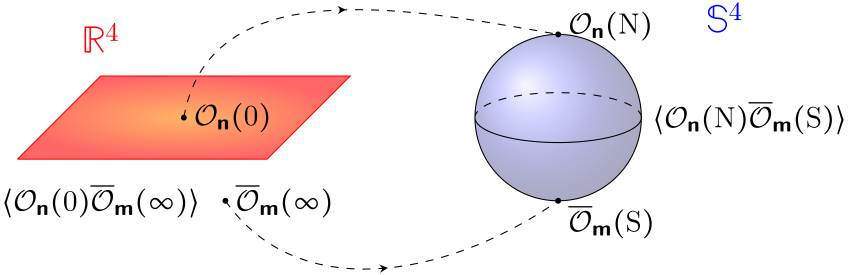
$$x(\tau) = \mathcal{R}\,(\cos\tau, \sin\tau, 0, 0)$$
$$0 \leq \tau < 2\pi$$

$$\langle W_{\mathcal{C}}\, \mathcal{O}_{\mathbf{n}}(x) \rangle = \frac{w_{\mathbf{n}}(g, N)}{(2\pi \|x\|_{\mathcal{C}})^{\Delta_{\mathbf{n}}}}$$



$\mathcal{R} \equiv 1$ and **single**-trace $\mathcal{O}_n(x)$

[E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)]



Large N-limit

Matrix Model $a \equiv a_b T^b$ [V.Pestun (2012)]

$$\mathcal{Z} = \int da\ \mathrm{e}^{-\mathrm{tr}a^2 - S_{\mathrm{int}}(a) - \cancel{S_{\mathrm{inst}}(a)}}\ , \langle f(a)\rangle = \mathcal{Z}^{-1}\int da f(a) \mathrm{e}^{-\mathrm{tr}a^2 - S_{\mathrm{int}}(a)}$$

[M. Billò, F. Fucito, A. Lerda, J.F. Morales, Ya.S. Stanev and Congkao Wen (2017)]

Gram-Schmidt Orthogonalization

Mixing with lower dimensional operators

$$\mathcal{O}_\Delta \;\rightarrow\; \mathcal{O}_\Delta + c_1\mathcal{O}_{\Delta-2} + c_2\mathcal{O}_{\Delta-4} + \cdots$$

$\mathbb{S}^4$ **Matrix Model** ⇒ $\mathbb{R}^4$ **CFT**

Orthogonalization procedure

$$\mathcal{O}_n = \operatorname{tr} a^n - \sum_{m<n} M_{n,m}\,\mathcal{O}_m,$$

$$\langle \mathcal{O}_n\,\mathcal{O}_m \rangle = 0 \;\;\forall\;\; m < n \quad\Rightarrow\quad M_{n,m}$$

[E. Gerchkovitz. J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski and S. Pufu (2016)]

Type-E theory properties [M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

$$S_{\text{int}}(a) = \sum_{\ell,m=1}^{\infty} g_{\ell,m}(\lambda) \; \boxed{\operatorname{tr} a^{2\ell+1} \operatorname{tr} a^{2m+1}}$$

only odd powers

Large N ⇓ limit

$$\mathcal{Z} = \int da \; \mathrm{e}^{-\operatorname{tr} a^2 - S_{\text{int}}(a)} = \int D\omega \; \mathrm{e}^{-\frac{1}{2}\omega^T(\mathbb{1}-\mathsf{X}^{\text{odd}})\omega} = \det{}^{-\frac{1}{2}}(\mathbb{1} - \mathsf{X}^{\text{odd}})$$

X^{odd}-matrix

$$(\mathsf{X}^{\text{odd}})_{i,j} \propto \int_0^\infty \frac{dt}{t} \frac{\mathrm{e}^t}{(\mathrm{e}^t - 1)^2} \; \boxed{J_{2i+1}(t\sqrt{\lambda}) \; J_{2j+1}(t\sqrt{\lambda})}$$

convolution

Even - planar equivalent

[V. Pestun (2012)]

$$\langle W_{\mathcal{C}}\, \mathcal{O}_{2n} \rangle \simeq \langle W_{\mathcal{C}}\, \mathcal{O}_{2n} \rangle_0 \,, \qquad \mathcal{O}_3 \equiv \operatorname{tr} a^3 \,, \quad W_{\mathcal{C}} \equiv \frac{1}{N} \sum_{k=0}^{\infty} \left(\frac{\lambda}{2N} \right)^k \frac{1}{k!} \operatorname{tr} a^k$$

$$\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle \simeq \frac{\sqrt{\mathcal{G}_3}}{N} \sum_{n=1}^{\infty} \sqrt{2n+1}\, \mathsf{D}_{n,1}(\lambda)\, I_{2n+1}(\sqrt{\lambda})$$

$$\mathsf{D}_{n,\ell}(\lambda) \equiv \delta_{n,\ell} + (\mathsf{X}^{\mathsf{odd}})_{n,\ell} + (\mathsf{X}^{\mathsf{odd}})^2_{n,\ell} + \cdots$$

Type-E theory

$$\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle_0 \simeq \frac{\sqrt{3\, \mathcal{G}_3}}{N} I_3(\sqrt{\lambda}), \quad \mathcal{G}_3 \equiv \langle \mathcal{O}_3\, \overline{\mathcal{O}}_3 \rangle_0$$

$\mathcal{N} = 4$ **SYM**

[G. W. Semenoff and K. Zarembo (2001)]

Warm-up $\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle$ at strong coupling

[AP and P. Vallarino (2023)]

$$1 + \Delta w_3(\lambda) \equiv \frac{\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle}{\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle_0} \simeq \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \sqrt{2n+1}\, \frac{I_{2n+1}(\sqrt{\lambda})}{I_3(\sqrt{\lambda})}\, \mathsf{D}_{1,n}(\lambda)$$

Asymptotic expansion

$$\frac{I_{2n+1}(\sqrt{\lambda})}{I_3(\sqrt{\lambda})} \underset{\lambda\to\infty}{\sim} 1 - \frac{2n^2+2n-4}{\sqrt{\lambda}} + \frac{2n^4+4n^3-7n^2-9n+10}{\lambda} + \cdots \equiv \sum_{s=0}^{\infty} \frac{Q_{2s}(n)}{\lambda^{s/2}}$$

$$\frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \sqrt{2n+1}\, \mathsf{D}_{1,n}(\lambda) + R_3(\lambda) \underset{\lambda\to\infty}{\sim} \frac{4\pi}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) + R_3(\lambda)$$

$s = 0$ $\quad s \geq 1$ $\quad s = 0$ $\quad s \geq 1$

[M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (2022)]

$R_3(\lambda)$ numerical evaluation

[AP and P. Vallarino (2023)]

$$\frac{\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle}{\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle_0} = \sum_k d_k \left(\frac{\lambda}{\pi^2} \right)^k \quad \text{for} \quad |\lambda| < \lambda_c$$

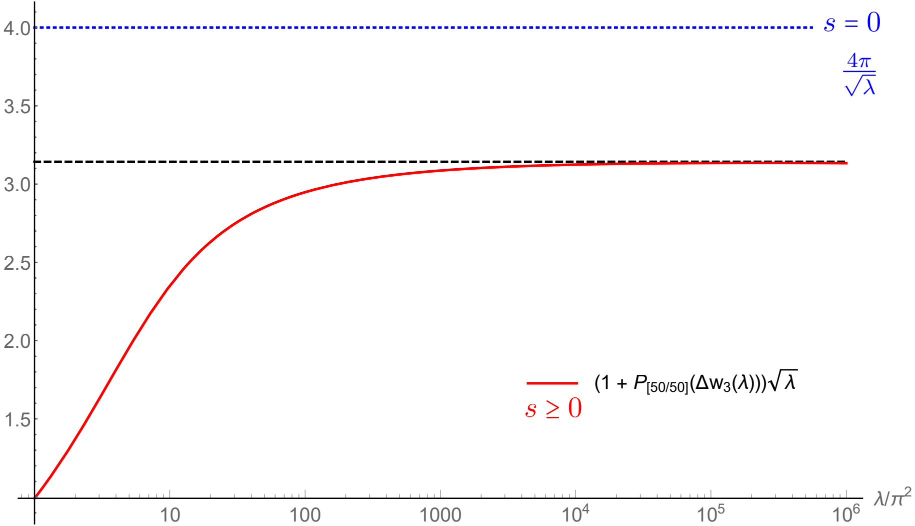
Radius of convergence $\lambda_c = \pi^2 \quad \Rightarrow \quad$ Behaviour outside λ_c ??

Numerical Method: Conformal **Padé approximant** $P_{[N,M]}$

[O.Costin G.V.Dunne (2019)]

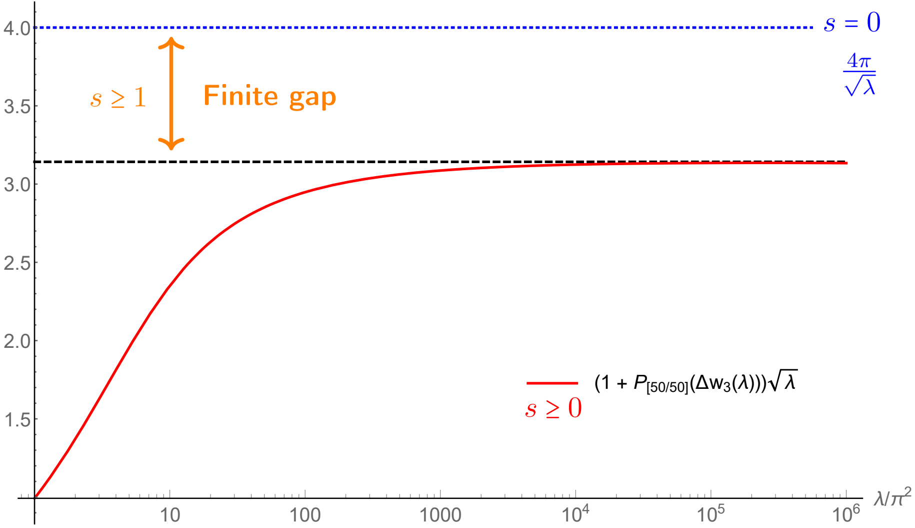
$R_3(\lambda)\sqrt{\lambda}$ numerical evaluation

[AP and P.Vallarino (2023)]



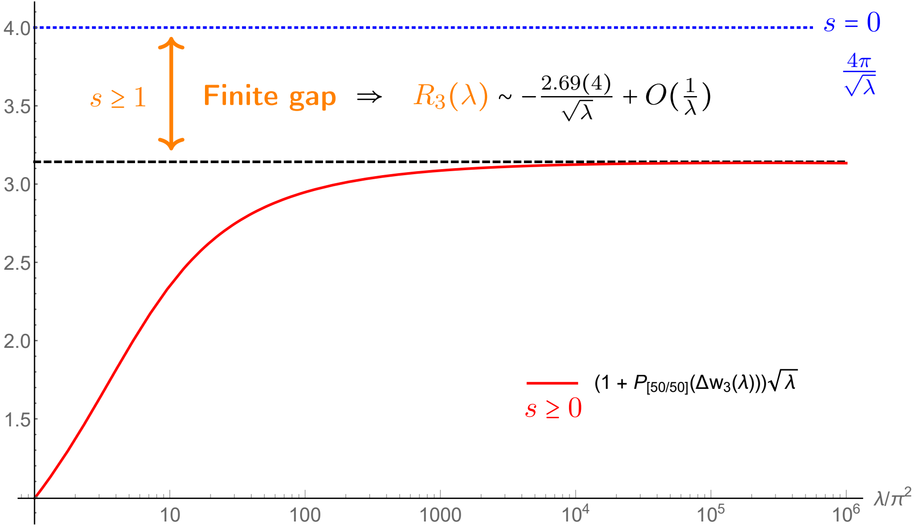
$R_3(\lambda)\sqrt{\lambda}$ numerical evaluation

[AP and P.Vallarino (2023)]



$R_3(\lambda)\sqrt{\lambda}$ numerical evaluation

[AP and P.Vallarino (2023)]



Warm-up $\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle$ at strong coupling

[AP and P. Vallarino (2023)]

Strong coupling prediction

$$\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle \underset{\lambda\to\infty}{\sim} \frac{\langle W_{\mathcal{C}}\, \mathcal{O}_3 \rangle_0}{\sqrt{\lambda}} \left(4\pi + R_3^{(1)} \right) + O\left(\frac{1}{\lambda}\right)$$

$s = 0$ $\quad$ $s \geq 1$

$$R_3(\lambda) \underset{\lambda\to\infty}{\sim} \frac{R_3^{(1)}}{\sqrt{\lambda}} + \frac{R_3^{(2)}}{\lambda} + \cdots, \qquad R_3^{(1)} = -2.69(4)$$

$\langle W_{\mathcal{C}}\,\mathcal{O}_{2p+1}\rangle$ Planar limit

[AP and P. Vallarino (2023)]

$\langle W_{\mathcal{C}}\,\mathcal{O}_{2p+1}\rangle$ **Type-E theory**

$$\frac{1}{N}\sum_{\ell=1}^{p}\sqrt{\mathcal{G}_{2\ell+1}}M_{2p+1,2\ell+1}(\lambda)\sum_{n=1}^{\infty}\sqrt{2n+1}\left(I_{2n+1}(\sqrt{\lambda})\sum_{m=1}^{\ell}h_m^{(\ell)}\mathsf{D}_{n,m}(\lambda)\right)$$

Gram-Schmidt coefficients

Numerical coefficients

$\mathcal{N}=4$ **SYM**

$$\langle W_{\mathcal{C}}\,\mathcal{O}_{2p+1}\rangle_0 \simeq \frac{\sqrt{(2p+1)\,\mathcal{G}_{2p+1}}}{N}I_{2p+1}(\sqrt{\lambda}), \qquad \langle \mathcal{O}_n\,\overline{\mathcal{O}}_n\rangle_0 \equiv \mathcal{G}_n$$

[G. W. Semenoff and K. Zarembo (2001)]

Strong coupling prediction

$$\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle \underset{\lambda\to\infty}{\sim} \langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle_0 \frac{(\Delta_{2p+1} - 1)}{\sqrt{\lambda}} \left(2\pi + \frac{R_3^{(1)}}{2} \right) + O\left(\frac{1}{\lambda}\right)$$

Conformal dimension

$$R_3^{(1)} = -2.69(4)$$

$\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$ at strong coupling

[AP and P. Vallarino (2023)]

Strong coupling prediction

$$\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle \underset{\lambda\to\infty}{\sim} \langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle_0 \frac{(\Delta_{2p+1} - 1)}{\sqrt{\lambda}} \left(2\pi + \frac{R_3^{(1)}}{2} \right) + O\left(\frac{1}{\lambda}\right)$$

Conformal dimension

$$R_3^{(1)} = -2.69(4)$$

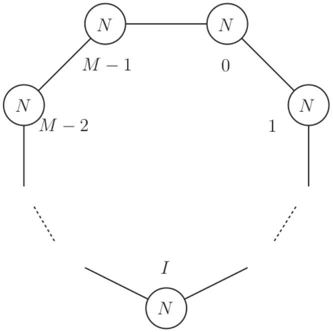
Systematic and analytic large λ expansion

[G.P. Korchemsky, AP and P. Vallarino TO APPEAR]

$$R_3^{(1)} = \pi^2 - 4\pi \simeq -2.6967\ldots \checkmark ☺$$

Contents of the talk

1. $\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$ in the type-E theory
2. $\langle \underbrace{W \cdots W}_{n} \rangle$ in the $4d$ $\mathcal{N} = 2$ circular quiver
3. Conclusions



- $I = 0, \cdots, M-1$
- **node** $\mapsto$ $\mathcal{V}^{(I)}_{\mathcal{N}=2}$ $SU(N)_I$
- **line** $\mapsto$ $\mathcal{H}_{\mathcal{N}=2}$
- $g_I \equiv g \quad \forall I \quad \Rightarrow \quad \lambda \equiv g^2 N$

Single trace chiral operators

$$\varphi^{(I)} \in \mathcal{V}^{(I)}_{\mathcal{N}=2} \qquad \mathcal{O}^{(I)}_n(\vec{x}) = \mathrm{tr}\ \varphi^{(I)}(\vec{x})^n$$

Untwisted/Twisted Wilson loops

Half-BPS Circular Wilson loop

$$W^{(I)}(x) = \frac{1}{N} \operatorname{tr} \mathcal{P} \exp\left[g \oint_{\mathcal{C}} d\tau \left(i\, A_\mu^{(I)}(x) \dot{x}^\mu(\tau) + \frac{1}{2}(\varphi^{(I)}(x) + \overline{\varphi}^{(I)}(x)) \right) \right]$$

[F. Galvagno and M.Preti (2021)], [S. -J. Rey and T. Suyama (2011)], [H. Ouyang (2021)], [K.Zarembo (2020)]

Change of basis

$$W_\alpha = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{I\alpha} W^{(I)} \qquad \rho \equiv \mathrm{e}^{\frac{2\pi i}{M}}$$

Untwisted $\alpha = 0$

$$W_0 \equiv \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} W^{(I)}$$

Twisted $\alpha \neq 0$

$$W_\alpha \equiv \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{\alpha I} W^{(I)}$$

twisted sector $\alpha = 1, \cdots, M-1$

The correlators

n-coincident Wilson loops **planar limit**

$$\langle W_{\alpha_1} W_{\alpha_2} \cdots W_{\alpha_n} \rangle \qquad \sum_{i=1}^{n} \alpha_i = 0 \mod M$$

$$\langle W_0 \rangle \simeq \sqrt{M} \langle W \rangle_0 = \frac{2\sqrt{M}}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \qquad \langle W_\alpha \rangle = 0$$ **1-point**

[F.Galvagno and M.Preti (2021)], [S.-J. Rey and T. Suyama (2011)]

$$\langle \underbrace{W_0 W_0 \cdots W_0}_{n} \rangle \simeq (\sqrt{M} \langle W \rangle_0)^n$$ **n-point untwisted**

The matrix model

[M. Billò, M. Frau, A. Lerda, AP and P. Vallarino (2022)]

$$A_{\alpha,k} = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{\alpha\, I} \operatorname{tr} a_I^k \qquad A^\dagger_{\alpha,k} = A_{M-\alpha,k} \qquad a_I \equiv a_I^b T_b$$

$\mathfrak{su}(N)$ generators

Wilson loop ⇓ **representation** [V. Pestun (2012)]

$$W_\alpha = \frac{1}{N} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\lambda}{2N}\right)^{\frac{k}{2}} A_{\alpha,k}$$

$$\langle W_{\alpha_1} \cdots W_{\alpha_n} \rangle \Rightarrow \langle A_{\alpha_1,k_1} \cdots A_{\alpha_n,k_n} \rangle$$

S_{int} can be **diagonalized** $\Rightarrow$ **Generalization** of the X^{odd}-matrix

$$\hat{A}_{\alpha,n} \equiv A_{\alpha,n} - \langle A_{\alpha,n} \rangle_0 \qquad \langle \hat{A}_{\alpha,n} \, \hat{A}^\dagger_{\alpha,m} \rangle \simeq \sum_{i,j} c_{i,n} \, c_{j,m} \, \mathsf{D}^{(\alpha)}_{n-2i,m-2j}$$

$$\mathsf{D}^{(\alpha)}_{i,j} = \delta_{k,\ell} + s_\alpha \, \mathsf{X}_{k,\ell} + s_\alpha^2 \, \mathsf{X}^2_{k,\ell} + s_\alpha^3 \, \mathsf{X}^3_{k,\ell} + \cdots \qquad s_\alpha \equiv \sin\left(\frac{\pi\alpha}{M}\right)^2$$

$S_{\text{int}} = 0$ $\qquad$ $S_{\text{int}} \neq 0$

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}} \sqrt{k\ell} \int_0^\infty \frac{dt}{t} \frac{\mathrm{e}^t}{(\mathrm{e}^t - 1)^2} J_k\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_\ell\left(\frac{t\sqrt{\lambda}}{2\pi}\right), \quad \mathsf{X}_{2k,2\ell+1} = 0$$

Warm-up $\langle W_\alpha W_\alpha^\dagger \rangle$

[AP and P. Vallarino (2023)]

$$\langle W_\alpha W_\alpha^\dagger \rangle \simeq \frac{1}{N^2} \sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} I_k(\sqrt{\lambda}) I_\ell(\sqrt{\lambda}) \sqrt{k\,\ell}\, \mathsf{D}_{k,\ell}^{(\alpha)} \qquad S_{\text{int}} \neq 0$$

$S_{\text{int}} \to 0$ $\quad \mathcal{N} = 4$ **SYM** [K. Okuyama (2018)]

$$W_{\text{conn}}^{(2)} \equiv \langle W\, W \rangle_0 - \langle W \rangle_0^2 = \frac{\sqrt{\lambda}}{2N} I_1(\sqrt{\lambda}) I_2(\sqrt{\lambda})$$

Ratio

$$\frac{\langle W_\alpha W_\alpha^\dagger \rangle}{W_{conn}^{(2)}(\lambda)} = 1 + \Delta w^{(\alpha)}(M, \lambda)$$

$\langle W_\alpha W_\alpha^\dagger \rangle$ at strong coupling

[AP and P. Vallarino (2023)]

$$\frac{I_k(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \quad \Rightarrow \quad \Delta w^{(\alpha)}(M,\lambda) \underset{\lambda\to\infty}{\sim} \frac{2}{\sqrt{\lambda}} \sum_{p=0}^{\infty} \frac{\mathcal{S}^{(p)}(s_\alpha)}{\lambda^{p/2}}$$

Large λ expansion of **Fredholm determinant** of Bessel operators
[M.Beccaria, G. P. Korchemsky and A. A. Tseytlin (2023)]

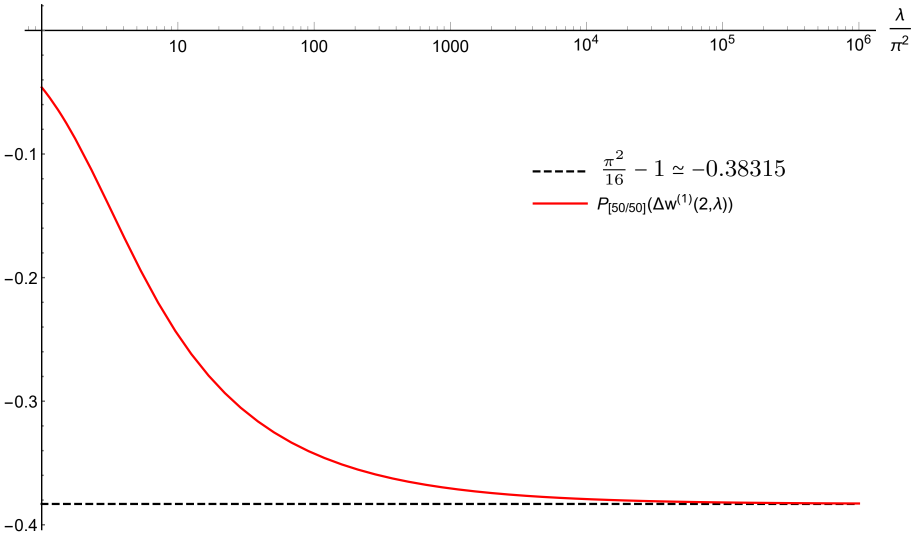
analytic ⇓ computation

$$1 + \Delta w^{(\alpha)}(M,\lambda) \underset{\lambda\to\infty}{\sim} \frac{\mathcal{I}_0(s_\alpha)}{2\sqrt{s_\alpha}} \times \frac{\mathcal{I}_0(s_{M-\alpha})}{2\sqrt{s_{M-\alpha}}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$\mathcal{I}_0(s_\alpha) \equiv \int_0^\infty \frac{dz}{\pi}\, z\, \partial_z \log\left(1 + s_\alpha \sinh\left(\frac{z}{2}\right)^{-2}\right), \quad \text{e.g.} \;\; \mathcal{I}_0(1) = \frac{\pi}{2}$$

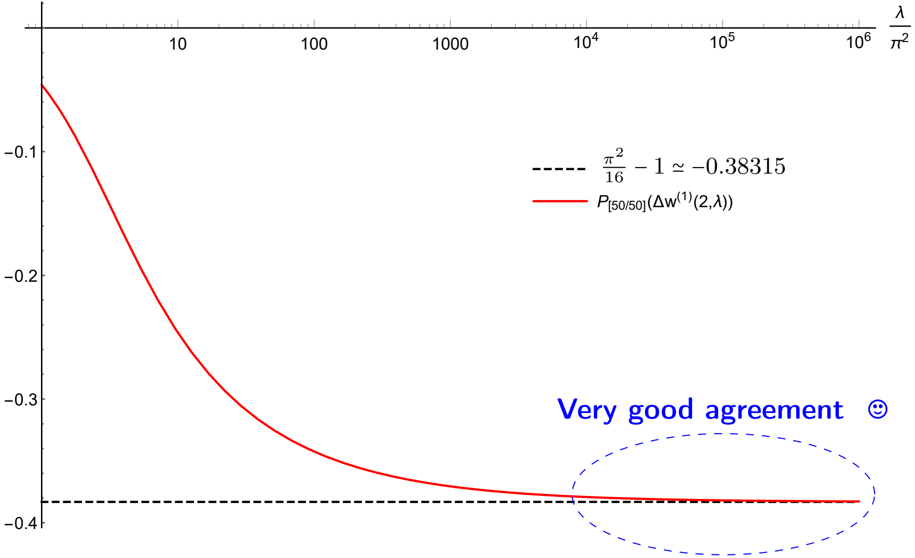
Numerical check for $M = 2$ $s_\alpha = 1$

[AP and P. Vallarino (2023)]

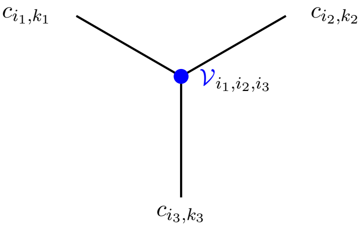


Numerical check for $M = 2$ $s_\alpha = 1$

[AP and P. Vallarino (2023)]



Very good agreement ☺



$$\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_3,k_3} \rangle \simeq$$

$$\sum_{i_1,i_2,i_3} c_{i_1,k_1} c_{i_2,k_2} c_{i_3,k_3} \mathcal{V}_{i_1,i_2,i_3}$$

The Vertex

$$\mathcal{V}_{i_1,i_2,i_3} \simeq \frac{1}{\sqrt{M}} \frac{1}{N} \mathsf{d}^{(\alpha_1)}_{k_1-2i_1} \mathsf{d}^{(\alpha_2)}_{k_2-2i_2} \mathsf{d}^{(\alpha_3)}_{k_3-2i_3} \qquad \mathsf{d}^{(\alpha)}_k \equiv \sum_{\ell=2}^{\infty} \mathsf{D}^{(\alpha)}_{\ell,k} \sqrt{\ell}$$

Factorization at large N

3-point $\langle W_{\alpha_1} W_{\alpha_2} W_{\alpha_3} \rangle$

$\langle W_{\alpha_1} W_{\alpha_2} W_{\alpha_3} \rangle$ **analytic** expression $\forall \lambda$ $\quad S_{\text{int}} \neq 0$

[AP and P. Vallarino (2023)]

$S_{\text{int}} \to 0 \qquad \mathcal{N} = 4$ **SYM**

$$W^{(3)}_{\text{conn}} \equiv \langle W\, W\, W \rangle_0 - 3\langle W \rangle_0 \,\langle W\, W \rangle_0 + 2\langle W \rangle_0^3$$

[K. Okuyama (2018)]

Ratio

$$\frac{\langle W_{\alpha_1}\, W_{\alpha_2}\, W_{\alpha_3} \rangle}{\sqrt{M}\, W^{(3)}_{\text{conn}}(\lambda)} = 1 + \Delta w^{(\alpha_1,\alpha_2)}(M, \lambda)$$

3-point $\langle W_{\alpha_1} W_{\alpha_2} W_{\alpha_3} \rangle$ at strong coupling [AP and P. Vallarino (2023)]

$$\langle W_0 \, W_\alpha \, W_\alpha^\dagger \rangle \simeq \sqrt{M} \, \langle W \rangle_0 \, \langle W_\alpha \, W_\alpha^\dagger \rangle$$ **mixed correlator**

Large λ expansion of **Fredholm determinant** of Bessel operators
[M.Beccaria, G. P. Korchemsky and A. A. Tseytlin (2023)]

analytic ⇓ computation

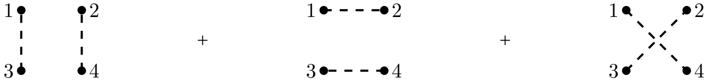
$$1 + \Delta w^{(\alpha_1,\alpha_2)}(M,\lambda) \underset{\lambda\to\infty}{\sim} -\frac{\delta_{\alpha_1+\alpha_2, M-\alpha_3}}{8} \prod_{p=1}^{3} \frac{\mathcal{I}_0(s_{\alpha_p})}{\sqrt{s_{\alpha_p}}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

2pt $\langle \hat{A}_{\alpha_1,k_1}\, \hat{A}_{\alpha_2,k_2} \rangle_0 \simeq N^{\frac{k_1+k_2}{2}}\, \delta_{\alpha_1, M-\alpha_2}$

3pt $\langle \hat{A}_{\alpha_1,k_1}\, \hat{A}_{\alpha_2,k_2}\, \hat{A}_{\alpha_3,k_3} \rangle_0 \simeq N^{\frac{k_1+k_2+k_3}{2}-1}\, \delta_{\alpha_1+\alpha_2, M-\alpha_3}$

$n \geq 4$ points $\Rightarrow$ **Factorization à la Wick**

Example $\langle \hat{A}_{\alpha_1,k_1}\, \hat{A}_{\alpha_2,k_2}\, \hat{A}_{\alpha_3,k_3}\, \hat{A}_{\alpha_4,k_4} \rangle_0 \simeq$



$\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_3,k_3} \rangle_0 \langle \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_4,k_4} \rangle_0$ $\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_2,k_2} \rangle_0 \langle \hat{A}_{\alpha_3,k_3} \hat{A}_{\alpha_4,k_4} \rangle_0$ $\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_4,k_4} \rangle_0 \langle \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_3,k_3} \rangle_0$

r.h.s $\neq 0$ $\Rightarrow$ **leading order** reducible

n-pts free theory $\Rightarrow$ **Factorization à la Wick** interacting theory
[M. Billò, M. Frau, A. Lerda, AP and P.Vallarino (2022)]

$\langle W_{\alpha_1} \cdots W_{\alpha_p} \rangle$ **exact** expressions in λ in the planar limit

$p = 2n$ **even** $\quad \langle W_{\alpha_1} W_{\alpha_2} \cdots W_{\alpha_{2n}} \rangle$ **large λ**

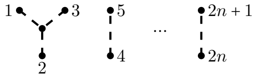
$$\simeq (W^{(2)}_{\mathrm{conn}})^n \left(\prod_{j=1}^{n} \frac{\mathcal{I}_0(s_{\alpha_{2j-1}})}{2\sqrt{s_{\alpha_{2j-1}}}} \frac{\mathcal{I}_0(s_{\alpha_{2j}})}{2\sqrt{s_{\alpha_{2j}}}} \delta_{\alpha_{2j-1}, M-\alpha_{2j}} + \cdots \right)$$

Higher correlators

[AP and P. Vallarino (2023)]

$$p = 2n+1 \ \textbf{odd} \qquad \langle W_{\alpha_1} W_{\alpha_2} \cdots W_{\alpha_{2n+1}} \rangle \quad \textbf{large } \lambda$$

$$\simeq \sqrt{M} W^{(3)}_{\text{conn}} (W^{(2)}_{\text{conn}})^{n-1} \left(-\frac{\delta_{\alpha_1+\alpha_2, M-\alpha_3}}{8} \prod_{i=1}^{3} \frac{\mathcal{I}_0(s_{\alpha_i})}{\sqrt{s_{\alpha_i}}} \prod_{j=2}^{n} \frac{\mathcal{I}_0(s_{\alpha_{2j+1}})}{2\sqrt{s_{\alpha_{2j+1}}}} \frac{\mathcal{I}_0(s_{\alpha_{2j}})}{2\sqrt{s_{\alpha_{2j}}}} \delta_{\alpha_{2j+1}, M-\alpha_{2j}} + \cdots \right)$$



$$W_{\alpha_i} \mapsto -\frac{\mathcal{I}_0(s_{\alpha_i})}{2\sqrt{s_{\alpha_i}}}$$

simple rule at large λ ☺

Contents of the talk

Summary and outlook

- Exploiting the properties of the X-matrix we summed up the perturbative series for
  1. $\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$ in the Type-E theory.
  2. $\langle W_{\alpha_1} \cdots W_{\alpha_n} \rangle$ in the $\mathbb{Z}_M$ orbifold of $\mathcal{N} = 4$ SYM.

  Exact expressions in the planar limit.

- Numerical and analytical methods ⇒ QFT **strong coupling predictions** ⇒ Test with the AdS/CFT correspondence.

- **Analytic** derivation of the strong coupling expansion for $\langle W_{\mathcal{C}}\, \mathcal{O}_{2p+1} \rangle$.

[G. Korchemsky, AP and P. Vallarino TO APPEAR]

THANKS FOR YOUR ATTENTION

The X-matrix

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

The interaction action $S_{\text{int}}(a)$

$$4\sum_{\ell,m=1}^{\infty}(-1)^{\ell+m}\left(\frac{g^2}{8\pi^2}\right)^{\ell+m+1}\frac{(2\ell+2m+1)!}{(2\ell+1)!(2m+1)!}\zeta(2\ell+2m+1)\operatorname{tr}a^{2\ell+1}\operatorname{tr}a^{2m+1}$$

The matrix element $X_{k,\ell}$

$$-8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_0^\infty\frac{dt}{t}\frac{e^t}{(e^t-1)^2}J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

The full Lie algebra approach [B. Fiol, J. Martinez-Montoya and A. Rios Fukelman (2019)]

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

$$\mathcal{Z}_{\mathbb{S}^4} = \int \prod_{u=1}^{N} dm_u \ \Delta(m) |Z(im, g)|^2 \delta\left(\sum_u m_u\right) = \int dM \ \mathrm{e}^{-S(M)} \delta(\mathrm{tr} M)$$

Hermitean traceless matrix M with eigenvalues m_u

We introduce the matrix a, tr $T_b T_c = \frac{1}{2}\delta_{bc}$ $\quad b, c = 1, \cdots, N^2 - 1$

$$a \equiv \sqrt{\frac{8\pi^2}{g^2}} M \quad \Rightarrow \quad \mathcal{Z}_{\mathbb{S}^4} = \left(\frac{g^2}{8\pi^2}\right)^{\frac{N^2-1}{2}} \int da \ \mathrm{e}^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}$$

The full Lie algebra approach [B. Fiol, J. Martinez-Montoya and A. Rios Fukelman (2019)]

$$da \equiv \prod_b \frac{da^b}{\sqrt{2\pi}} \qquad \langle f(a) \rangle_{(0)} \equiv \int da \ \mathrm{e}^{-\mathrm{tr} a^2} f(a)$$

Expectation value in the interacting model

$$\langle f(a) \rangle \equiv \frac{\int da \ f(a) e^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}}{\int da \ e^{-\mathrm{tr} a^2 - S_{\mathrm{int}}(a)}} = \frac{\langle f(a) \mathrm{e}^{-S_{\mathrm{int}}(a)} \rangle_{(0)}}{\langle \mathrm{e}^{-S_{\mathrm{int}}(a)} \rangle_{(0)}}$$

The free-variables representation in the large N-limit

single trace operator on $\mathbb{R}^4$ for $\mathcal{N} = 4$ SYM

$$\langle O_n^{(0)} O_m^{(0)} \rangle_{(0)} = n \left(\frac{N}{2} \right)^n \delta_{n,m} \equiv G_n^{(0)} \delta_{n,m}$$

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

the free variables ω

$$\omega_i(a) \equiv \frac{O_{2i+1}^{(0)}(a)}{\sqrt{G_{2i+1}^{(0)}}} \quad \Rightarrow \quad \langle \omega_i(a) \omega_j(a) \rangle_{(0)} = \delta_{i,j}$$

The free-variables representation in the large N-limit

[M. Beccaria, M. Billò, F. Galvagno, A. Hassan and A. Lerda (2020)]

Wick's theorem for odd correlation function ⇒

$$\langle \omega_{i_1}(a)\omega_{i_2}(a)\cdots\omega_{i_n}(a)\rangle_{(0)} = \int D\omega\ \omega_{i_1}\omega_{i_2}\cdots\omega_{i_n}\mathrm{e}^{-\frac{1}{2}\omega^T\omega} \qquad D\omega \equiv \prod_{i=1}^{\infty}\frac{d\omega_i}{\sqrt{2\pi}}$$

interacting theory

$$S_{\text{int}} = -\frac{1}{2}\omega^T \mathrm{X}\omega$$

$$\langle f(\omega)\rangle = \frac{1}{\mathcal{Z}}\int D\omega\ f(\omega)\mathrm{e}^{-\frac{1}{2}\omega^T(\mathbb{1}-\mathrm{X})\omega}, \qquad \mathcal{Z} = \det{}^{-\frac{1}{2}}(\mathbb{1}-\mathrm{X})$$

3-point function at Large-N

[M. Baggio, V. Niarchos and K.Papadodimas (2015)]

Planar limit $\Rightarrow$ Gram-Schmidt **simpler** set **single**-trace $\{O_n\}$

$$O_n = \Omega_n - \sum_{m<n} C_{n,m} O_m$$

$$\mathcal{O}_n = O_n + \frac{1}{N}(\text{single and multi trace})$$

$\mathcal{O}_n$: orthogonal to **all** lower

O_n: orthogonal **only** to single-trace

3-point

$$G_{n_1,n_2} = \langle \mathcal{O}_{n_1} \mathcal{O}_{n_2} \mathcal{O}_{n_1+n_2} \rangle = \langle O_{n_1} O_{n_2} O_{n_1+n_2} \rangle + O\left(1/N\right)$$

Results at strong coupling 1

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Product of Bessel functions $\Rightarrow$ **inverse Mellin transform**

$$J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$= \int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\,\frac{\Gamma(-s)\Gamma(2s+2k+2\ell+3)}{\Gamma(s+2k+2)\Gamma(s+2\ell+2)\Gamma(s+2k+2\ell+3)}\left(\frac{t\sqrt{\lambda}}{4\pi}\right)^{2(s+k+\ell+1)}$$

$$\mathrm{X}_{k\ell} = -8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_{-i\infty}^{+i\infty}\frac{ds}{2\pi i}\left[\left(\frac{t\sqrt{\lambda}}{4\pi}\right)^{2(s+k+\ell+1)}\right.$$

$$\left.\frac{\Gamma(-s)\Gamma(2s+2k+2\ell+3)\Gamma(2s+2k+2\ell+2)}{\Gamma(s+2k+2)\Gamma(s+2\ell+2)\Gamma(s+2k+2\ell+3)}\zeta(2s+2k2\ell+1)\right]$$

Results at strong coupling 2

[M. Beccaria, M. Billò, M. Frau, A. Lerda and AP (2021)]

Contributions ⇒ **poles** on the negative real axis

$$\mathrm{X}_{k\ell} = -8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\Bigg[\frac{\lambda}{16\pi^2}\left(\frac{\delta_{k-1,\ell}}{2(2k-1)2k(2k+1)}+\frac{\delta_{k,\ell}}{2k(2k+1)(2k+2)}+\frac{\delta_{k+1,\ell}}{2(2k+1)(2k+2)(2k+3)}\right)-\frac{\delta_{k\ell}}{24(2k+1)}+O(\lambda^{-1/2})\Bigg]$$

At strong coupling [M.Beccaria, A.A.Tseytlin and G.V. Dunne (2021)]

$$\mathrm{X} \underset{\lambda\to\infty}{\sim} -\frac{\lambda}{2\pi^2}\mathrm{S}$$

$$\mathrm{S}_{k\ell} = \frac{1}{4}(-1)^{k+\ell}\sqrt{\frac{2\ell+1}{2k+1}}\left(\frac{\delta_{k-1,\ell}}{k(2k-1)}+\frac{\delta_{k,\ell}}{k(k+1)}+\frac{\delta_{k+1,\ell}}{(k+1)(2k+3)}\right)$$

[Y. Ikebe, Y.Kikuchi and I.Fujishiro (1991)]

$$U_n \mapsto A_n = \frac{1}{\sqrt{M}} \left(\mathrm{tr} a_0^n + \mathrm{tr} a_1^n + \cdots + \mathrm{tr} a_{M-1}^n \right)$$

$$T_{\alpha,n} \mapsto A_{\alpha,n} = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I} \mathrm{tr} a_I^n$$

Interaction action

$$S_{int} = \sum_{I=0}^{M-1} \left[\sum_{m=2}^{\infty} \sum_{k=2}^{2m} \left(\frac{\lambda}{8\pi^2 N} \right)^m f_{m,k} (\mathrm{tr} a_I^{2m-k} - \mathrm{tr} a_{I+1}^{2m-k})(\mathrm{tr} a_I^k - \mathrm{tr} a_{I+1}^k) \right]$$

$$f_{m,k} = (-1)^{m+k} \binom{2m}{k} \frac{\zeta(2m-1)}{2m}$$

The X-matrix

[M. Billò, M. Frau, F. Galvagno, A. Lerda and AP (2021)]

$$\mathrm{X}_{k,\ell} = \mathrm{X}_{\ell,k}, \qquad \mathrm{X}_{2k+1,2\ell} = 0$$

$$\mathrm{X}^{odd}_{k,\ell} \equiv \mathrm{X}_{2k+1,2\ell+1}$$

$$-8(-1)^{k+\ell}\sqrt{(2k+1)(2\ell+1)}\int_0^\infty \frac{dt}{t}\frac{e^t}{e^t-1}J_{2k+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell+1}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathrm{X}^{even}_{k,\ell} \equiv \mathrm{X}_{2k,2\ell}$$

$$-8(-1)^{k+\ell}\sqrt{(2k)(2\ell)}\int_0^\infty \frac{dt}{t}\frac{e^t}{e^t-1}J_{2k}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)J_{2\ell}\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

The X-matrix

[M. Billò, M. Frau, F. Galvagno, A. Lerda and AP (2021)]

Gram-Schimdt $\Rightarrow$ Change of basis $A_{\alpha,n} \mapsto \mathcal{P}_{\alpha,n}$

$$\mathcal{P}_{\alpha,n} \equiv \sum_k c_{n,k} A_{\alpha,n-2k} \quad \Rightarrow \quad \langle \mathcal{P}_{\alpha,n} \mathcal{P}_{\beta,m} \rangle_0 = \delta_{n,m} \delta_{\alpha+\beta,0}$$

Explicit expression

Expansion with the X-matrix $\qquad s_\alpha \equiv \sin^2\left(\frac{\pi\alpha}{M}\right)$

$$\langle \mathcal{P}_{\alpha,2k} \mathcal{P}^\dagger_{\alpha,2\ell} \rangle = \left(\frac{1}{1 - s_\alpha \mathrm{X}^{\text{even}}} \right)_{k,\ell}$$

$$\langle \mathcal{P}_{\alpha,2k+1} \mathcal{P}^\dagger_{\alpha,2\ell+1} \rangle = \left(\frac{1}{1 - s_\alpha \mathrm{X}^{\text{odd}}} \right)_{k,\ell}$$

The correlator $\langle W_\alpha \, W_\beta \, W^\dagger_{\alpha+\beta} \rangle$

[AP and P. Vallarino (2023)]

$$\langle W_\alpha \, W_\beta \, W^\dagger_{\alpha+\beta} \rangle \simeq \frac{1}{\sqrt{M}\, N^4} \Bigg[\prod_{p=1}^{3} \Bigg(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} I_{2k}(\sqrt{\lambda}) \sqrt{(2k)\,(2\ell)}\, \mathsf{D}^{(\alpha_p)}_{2k,2\ell} \Bigg) +$$

$$\sum_{\sigma \in \mathcal{Q}_3} \Bigg[\Bigg(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} I_{2k}(\sqrt{\lambda}) \sqrt{(2k)\,(2\ell)}\, \mathsf{D}^{(\alpha_{\sigma(1)})}_{2k,2\ell} \Bigg) \times$$

$$\Bigg(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} I_{2k+1}(\sqrt{\lambda}) \sqrt{(2k+1)\,(2\ell+1)}\, \mathsf{D}^{(\alpha_{\sigma(2)})}_{2k+1,2\ell+1} \Bigg) \times$$

$$\Bigg(\sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} I_{2k+1}(\sqrt{\lambda}) \sqrt{(2k+1)\,(2\ell+1)}\, \mathsf{D}^{(\alpha_{\sigma(3)})}_{2k+1,2\ell+1} \Bigg) \Bigg] \Bigg],$$

where

$$\mathcal{Q}_3 = \{ (1,2,3),\, (3,1,2),\, (2,3,1) \} .$$