1. **Gravitational waves from pulsars:**

(a) *Power emitted in GWs:*

A set of coordinates $\mathbf{x}'$ rotating with the object is related to an inertial coordinate system $\mathbf{x}$ with common origin at the star's center of mass by a rotation matrix

$$x'_i = R_{ij}x^j, \qquad R_{ij} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $\phi = \Omega t$ and Ω is the constant rotation frequency. The components of the inertia tensor in the inertial coordinates are therefore obtained by the transformation

$$I_{ij} = R_{ik}I'_{kl}R_{jl}, \tag{2}$$

where $I' = \mathrm{diag}(I_1, I_2, I_3)$. Explicitly,

$$\begin{aligned} I_{xx} &= I_1(\cos\phi)^2 + I_2\sin(\phi)^2 = \frac{1}{2}(I_1 - I_2)\cos(2\phi) + \text{const}, && (3)\\ I_{yy} &= I_1(\sin\phi)^2 + I_2\cos(\phi)^2 = \frac{1}{2}(I_1 - I_2)\cos(2\phi) + \text{const}, && (4)\\ I_{xy} &= I_{yx} = (I_1 - I_2)\sin\phi\cos\phi = \frac{1}{2}(I_1 - I_2)\sin(2\phi) && (5)\\ I_{zz} &= \text{const}, \qquad I_{xz} = I_{yz} = 0. && (6) \end{aligned}$$

Since $\mathrm{Tr}I' = \mathrm{Tr}I = I_1 + I_2 + I_3 = \text{const}$ we can use (6) directly in place of the quadrupole moment in the quadrupole formula for the energy loss:

$$\begin{aligned} \frac{dE_{\mathrm{GW}}}{dt} &= -\frac{1}{5}\frac{G}{c^5}\langle \dddot{I}^2_{xx} + \dddot{I}^2_{yy} + 2\dddot{I}^2_{xy}\rangle && (7)\\ &= -\frac{1}{5}\frac{G}{c^5}\frac{1}{4}(2\Omega)^6(I_1 - I_2)^2\langle(\cos 2\phi)^2 + (\cos 2\phi)^2 + 2(\sin 2\phi)^2\rangle && (8)\\ &= -\frac{32}{5}\frac{G}{c^5}(I_1 - I_2)^2\Omega^6 && (9) \end{aligned}$$

Defining the ellipticity $\epsilon = (I_1 - I_2)/I_3$ we obtain

$$\frac{dE}{dt} = -\frac{32}{5}\epsilon^2 I_3^2\Omega^6. \tag{10}$$

(b) *Spindown due to GW emission*

We use the energy balance equation $\dot{E}_{\mathrm{rot}} = -\dot{E}_{\mathrm{GW}}$ with $E_{\mathrm{rot}} = I\Omega^2/2$ for a uniform sphere to obtain

$$\dot{\Omega} = \frac{32}{5}\epsilon^2 I\Omega^5 \tag{11}$$

Substituting the values for the Crab pulsar we find that

$$\frac{\dot{\Omega}}{\Omega} \approx 2\times 10^{-11}\frac{1}{s}. \tag{12}$$

Over an observation time of $\sim 3\mathrm{yr}\sim 10^8\mathrm{s}$ the change in the frequency due to GW losses is very small and the signal remains nearly monochromatic.

(c) *Upper limit on the ellipticity*

Solving Eq. (11) and $\dot{\Omega}/\Omega = -\dot{P}/P$ for ϵ, using $\Omega = 2\pi/(0.033s)$ and assuming that the pulsar has $M = 1.4M_\odot$, $R = 10$km we find that

$$\epsilon \lesssim 5.5 \times 10^{-7}. \tag{13}$$

In reality, the mass, radius, and moment of inertia of the Crab pulsar are uncertain and could differ from the fiducial values given above, which changes the upper limit on ϵ.

The braking index for GW emission is $n = 5$ which is much higher than the observed values for the Crab and Vela pulsars. Pulsars also spin-down due to electromagnetic emission through magnetic dipole radiation, for example, the Crab pulsar radiates a huge amount of power $\sim 10^5 L_\odot$ that is absorbed by and powers the Crab nebula. The small braking index of the Vela pulsar cannot be attributed entirely to radiation from a constant magnetic dipole but might be due to a changing, magnetic moment or effective moment of inertia.

2. **Tidal signature in the gravitational wave phasing:**

(a) *Orbital dynamics for adiabatic quadrupolar tides*

Inserting the expressions for $\mathcal{E}_{ij}$ and the adiabatic relation $Q_{ij} = -\lambda\mathcal{E}_{ij}$ into the Lagrangian yields an effective Lagrangian that involves only the orbital variables

$$L^{\rm eff} = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\mu M}{r} + \frac{3m_{\rm BH}\lambda}{2r^6}. \tag{14}$$

From the Euler-Lagrange equations we obtain

$$\ddot{r} = r\dot{\phi}^2 - \frac{M}{r^2} - \frac{9\lambda m_{\rm BH}^2}{\mu r^7}, \qquad\qquad r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0. \tag{15}$$

(b) *Radius-frequency relationship*

In the expression from (a) we set $\ddot{r} = 0$ and expand for $r = M^{1/3}\Omega^{-2/3}(1 + \delta r)$, with $\delta r \ll 1$. We solve this equation at each order in the tidal terms. At zeroth tidal order, the equation is already satisfied since we assumed Kepler's law as the leading order term in $r(\Omega)$. At linear order in the tidal effects we obtain

$$\delta r = \frac{3m_{\rm BH}^2\lambda\Omega^{10/3}}{\mu M^{7/3}}. \tag{16}$$

(c) *Binding energy*

The energy associated with the system is given by

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - \frac{\mu M}{r} - \frac{3m_{\rm BH}^2\lambda}{2r^6}. \tag{17}$$

Specializing to circular orbits and using the radius-frequency relationship from (b) we find that to linear order in the tidal terms

$$E(\Omega) = -\frac{1}{2}\mu(M\Omega)^{2/3}\left[1 - \frac{9m_{\rm BH}^2\lambda\Omega^{10/3}}{\mu M^{8/3}}\right]. \tag{18}$$

(d) *Energy loss*

From the quadrupole formula $\dot{E} = -\frac{1}{5}\langle \dddot{Q}_{ij}^{\mathrm{T}} \dddot{Q}_{ij}^{\mathrm{T}} \rangle$. Inserting the total quadrupole (orbit plus neutron star deformation), computing the time derivatives, and linearizing the results in the tidal effects gives

$$\dot{E} = -\frac{32}{5}\mu^2 r^4 \Omega^6 - \frac{192 m_{\mathrm{BH}} \lambda \mu \Omega^6}{5r}. \tag{19}$$

Substituting $r(\Omega)$ and truncating at linear tidal order gives

$$\dot{E} = -\frac{32}{5}\mu^2 M^{4/3}\left[1 + \frac{6 m_{\mathrm{bH}} \lambda \Omega^{10/3}}{\mu M^{5/3}}\left(1 + 2\frac{m_{\mathrm{BH}}}{M}\right)\right] \tag{20}$$

(e) *Phasing*

Using (c) and (d) in the formula for $d^2\Psi_{\mathrm{SPA}}/d\Omega^2$ gives

$$\frac{d^2\Psi_{\mathrm{SPA}}}{d\Omega^2} = \frac{5}{48 M^{2/3}\mu\Omega^{11/3}} - \frac{5 m_{\mathrm{BH}}\lambda}{8 M^{7/3}\mu^2\Omega^{1/3}}\left(1 + \frac{11 m_{\mathrm{BH}}}{M}\right). \tag{21}$$

Integrating twice with respect to Ω gives

$$\Psi_{\mathrm{SPA}} = \frac{3}{128 M^{2/3}\mu\Omega^{5/3}} - \frac{9 m_{\mathrm{BH}}\lambda\Omega^{5/3}}{16\mu^2 M^{7/3}}\left(1 + \frac{11 m_{\mathrm{BH}}}{M}\right). \tag{22}$$

Introducing $x = (M\Omega)^{2/3}$ leads to

$$\Psi_{\mathrm{SPA}} = \frac{3M}{128\mu x^{5/2}}\left[1 - \frac{24 m_{\mathrm{BH}}\lambda}{\mu M^5} x^5 \left(1 + \frac{11 m_{\mathrm{BH}}}{M}\right)\right]. \tag{23}$$

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- $\mathcal{L} = \mathcal{L}_0 + \frac{1}{c^2}\mathcal{L}_2$

$$\mathcal{L}_0 = \frac{m_1}{2}\vec{v}_1^{\,2} + \frac{m_2}{2}\vec{v}_2^{\,2} + \frac{G m_1 m_2}{r}$$

$$\mathcal{L}_2 = \frac{m_1(\vec{v}_1^{\,2})^2}{8} + \frac{m_2}{8}(\vec{v}_2^{\,2})^2 + \frac{G m_1 m_2}{2r}\left[3(\vec{v}_1^{\,2}+\vec{v}_2^{\,2}) - 7\,\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) - \frac{G(m_1+m_2)}{r}\right]$$

$$\vec{p}_1 = \frac{\partial\mathcal{L}}{\partial\vec{v}_1} = m_1\vec{v}_1 + \frac{1}{c^2}\left\{\frac{m_1}{2}(\vec{v}_1^{\,2})\vec{v}_1 + \frac{G m_1 m_2}{2r}\left[6\vec{v}_1 - 7\vec{v}_2 - \hat{r}(\vec{v}_2\cdot\hat{r})\right]\right\}$$

$$\vec{p}_2 = \frac{\partial\mathcal{L}}{\partial\vec{v}_2} = m_2\vec{v}_2 + \frac{1}{c^2}\left\{\frac{m_2}{2}(\vec{v}_2^{\,2})\vec{v}_2 + \frac{G m_1 m_2}{2r}\left[6\vec{v}_2 - 7\vec{v}_1 - \hat{r}(\vec{v}_1\cdot\hat{r})\right]\right\}$$

Let $\vec{r}_1 = \frac{1}{2}(\vec{R}-\vec{r})$, $\vec{r}_2 = \frac{1}{2}(\vec{R}+\vec{r})$, $\vec{p}_1 = \vec{P} - \vec{p}$, $\vec{p}_2 = \vec{P}+\vec{p}$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{p}_1+\vec{p}_2) \underset{\text{Euler-Lagrange eqs.}}{=} \frac{\partial\mathcal{L}}{\partial\vec{r}_1} + \frac{\partial\mathcal{L}}{\partial\vec{r}_2} = \frac{\partial R^i}{\partial\vec{r}_1}\frac{\partial\mathcal{L}}{\partial R^i} + \frac{\partial r^i}{\partial\vec{r}_1}\frac{\partial\mathcal{L}}{\partial r^i} + \frac{\partial R^i}{\partial\vec{r}_2}\frac{\partial\mathcal{L}}{\partial R^i} + \frac{\partial r^i}{\partial\vec{r}_2}\frac{\partial\mathcal{L}}{\partial r^i} =$$

$$= \frac{\partial\mathcal{L}}{\partial\vec{R}} - \cancel{\frac{\partial\mathcal{L}}{\partial\vec{r}}} + \frac{\partial\mathcal{L}}{\partial\vec{R}} + \cancel{\frac{\partial\mathcal{L}}{\partial\vec{r}}} = 2\frac{\partial\mathcal{L}}{\partial\vec{R}}$$

Note that $\mathcal{L} = \mathcal{L}(r, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$, then it does NOT depend on $\vec{R}$:

$$\frac{\partial\mathcal{L}}{\partial\vec{R}} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} \text{ is conserved}$$

- $H = H_0 + \frac{1}{c^2}H_2 = \vec{p}_1\cdot\dot{\vec{r}}_1 + \vec{p}_2\cdot\dot{\vec{r}}_2 - \mathcal{L}$

At 0PN order we have $\vec{p}_1 = m_1\vec{v}_1 + \mathcal{O}(c^{-2})$

$$\vec{p}_2 = m_2\vec{v}_2 + \mathcal{O}(c^{-2})$$

then

$$H_0 = \vec{p}_1\cdot\frac{\vec{p}_1}{m_1} + \vec{p}_2\cdot\frac{\vec{p}_2}{m_2} - \mathcal{L}_0 = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{G m_1 m_2}{r}$$

At 1PN order I can replace $\vec{v}_{1,2}$ with $\vec{p}_{1,2}/m_{1,2}$ inside $\mathcal{L}_2$:

$$\mathcal{L}_2 = \frac{1}{8}\frac{(\vec{p}_1^{\,2})^2}{m_1^3} + \frac{1}{8}\frac{(\vec{p}_2^{\,2})^2}{m_2^3} + \frac{G m_1 m_2}{2r}\left[3\left(\frac{\vec{p}_1^{\,2}}{m_1^2} + \frac{\vec{p}_2^{\,2}}{m_2^2}\right) - \frac{7\,\vec{p}_1\cdot\vec{p}_2}{m_1 m_2} - \frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{m_1 m_2} - \frac{G(m_1+m_2)}{r}\right]$$

$H_2 = -\mathcal{L}_2$ at 1PN

Move to a frame where $\vec{P} = 0 \Rightarrow \vec{p}_1 = -\vec{p}_2 = -\vec{p}$. Use μ and M:

$$H_0 = \frac{\vec{p}^{\,2}}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{G\mu M}{r} = \frac{\vec{p}^{\,2}}{2}\frac{m_1+m_2}{m_1 m_2} - \frac{G\mu M}{r} = \boxed{\frac{\vec{p}^{\,2}}{2\mu} - \frac{G\mu M}{r}}$$

$$H_2 = -\frac{1}{8}(\vec{p}^2)^2\underbrace{\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)} - \frac{G\mu M}{2r}\left[\vec{p}^2\underbrace{\left(\frac{3}{m_1^2}+\frac{3}{m_2^2}+\frac{7}{m_1 m_2}\right)} + \frac{(\vec{p}\cdot\hat{r})^2}{\underbrace{m_1 m_2}_{\mu M}} - \frac{GM}{r}\right]$$

$$= \frac{m_1^3+m_2^3}{(\mu M)^3} = \frac{M}{(\mu M)^3}(m_1^2+m_2^2-m_1m_2) = \frac{1}{\mu^3 M^2}(M^2-3\mu M) = \frac{1}{\mu^3}\left(1-3\frac{\mu}{M}\right) = \frac{1}{\mu^3}(1-3\nu)$$

$$= \frac{3(m_1^2+m_2^2)}{(\mu M)^2} + \frac{7}{\mu M} = \frac{3(M^2-2\mu M)+7\mu M}{(\mu M)^2} = \frac{1}{\mu^2}\left[3\left(1-2\frac{\mu}{M}\right)+\frac{7\mu}{M}\right] = \frac{1}{\mu^2}(3+\nu)$$

Then

$$\boxed{H_2 = \frac{1}{8\mu^3}(3\nu-1)(\vec{p}^2)^2 - \frac{G\mu M}{2r}\left[(3+\nu)\left(\frac{\vec{p}}{\mu}\right)^2 + \frac{(\vec{p}\cdot\hat{r})^2}{\mu M} - \frac{GM}{r}\right]}$$

We can get rid of μ completely: $\mu = \nu M$.

- For a <u>circular orbit</u>:

$\dot{r} = 0$, $\dot{\vec{p}}\cdot\hat{r} = \frac{d}{dt}(\vec{p}\cdot\hat{r}) = 0 \Rightarrow \vec{p}_{circ} = p_\varphi^{circ}/r\,\hat{\varphi}$ (where I introduced polar coords $\{r,\varphi\}$ in the orbital plane)

Note that the angular momentum is

$\vec{L} = \vec{r}\times\vec{p} = (r\hat{r})\times\left(\frac{p_\varphi}{r}\hat{\varphi}\right) = p_\varphi(\hat{r}\times\hat{\varphi}) = p_\varphi\hat{L}$ where $\hat{L}$ is $\perp$ orbital plane

Then $\vec{p}^2_{circ} = L^2_{circ}/r^2$. In general $\vec{p}^2 = p_r^2 + L^2/r^2$, where $p_r \equiv \vec{p}\cdot\hat{r}$.

$$E \equiv H|_{circ} = \frac{L^2_{circ}}{2\mu r^2} - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{(3\nu-1)}{8\mu^3}\frac{L^4_{circ}}{r^4} - \frac{G\mu M}{2r}\left[\frac{(3+\nu)}{\mu^2}\frac{L^2_{circ}}{r^2} - \frac{GM}{r}\right]\right.$$

Note that: $\dot{r} = \frac{\partial H}{\partial p_r}$ is automatically 0 if $p_r = 0$. Therefore the only eqn. left to exploit is: $\dot{p}_r = -\frac{\partial H}{\partial r} \stackrel{!}{=} 0$.

$$\left.\frac{\partial H_0}{\partial r}\right|_{p_r=0} = \frac{G\mu M}{r^2} - \frac{1}{2\mu}\left(\frac{+2L^2}{r^3}\right)$$

(note that this gives us the correct Newtonian result for L^2_{circ}, that is $G\mu^2 M r$, so that $E = -\frac{1}{2}\frac{G\mu M}{r}$)

$$\left.\frac{\partial H_2}{\partial r}\right|_{p_r=0} = \frac{3\nu-1}{8\mu^3}\,2\left(\frac{L^2}{r^2}\right)\left(\frac{-2L^2}{r^3}\right) + \frac{G\mu M}{2r^2}\left[\frac{3+\nu}{\mu^2}\frac{L^2}{r^2} - \frac{GM}{r}\right] - \frac{G\mu M}{2r}\left[\frac{3+\nu}{\mu^2}\left(\frac{-2L^2}{r^3}\right) + \frac{GM}{r^2}\right]$$

$$= -\frac{3\nu-1}{2\mu^3}\frac{L^4}{r^5} + \frac{G\mu M}{2r^3}\left[\frac{3+\nu}{\mu^2}\frac{L^2}{r} - GM + \frac{3+\nu}{\mu^2}\frac{2L^2}{r} - GM\right]$$

$$= -\frac{3\nu-1}{2\mu^3}\frac{L^4}{r^5} + \frac{G\mu M}{2r^3}\left[\frac{9+3\nu}{\mu^2}\frac{L^2}{r} - 2GM\right]$$

$L_{circ} = L_0 + \frac{1}{c^2} L_2$ where $L_0 = \mu\sqrt{GMr}$

$\left(\frac{\partial H_0}{\partial r} + \frac{1}{c^2}\frac{\partial H_2}{\partial r}\right)_{p_r=0} = 0$ at 1PN I can replace $L \to L_0$ in $\frac{\partial H_2}{\partial r}$

and get:

$$\frac{\partial H_2}{\partial r}\Big|_{p_r=0} = -\frac{3\nu-1}{2\cancel{\mu^3}}\,\frac{\cancel{\mu^4}\,G^2M^2\cancel{r^2}}{r^{\cancel{5}\,3}} + \frac{G\mu M}{2r^3}\left[\frac{9+3\nu}{\cancel{\mu^2}}\,\frac{\cancel{\mu^2}\,GM\cancel{r}}{\cancel{r}} - 2GM\right] =$$

$$= \frac{4\mu G^2M^2}{r^3}$$

Then

$$\frac{G\mu M}{r^2} - \frac{1}{\mu r^3}\left(L_0 + \frac{1}{c^2}L_2\right)^2 + \frac{1}{c^2}\,\frac{4\mu G^2M^2}{r^3} =$$

$$\simeq \underbrace{\left(\frac{G\mu M}{r^2} - \frac{L_0^2}{\mu r^3}\right)}_{0} + \frac{1}{c^2}\left(-\frac{2L_0L_2}{\mu r^3} + \frac{4\mu G^2M^2}{r^3}\right) \stackrel{!}{=} 0$$

$\Rightarrow L_2 = \frac{2G^2M^2\mu}{\sqrt{GMr}}$, so at 1PN we have:

$$\Rightarrow L_{circ} = \mu\sqrt{GMr} + \frac{1}{c^2}\,\frac{2G^2M^2\mu}{\sqrt{GMr}} =$$

$$= \mu\sqrt{GMr}\left[1 + \frac{1}{c^2}\,\frac{2G^2M^2}{GMr}\right] = \boxed{\mu\sqrt{GMr}\left[1 + \frac{1}{c^2}\,\frac{2GM}{r}\right]}$$

$\omega = \frac{\partial H}{\partial L}\Big|_{L=L_{circ},\ p_r=0}$ is the orb. freq. for circ. orbits

$$\frac{\partial H_0}{\partial L}\Big|_{L=L_{circ},\ p_r=0} = \frac{L_0 + \frac{1}{c^2}L_2}{\mu r^2}$$

$\frac{\partial H_2}{\partial L}\Big|_{L=L_0,\ p_r=0} = -\frac{GM(3+\nu)L_0}{r^3\mu} + \frac{(3\nu-1)L_0^3}{2r^4\mu^3}$ where I neglected the L_2 piece at 1PN

Substituting the explicit expression for L_0, L_2 and simplifying:

$$\omega = \frac{\sqrt{GMr}}{2r^3}\left[2r + \frac{1}{c^2}GM(\nu-3)\right] \Rightarrow \boxed{\omega = \frac{\sqrt{GM}}{r^{3/2}}\left[1 + \frac{1}{c^2}\,\frac{GM(\nu-3)}{2r}\right]}$$

Squaring it

$$\omega^2 = \frac{GM}{r^3}\left[1 + \frac{1}{c^2}\frac{GM}{r}(\nu-3)\right]$$

$$v^2 \equiv (GM\omega)^{2/3} = \frac{(GM)^{2/3}(GM)^{1/3}}{r}\left[1+\frac{1}{c^2}\frac{GM}{r}\left(\frac{\nu}{3}-1\right)\right]$$

$$= \frac{GM}{r}\left[1+\frac{1}{c^2}\frac{GM}{r}\left(\frac{\nu}{3}-1\right)\right]$$

$$r = r_0 + \frac{1}{c^2} r_2 \quad \text{where} \quad r_0 = \frac{GM}{v^2}$$

$$\frac{GM}{r_0}\left(1-\frac{1}{c^2}\frac{r_2}{r_0}\right)\left[1+\frac{1}{c^2}\frac{GM}{r_0}\left(\frac{\nu}{3}-1\right)\right] = v^2$$

$$\underbrace{\frac{GM}{r_0}}_{v^2} + \frac{1}{c^2}\left[\frac{GM}{r_0}\left(\frac{\nu}{3}-1\right) - \frac{r_2}{r_0}\right] = \cancel{v^2} \Rightarrow \frac{GM}{r_0}\left(\frac{\nu}{3}-1\right) - \frac{r_2}{r_0} = 0$$

$$\Rightarrow r_2 = GM\left(\frac{\nu}{3}-1\right)$$

$$\Rightarrow \boxed{r = \frac{GM}{v^2}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{3}-1\right)\right]}$$

$$L_{circ} = \mu\sqrt{GM}\,\frac{\sqrt{GM}}{v}\left[1+\left(\frac{v}{c}\right)^2\frac{1}{2}\left(\frac{\nu}{3}-1\right)\right]\left[1+\frac{1}{c^2}\frac{2\cancel{GM}}{\cancel{GM}}v^2\right]$$

$$= \frac{\mu MG}{v}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{6}-\frac{1}{2}+2\right)\right] =$$

$$= \boxed{\frac{\mu MG}{v}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{6}+\frac{3}{2}\right)\right]}$$

$$E_{circ} = \frac{L_0^2 + 2L_0L_2/c^2}{2\mu r^2} - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{(3\nu-1)}{8\mu^3}\frac{L_0^4}{r^4} - \frac{G\mu M}{2r}\left[\frac{(3+\nu)}{\mu^2}\frac{L_0^2}{r^2} - \frac{GM}{r}\right]\right.$$

= substitute L_0, L_2 and simplify

$$= -\frac{G\mu M}{8r^2}\left[+4r + \frac{GM}{c^2}(\nu-7)\right] = -\frac{G\mu M}{2r}\left[1+\frac{1}{c^2}\frac{GM}{4r}(\nu-7)\right] =$$

$$= -\frac{G\mu M}{2r_0}\left[1-\frac{1}{c^2}\frac{r_2}{r_0}\right]\left[1+\frac{1}{c^2}\frac{GM}{4r_0}(\nu-7)\right] =$$

$$= \boxed{-\frac{\mu v^2}{2}\left[1-\left(\frac{v}{c}\right)^2\frac{1}{12}(\nu+9)\right]}$$

Note: $x \equiv \frac{v^2}{c^2} = \frac{(GM\omega)^{2/3}}{c^2}$ is dimensionless

Also: $\frac{GM}{c^2 r}$ is dimensionless

• Start from H (which is already in the frame where $\vec{P}=0$)

$$\mathcal{L}_0 = \frac{1}{2}\mu\vec{v}^2 + \frac{G\mu M}{r} \quad \text{(just Newtonian)}$$

$$\mathcal{L}_2 = (-H_2)\Big|_{\vec{p}=\mu\vec{v}} = \frac{1}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\left[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 - \frac{GM}{r}\right]\Big\}$$

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\vec{v}} = \mu\vec{v} + \frac{1}{c^2}\left\{\frac{1}{2}\mu(1-3\nu)\vec{v}^3 + \frac{G\mu M}{r}\left[(3+\nu)\vec{v} + \nu\hat{r}(\hat{r}\cdot\vec{v})\right]\right\}$$

$$\frac{\partial\mathcal{L}}{\partial t} = 0 \Rightarrow E = \text{const.}$$

$$E = \vec{p}\cdot\vec{v} - \mathcal{L} = \mu\vec{v}^2 + \frac{1}{c^2}\left\{\underline{\underline{\frac{1}{2}\mu(1-3\nu)\vec{v}^4}} + \frac{G\mu M}{r}\left[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2\right]\right\} - \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{-\underline{\underline{\frac{1}{8}(1-3\nu)\mu\vec{v}^4}} - \frac{G\mu M}{2r}\left[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 - \frac{GM}{r}\right]\right\} =$$

$$= \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{3}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\left[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 + \frac{GM}{r}\right]\right\}$$ energy for generic orbit ✓

$$\frac{\partial\mathcal{L}}{\partial\varphi} = 0 \Rightarrow L = \text{const.} = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$$

Use $\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} \Rightarrow \vec{v}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$, then

$$L = \mu r^2\dot{\varphi} + \frac{1}{c^2}\left\{\frac{1}{2}\mu(1-3\nu)\vec{v}^2 r^2\dot{\varphi} + G\mu M(3+\nu)r\dot{\varphi}\right\}$$

Find $\dot{\varphi}$ to substitute into E. We want to invert $L = L(r, \dot{r}, \dot{\varphi})$

At 0PN

$$L = \mu r^2\dot{\varphi}_0 \Rightarrow \dot{\varphi}_0 = \frac{L}{\mu r^2}$$

At 1PN

$$L = \mu r^2\dot{\varphi}_0 + \frac{1}{c^2}\left\{\mu r^2\dot{\varphi}_2 + \frac{1}{2}\mu(1-3\nu)\vec{v}_0^2 r^2\dot{\varphi}_0 + G\mu M(3+\nu)r\dot{\varphi}_0\right\}$$

$$\Rightarrow \dot{\varphi}_2 = -\frac{1}{\mu r^2}\left[\frac{1}{2}\mu(1-3\nu)\vec{v}_0^2 r^2\dot{\varphi}_0 + G\mu M(3+\nu)r\dot{\varphi}_0\right]$$

where $\vec{v}_0^2 = \dot{r}^2 + r^2\dot{\varphi}_0^2$

Replace $\dot{\varphi} = \dot{\varphi}_0 + \frac{1}{c^2}\dot{\varphi}_2$ into E:

$$E = \frac{1}{2}\mu\left[\dot{r}^2 + r^2\left(\dot{\varphi}_0 + \frac{1}{c^2}\dot{\varphi}_2\right)^2\right] - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{3}{8}\mu(1-3\nu)(\dot{r}^2 + r^2\dot{\varphi}_0^2)^2 + \right.$$

$$\left. + \frac{G\mu M}{2r}\left[(3+\nu)(\dot{r}^2 + r^2\dot{\varphi}_0^2) + \nu\dot{r}^2 + \frac{GM}{r}\right]\right\} =$$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}_0^2) + \frac{1}{c^2}\left\{\mu r^2\dot{\varphi}_0\dot{\varphi}_2 + (\text{all the 1PN terms above})\right\} \Rightarrow$$

$$\frac{E}{\mu} = \dot{r}^4\,\frac{1}{c^2}\left(\frac{3}{8} - \frac{9}{8}\nu\right) + \dot{r}^2\left[\overbrace{\frac{1}{2}}^{\text{Newtonian}} + \frac{1}{c^2}\left(\frac{3GM}{2r} + \frac{L^2}{4M^2r^2\nu^2} - \frac{3L^2}{4M^2r^2\nu} + \frac{G\nu M}{r}\right)\right]$$

$$+\left[\underbrace{-\frac{GM}{r} + \frac{L^2}{2M^2r^2\nu^2}}_{\text{Newtonian}} + \frac{1}{c^2}\left(\frac{G^2M^2}{2r^2} - \frac{GL^2}{2Mr^3\nu^2}(3+\nu) + \frac{L^4}{8M^4r^4\nu^4}(3\nu-1)\right)\right]$$

Neglect the term $\dot{r}^4$. Solve for $\dot{r}^2$

$$\dot{r}^2 = 2\left(\frac{E}{\mu} + \frac{GM}{r} - \frac{L^2}{2M^2r^2\nu^2}\right) + \frac{1}{c^2}\left\{-\frac{2GM}{r}\frac{E}{\mu}(3+2\nu) + \frac{GL^2}{M\nu^2r^3}(5+6\nu)\right.$$

$$\left. + \frac{3L^4}{4M^4\nu^4r^4}(1-3\nu) + \frac{1}{r^2}\left[\frac{L^2}{M^2\nu^2}\frac{E}{\mu}(3\nu-1) - G^2M^2(7+4\nu)\right]\right\} \qquad (**)$$

Note that we have the following structure:

$$\frac{1}{2}\dot{r}^2 + V(r) = (\text{constants})$$

Therefore radial perturbation will have a freq

$$\Omega_r^2 = \left.\frac{\partial^2 V}{\partial r^2}\right|_{circ}$$

I use $-\frac{1}{2}$ of the RHS of eq. (**) as my radial potential.
I don't have to worry that I'm including constants into V, as they are killed by ∂_r^2

I also replace E, L, r with the circ. orbit values at 1PN:

$$\frac{E}{\mu} \to -\frac{v^2}{2}\left[1 - \left(\frac{v}{c}\right)^2\left(\frac{\nu}{12} + \frac{9}{12}\right)\right], \quad r \to \frac{GM}{v^2}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{3} - 1\right)\right]$$

$$L \to \frac{\nu M^2 G}{v}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{6} + \frac{3}{2}\right)\right]$$

Then at 1PN I get

$$\Omega_r^2 = \frac{v^6}{G^2M^2}\left[1 - 6\frac{v^2}{c^2}\right]$$

$$GM\Omega = v^3 \Rightarrow \Omega_r^2 = \Omega^2\left(1 - 6\frac{v^2}{c^2}\right) \Rightarrow \left(\frac{\Omega_r}{\Omega}\right)^2 = 1 - 6\frac{v^2}{c^2}$$

$$\Rightarrow K = \frac{\Omega}{\Omega_r} = 1 + 3\frac{v^2}{c^2}$$

• In polar coords.

$$H_0 = \frac{1}{2\mu}\left(p_R^2 + \frac{p_\varphi^2}{R^2}\right) - \frac{G\mu M}{R}$$

$$H_2 = \frac{1}{8\mu^3}(3\nu-1)\left(p_R^2 + p_\varphi^2/R^2\right) - \frac{G\mu M}{2R}\left[\frac{3+\nu}{\mu^2}\left(p_R^2 + \frac{p_\varphi^2}{R^2}\right) + \frac{p_R^2}{\mu M} - \frac{GM}{R}\right]$$

Let's use for simplicity $G=1$ and $\mu=1$ (see 5.8 in gr-qc/0209089)

$$\frac{H}{\mu} = \frac{\vec{P}^2}{2} - \frac{M}{R} + \frac{1}{c^2}\left\{\frac{3\nu-1}{8}(\vec{P}^2)^2 - \frac{M}{R}\left[\frac{3+\nu}{2}\vec{P}^2 + \frac{\nu P_R^2}{2}\right] + \frac{M^2}{2R^2}\right\}$$

with $\vec{P}^2 = P_R^2 + \frac{P_\varphi^2}{R^2}$, in polar form. Let $H \equiv H^{ADM}/\mu$:

$$\dot{R} = \frac{\partial H}{\partial P_R}, \quad \dot{P}_R = -\frac{\partial H}{\partial R} \qquad \text{note that } H = H(R, P_R, P_\varphi)$$

$$\Omega = \frac{\partial H}{\partial P_\varphi}, \quad \dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

For a circular orbit $R=R_0$, $P_R = 0$, $\underbrace{\left.\frac{\partial H}{\partial R}\right|_{R=R_0, P_R=0}}_{(\#)} = 0 \Leftrightarrow \dot{P}_R = 0$

Let
$$R = R_0 + \delta R$$
$$P_R = \delta P_R$$
$$\Omega = \Omega_0 + \delta\Omega$$
$$P_\varphi = P_{\varphi 0} + \delta P_\varphi$$

At linear order:

$$\delta\dot{R} = \frac{\partial H}{\partial P_R}(R_0+\delta R, \delta P_R, P_{\varphi 0}+\delta P_\varphi) \overset{\text{Taylor}}{=} \overbrace{\frac{\partial H}{\partial P_R}(R_0, 0, P_{\varphi 0})}^{\dot{R}|_{circ}=0} + \overbrace{\frac{\partial^2 H}{\partial P_R \partial R}(R_0, 0, P_{\varphi 0})\,\delta R}^{0 \text{ by } (\#)} +$$

$$+ \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi 0})\,\delta P_R + \underbrace{\frac{\partial^2 H}{\partial P_R \partial P_\varphi}(R_0, 0, P_{\varphi 0})}_{\frac{\partial}{\partial P_\varphi}\left(\frac{\partial H}{\partial P_R}\right)_{R=R_0, P_R=0, P_\varphi=P_{\varphi 0}}}\delta P_\varphi$$

$$= C_0\,\delta P_R$$

↳ this is linear in P_R, so it vanishes for $P_R = 0$

where

$$C_0 \equiv \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi 0}) = 1 + \frac{1}{c^2}\left\{\frac{3\nu-1}{2}\left(\frac{P_{\varphi 0}}{R_0}\right)^2 - (3+2\nu)\frac{M}{R_0}\right\}$$

$\delta\dot{P}_\varphi = 0$ since $\dot{P}_\varphi = 0$

$$\delta\dot{P}_R = -\frac{\partial H}{\partial R}(R_0+\delta R, \delta P_R, P_{\varphi 0}+\delta P_\varphi) \overset{\text{Taylor}}{=} -\cancel{\frac{\partial H}{\partial R}(R_0, 0, P_{\varphi 0})} - \frac{\partial^2 H}{\partial R^2}(R_0, 0, P_{\varphi 0})\,\delta R$$

0 by virtue of (#)

$$- \underbrace{\cancel{\frac{\partial^2 H}{\partial R \partial P_R}(R_0, 0, P_{\varphi 0}) \delta P_R}} - \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta P_\varphi =$$

again, we have a function linear in P_R which vanishes when $P_R = 0$

$$= - A_0 \, \delta R - B_0 \, \delta P_\varphi$$

where

$$A_0 \equiv + \frac{\partial^2 H}{\partial R^2}(R_0, 0, P_{\varphi 0}) = \frac{3 P_{\varphi 0}^2}{R_0^4} - \frac{2M}{R_0^3} + \frac{1}{c^2}\left\{ \frac{3M^2}{R_0^4} - 6M(3+\nu)\frac{P_{\varphi 0}^2}{R_0^5} + \frac{5}{2}(3\nu - 1)\frac{P_{\varphi 0}^4}{R_0^6} \right\}$$

$$B_0 \equiv + \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0}) = - 2 \frac{P_{\varphi 0}}{R_0^3} + \frac{1}{c^2}\left\{ 3M(3+\nu)\frac{P_{\varphi 0}}{R_0^4} - 2(3\nu - 1)\frac{P_{\varphi 0}^3}{R_0^5} \right\}$$

$$\Omega_0 + \delta\Omega = \frac{\partial H}{\partial P_\varphi}(R_0 + \delta R, \delta P_R, P_{\varphi 0} + \delta P_\varphi) = \underbrace{\frac{\partial H}{\partial P_\varphi}(R_0, 0, P_{\varphi 0})}_{\Omega_0} + \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta R +$$

$$+ \underbrace{\cancel{\frac{\partial^2 H}{\partial P_R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta P_R}} + \frac{\partial^2 H}{\partial P_\varphi^2}(R_0, 0, P_{\varphi 0})\, \delta P_\varphi$$

again $\frac{\partial H}{\partial P_R}$ is linear in P_R

$$\Rightarrow \delta\Omega = B_0 \, \delta R + D_0 \, \delta P_\varphi$$

where

$B_0 =$ (same as above)

$$D_0 \equiv \frac{\partial^2 H}{\partial P_\varphi^2}(R_0, 0, P_{\varphi 0}) = \frac{1}{R_0^2} + \frac{1}{c^2}\left\{ - \frac{M}{R_0^3}(3+\nu) + \frac{3}{2}(3\nu - 1)\frac{P_{\varphi 0}^2}{R_0^4} \right\}$$

Look for solution $\sim e^{i\sigma t}$

$$\delta P_R = \frac{\delta \dot{R}}{C_0} \Rightarrow \delta \dot{P}_R = \frac{\delta \ddot{R}}{C_0} = - A_0 \, \delta R - B_0 \, \delta P_\varphi$$

$\delta \dot{P}_\varphi = 0 \Rightarrow \delta P_\varphi = \text{const.}$, without loss of generality put $\delta P_\varphi = 0$

Then

$$\delta \ddot{R} = - A_0 C_0 \, \delta R \qquad \text{use } \delta R \sim e^{i\sigma t}$$

$$-\sigma^2 = - A_0 C_0 \Rightarrow \sigma = \pm\sqrt{+A_0 C_0}$$

In order to have stability, the perturbation δR must be oscillatory and not divergent as $t \to \infty$, so σ must be real, thus the condition is:

$$\boxed{A_0 C_0 > 0}$$

Compute $A_0 C_0$ and replace $R_0 \rightsquigarrow r_{circ} = \frac{M}{v^2}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{3} - 1\right)\right]$

$$P_{\varphi 0} \rightsquigarrow l_{circ} = \frac{M}{v}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{6} + \frac{3}{2}\right)\right]$$

and keep only up to 1PN:

$$A_0 C_0 = \frac{v^6}{M^2}\left[1 - 6\frac{v^2}{c^2}\right] \underset{\substack{\uparrow \\ M\Omega = v^3}}{=} \boxed{\Omega^2\left(1 - 6\,v^2/c^2\right)}$$

$A_0 C_0 = 0 \iff \frac{v^2}{c^2} = \frac{1}{6}$ that is $R = 6M$ in geom. units, the Schw. ISCO

From previous results on periastron

$$\Omega_r^2 = \Omega^2\left(1 - 6\,v^2/c^2\right) = A_0 C_0$$

So at the ISCO also $\Omega_r = 0$. This means that a perturbation of the ISCO has ∞ radial period, i.e. it's a divergent perturbation, and means that we plunge.

The ISCO is the last stable circ. orbit.