Supergravity techniques and the CFT bootstrap.

Introduction to consistent truncations

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1. Introduction

The idea is to give a pedestrian introduction to consistent truncations in supergravity focusing on AdS-CFT.

Ur-example :

$\mathcal{N}=4,\ d=4 \longleftrightarrow AdS_5 \times S_5$

chiral primary + desc. $\longleftrightarrow$ KK modes on S_5

$\cup \qquad\qquad \cup$

subset closed under OPE $\longleftrightarrow$ consistent truncation

• most famous case:

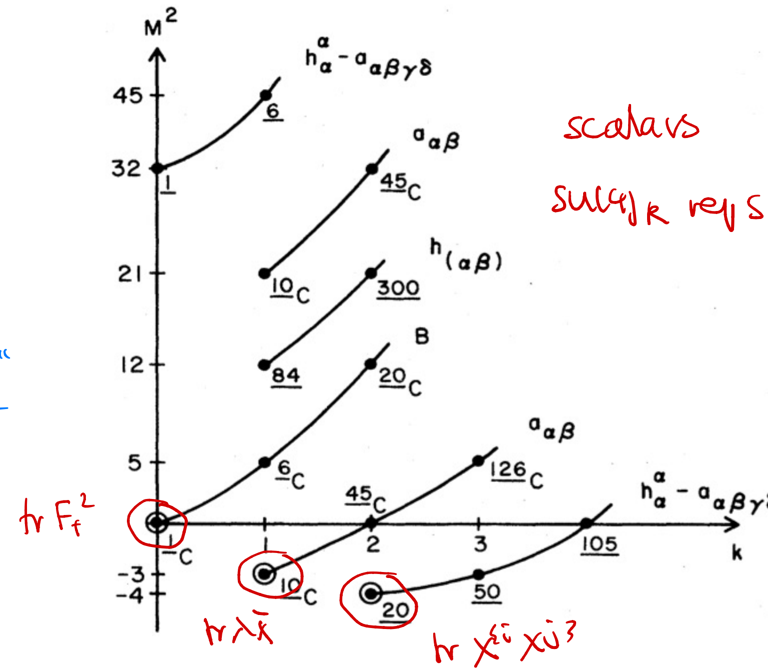
- N=4 stress tensor multiplet T

- TT ~ T "higher spin gap"

$\updownarrow$

- "mysterious" CT on S^5

- gives gauged $d=5$, $N=4$ sugra.



scalars

$SU(4)_R$ reps

expansion in spherical harmonics

NOT MASS HIERARCHY.

- CT is off-shell and allows finite deformations, so can study flows etc.

- And there are many more examples many w/ less supersymmetry

  unclear why OPE closes...

- Here we will focus on $CFT_4 \sim AdS_5 \times M$

  $M = S^5$ or Sasaki-Einstein

  $N=4$ $\qquad$ $N=1$

2. Building consistent truncations I

- How define? Off-shell theories:

solution of truncated theory is solution of full theory

example:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\lambda)^2 - \frac{1}{2}m^2\lambda^2 - \frac{1}{2}g\lambda\varphi^2$$

$$\partial^2\varphi = -g\lambda\varphi \,, \qquad (\partial^2+m^2)\lambda = -\frac{1}{2}g\varphi^2$$

$\lambda = 0$ $\;\swarrow$ $\qquad\qquad$ $\searrow\;$ $\varphi = 0$

inconsistent (λ sourced by φ)

ok if hierarchy $E \ll m$

consistent $\mathcal{L}' = \frac{1}{2}(\partial\lambda)^2 - \frac{1}{2}m^2\lambda^2$

no need for hierarchy.

- Why? Symmetry $= \mathbb{Z}_2$

$\varphi \to -\varphi \qquad \lambda \to \lambda$

$\rightsquigarrow$ no $\varphi\chi^2$ coupling.

$\rightsquigarrow$ truncate to singlets.

* singlets can't source non-singlets ~~s.s.~~ ns *

- What about in a gravity theory? Again try and use symmetry:

$$\underset{D\text{-dim}}{\mathcal{M}} = \underset{d\text{-dim}}{X} \times M \qquad \text{compactification}$$

compact Lie group $M = G \quad \dim M = n$
(or G/Γ)

expand everything in terms of left-invariant fields (singlets).

- eg internal metric: $X \times M \ni (y^\mu, x^m)$ coords:

$$ds^2(M) = m_{ij}(y)\,\big(e^i(x) + A^i(y)\big)\big(e^j(x) + A^j(y)\big)$$

scalar on X | L-invariant 1-form $e^i{}_m dx^m$ | vector on X $A^i{}_\mu dy^\mu$

Reducing Einstein-Hilbert: Scherk-Schwarz reduction.

$$S_D = \int_M \sqrt{-g}\,R \quad \rightsquigarrow$$

$$S_d = \int_X \sqrt{-g_X}\,\big(R_X + m^{-1} m^{-1}\, Dm \cdot Dm + \operatorname{tr} F^2 - V(m)\big)$$

"gauged supergravity form" gauge group G.

where:

- $m \in GL(n, \mathbb{R}) / O(n)$ coset space.

- gauging / potential controlled by structure constants $f_{ij}{}^k$

  $[\hat{e}_i, \hat{e}_j] = f_{ij}{}^k \, \hat{e}_k$ L-inv. vector fields. $\hat{e}_i{}^\mu e^j{}_\mu = \delta_i^j$

  "embedding tensor"

- can extend to other fields: eg:

3-form potential

$$
\begin{aligned}
A_3 = \; & a^{(0)}_{ijk}(y) \; e^i(x) \wedge e^j(x) \wedge e^k(x) \\
& + a^{(1)}_{ij}(y) \wedge e^i(x) \wedge e^j(x) \qquad \text{etc.} \\
& + a^{(2)}_i(y) \wedge e^i(x) \\
& + a^{(3)}(y)
\end{aligned}
$$

then:

$d=11$ sugra, type IIA/B

Lie group G s.s. reduction

$$S_d = \int \sqrt{-g_d}\, R_d + \dots$$

"gauged maximal supergravity"

- scalar fields $\sim$ coset $E_{11-d\,(11-d)} / H_{11-d}$ ← maximal compact

exceptional gr

- gauging + potential $\sim$ embedding tensor

($f_{ij}{}^k$ + constant fluxes

eg $F_4 = f_{ijkl}\, e^i \wedge e^j \wedge e^k \wedge e^l$)

3. Building consistent truncation II

- But S^5 is not a Lie group .... need to change perspective.

  G-structure $\doteq$ topological condition on M

  = TM actually only patched by $G \subset GL(n,\mathbb{R})$

  = $\exists$ global G-invariant tensors.

- for example:

  - $SL(n,\mathbb{R})$ structure = orientable M

    = $\exists$ global volume form $\varepsilon = dx^1 \wedge dx^2 \cdots \wedge dx^n$

* $O(d)$ structure $=$ $\exists$ global metric g_{mn} (no topological condition)

- Given a G-structure:

  tensors decompose into $G \subset GL(n,\mathbb{R})$ representations:

For example for $O(d)$

$$\text{symmetric tensor} \quad \alpha_{(mn)} = \underbrace{\mathring{\alpha}_{mn}}_{\text{traceless}} + \underbrace{\lambda g_{mn}}_{\text{trace}}$$

maybe we can use this for consistent truncation?? singlets??

• What structure does a Lie group define?

set of globally defined vectors $\{\hat{e}_i = \hat{e}_i{}^m \partial/\partial x^m\}$ $i = 1, \dots, n$

= "parallelization" = "identity structure" $G = \mathbb{1}$

$\hat{e}_i$ are invariant only under $\mathbb{1} \subset GL(n, \mathbb{R})$

furthermore: for Levi-Civita connection ∇ defined by

$g_{mn} = e_m{}^i e_n{}^j \delta_{ij}$ inv. metric

we have extra condition

$\nabla_m e^i_n = f_{jk}{}^i e^j_m e^k_n$ "constant singlet intrinsic torsion"

constants.

so

Lie group G = identity structure w/ constant singlet intrinsic torsion. (CSIT)

(eg: S^7 is parallelizable but not a group).

- then we can reinterpret:

  - Scherk-Schwarz = expand in G-structure singlets.

  - CSIT $\Leftrightarrow$ derivatives don't generate non-singlets.

therefore consistent by singlet argument.

- what about other cases? Sasaki-Einstein 5d in IIB

metric: $ds^2(M) = ds^2_4 + (d\psi + a)^2$ flux: $F = 4\,\mathrm{vol}_5$

with G-structure:

$$SU(2) \subset \frac{SU(2)\times SU(2)}{\mathbb{Z}_2} \simeq SO(4) \subset SO(5) \subset GL(5,\mathbb{R})$$

- Invariant tensors:

$\sigma = d\psi + a$ 1-form. $\qquad \underline{5} = \underline{1} \oplus \underline{2} \oplus \underline{2}$

ω real 2-form on ds^2_4 $\qquad \underline{10} = \underline{1} \oplus \underline{1} \oplus \underline{1} \oplus \underline{3} \oplus \underline{2} \oplus \underline{2}$

Ω complex 2-form on ds^2_4

together define metric $ds^2(M)$. $\quad 2\omega\wedge\omega = \Omega\wedge\bar{\Omega}$

- CSIT?

$$\nabla_m \sigma_n = \omega_{mn}$$

$$\nabla_m \omega_{np} = 0 \qquad\qquad \text{YES!!}$$

$$\nabla_m \Omega_{np} = 3i\, \sigma_m \Omega_{np}$$

- Consistent truncation of type IIB: expand in singlets

$$ds^2 = e^{2u} ds_4^2 + e^{2v}(\sigma + a)^2$$

$$B_2^i = p^i \omega + \xi^i \Omega + \bar{\xi}^i \bar{\Omega} + b^i \wedge \sigma + \alpha^i$$

$$C_4 = \lambda\, \omega\wedge\omega + c \wedge \omega \wedge \sigma + e \wedge \Omega \wedge \sigma + \bar{e} \wedge \bar{\Omega} \wedge \sigma$$

$$e^{2\varphi}, \qquad \chi .$$

scalars : e^{2u}, e^{2w}, ρ^i, ξ^i, $\bar{\xi}^i$, λ, $e^{2\varphi}$, χ 11

dual vector : a, b^i, e, $\bar{e}$ 6

dual 2-forms : α^i 2

$\left.\begin{matrix} 6 \\ 2 \end{matrix}\right\} 8$

- gravity : [Cassani, Dall'Agata, Faedo ; Gauntlett, Varela]

S_α = gauged $\frac{1}{2}$-maximal supergravity.

gravity multiplet + 2 vector multiplets:

$$\text{scalars} = \mathbb{R}^+ \times \frac{SO(5,2)}{SO(5)\times SO(2)} \qquad \text{coset.}$$

$$\text{gauge group} = Heis_3 \times U(1) \qquad \text{fixed by CSIT}$$

- Why $\frac{1}{2}$-maximal? $AdS_5 \times SE_5$ is $\frac{1}{4}$-maximal !!!

  theory is $\frac{1}{2}$-maximal, $AdS_5 \times SE_5$ vacuum breaks $\frac{1}{2}$ susy.

  this is not atypical — in fact CT might have no SUSY vacuum.

- quite an intricate truncation: what modes are we keeping?

  expand around $AdS_5 \times SE_5$ in $SU(2,2|1)$ multiplets

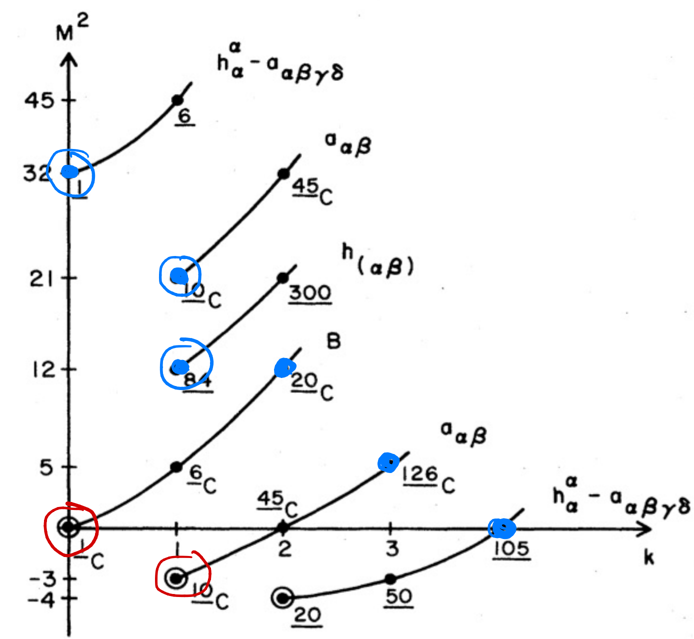
  - massless gravity multiplet. $\longleftrightarrow$ T + $U(1)_R$ current
  - hypermultiplet $\longleftrightarrow$?
  - long vector multiplet $\longleftrightarrow$?

• what are the other multiplets dual to? view S^5 as SE_5

$S^1 \to S^5 \to \mathbb{CP}^2$ Hopf fibration.

- massless graviton
  no scalars
- "universal" hypermultiplet
  for $W_\alpha W^\alpha$

part of $N=4$



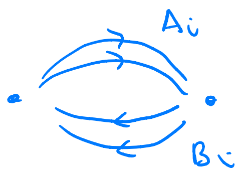
- long vector
  for $W_\alpha W^\alpha \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}$

part of $N=4$ breathing mode.

read off OPE from potential — $\bullet\bullet\bullet$, $\bullet\bullet\bullet$, $\bullet\bullet\bullet$ 3pt fns.

- this consistent truncation is shared by a v. large family of SCFT (quivers w/ CY cones) why long vector??

- can go further: special case with U(1) structure



$\xleftrightarrow{\text{dual}}$ $T^{1,1}$ (conifold)

gives $\frac{1}{2}$-maximal w/ 3 vectors: [Cassani-Faedo]

scalars $\sim \mathbb{R}^+ \times SO(5,3)/SO(5)\times SO(3)$

- gravity
- univ. hyper $\mathrm{tr}(\omega_1^2+\omega_2^2)$
- long vector $\mathrm{tr}(\omega_1^2\bar{\omega}_1^2+\omega_2^2\bar{\omega}_2^2)$
- new hyper $\mathrm{tr}(\omega_1^2-\omega_2^2)$
- new vector $\mathrm{tr}(A^i\bar{A}_i-\bar{B}_iB^i)$

4. Building consistent truncations III

- But this still doesn't give S^5! And where does structure (cosets etc) come from?

- Need new bit of geometry — generalised geometry.
  - reformulate supergravity unifying diffeomorphism and gauge symmetry.
  - repackages degrees of freedom into larger representations.

For type IIB on $\mathcal{M} = X \times M$

$GL(5,\mathbb{R})$ structure group on M $\mapsto$ $E_{6(6)}$ structure group.

can look for generalised G-structure $G \subset E_{6(6)}$

- Basic ingredient 1: generalised tangent space: 27-dimensional

$$E \simeq TM \oplus 2T^*M \oplus \Lambda^3 T^*M \oplus 2\Lambda^5 T^*M$$

diffeos

$\delta B^i = d\lambda^i$

$\delta C = d\rho$

$\delta \tilde{B}^i = d\tilde{\lambda}^i$

dual of B^i

rep. of $E_{6(6)} \times \mathbb{R}^+$

overall scaling.

$$V^M = (\xi, \lambda^i, \rho, \tilde{\lambda}^i)$$

combines symmetries

- Basic ingredient 2: generalized metric

$G_{MN} \in S^2 E^*$ encodes internal fields $g_{mn}, B^i_{mn}, C_{mnpq}, \varphi, \chi$

invariant under $USp(8) \subset E_{6(6)} \times \mathbb{R}^+$ max compact subgp.

- Basic ingredient 3: "exceptional field theory" mixed index fields also form $E_{6(6)}$ rep:

$$(g_\mu{}^m,\ B^i_{\mu m},\ C_{\mu mn},\ \tilde{B}^i_{\mu m_1\dots m_5}) \in T^*X \otimes E \qquad V_\mu{}^M$$

$$(B^i_{\mu\nu},\ C_{\mu\nu m},\ \tilde{B}^i_{\mu\nu m_1\dots m_4},\ \dots) \in \Lambda^2 T^*X \otimes N \qquad (N \sim E^*)\ B_{\mu\nu M}$$

- Can reformulate $d=10$ IIB with $GL(5,\mathbb{R}) \times E_{6(6)}$ structure group

X $\quad$ M

furthermore,

internal derivatives: $\nabla^{LC}_m \rightsquigarrow D_M$ "gen. LC"

analogue of Levi-Civita. (torsion-free, compatible)

- What is analogue of Scherk-Schwarz? $G = \mathbb{1} \subset E_{6(6)} \times \mathbb{R}^+$
  - generalised parallelisable = generalised identity structure
  - global frame $\{\hat{K}^A\} \in E$ $\quad A = 1, 2, \ldots, 27$

remarkably

S^5 is generalised parallelisable.

split gen. tangent space:

$$E \simeq \underbrace{(TM \oplus \Lambda^3 T^*M)}_{15\text{-dim}} \oplus \underbrace{2(T^*M \oplus \Lambda^5 T^*M)}_{2\times\, 6\text{-dim.}}$$

$\hat{K}^A \sim (v_{ij},\ * dy_i \wedge dy_j)$ $\qquad \hat{K}^A \sim (\ldots)$

$SO(6)$ isometries $\qquad y_i y^i = 1$

• CSIT ?

$$D_M K_N{}^A \sim T_{BC}{}^A K^B_M K^C_N$$

constants related to SO(6) structure constants. embedding tensor.

origin of mysterious S^5 consistent truncation

truncation is complicated eg scalars

$$G_{MN} = H_{AB}(y) K^A_M K^B_N$$

mixes g_{mn} + C_{mnpq} etc.

- This formalism gives systematic description of CT:

    - gen G-structure $G \subset USp(8) \subset E_{6(6)}$

    - scalars: coset $= \dfrac{C_{E_{6(6)}}(G)}{C_{USp(8)}(G)}$

    - vectors: singlets of E

    - embedding tensor: intrinsic torsion.

  may or may not preserve supersymmetry.

- Extends to other dimensions, $d=11$ sugra, heterotic etc.

5. Outlook

- We now have plenty of examples with different amounts of supersymmetry:

  - Sasaki-Einstein$_7$
  - wrapped M5-branes
  - β-deformed
  - massive IIA ...

  but typically based on known solution

  CSIT condition is hard since is differential

- Algebraic for $G = 1\!\!1$ [Inverso; Bugden et al] $\rightsquigarrow$ classification

  all compact CT w/ compact gauge groups (spheres...)

- Recent extensions to AdS_3 (dual CFT_2) and $d=1,2$ (matrix model)

- Are there ways of classifying w/ less susy?

- Higher derivatives: integrate out string modes & loops

$$S_{IIB} = \int (R + \dots) + \alpha'^3 (R^4 + \dots) + \dots$$

but, for conventional G-structure, singlet argument still holds since everything depends on ∇.

Also holds for generalized G-structure if can be written using D. Meaning? Beyond large N? Higher spin gap?

- What is CFT content of G-structure? Why these operators?

  - singlet fields arranged in $\mathcal{G} = C_{E_{6(6)}}(G)$ multiplets
  - not symmetry of spectrum (that is something group)

eg: $\mathcal{G} = E_{6(6)}$ for S^5, $\quad \mathcal{G} = \mathbb{R}^+ \times SO(5,2)$ for SE_5

global property of topology.

- topology of M : fixes set of operators? = Hilbert space?
- choice of metric G : fixes dynamics, spectrum