HW#5 —Phys879—Spring 2017
Due before class, Monday, May 1, 2017
Course webpage: http://www.aei.mpg.de/2000472

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Recommended readings:

1. A. Buonanno and T. Damour, Phys. Rev. **D59** (1999) 084006.
2. A. Buonanno and T. Damour, Phys.Rev. **D62** (2000) 064015.

Exercises (prepared by A. Buonanno, T. Hinderer, J. Steinhoff & J. Vines):

1. **Black-hole quasi-normal modes** [10 points]

   Use Table I from arXiv:gr-qc/0411025 to make plots of ω_R and ω_I versus n, where these are the real and imaginary parts of the complex frequencies $\omega = \omega_R + i\omega_I$ of the quadrupolar ($l = 2$) quasinormal modes (QNMs) of a Schwarzschild black hole, and n is the overtone number, i.e., the number of nodes in the radial wavefunction (plus 1 in that reference's conventions). [Note that the values are given in Table I as $(\omega_R, -\omega_I)$ in our conventions, with the time-dependence of the QNMs given by $e^{i\omega t}$.] Use n = 1–12, 20, 30, 40.

   Your plot should exhibit some features which could be considered strange according to certain intuition, interpreting ω_R as an oscillation frequency and ω_I as a decay rate. For typical systems with a set of vibrational modes, like a string or an elastic body, both the oscillation frequency and the decay rate increase with increasing overtone number, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that ω_R is first decreasing with n, then has a zero, and then increases to an asymptotically constant value.

   This behavior can be seen as somewhat less mysterious by reinterpreting ω_R and ω_I as follows.

   Consider a simple damped oscillator with amplitude $\psi(t)$ obeying

$$\ddot{\psi} + \gamma_0 \dot{\psi} + \omega_0^2 \psi = 0. \tag{1}$$

   Writing the two linearly independent solutions as $\exp((\pm i\omega_R - \omega_I)t)$, find the relationship between ω_R, ω_I and ω_0, γ_0. Then make plots of ω_0 and γ_0 versus n for the Schwarzschild QNMs, and comment on the behavior vis à vis the above discussion.

2. **On the effective-one-body Hamiltonian and dynamics** [20 points]

   We have derived in class the mapping between the *real* PN Hamiltonian and the *effective* Hamiltonian using the Hamilton-Jacobi formalism. Here we want to construct the effective-one-body (EOB) Hamiltonian using a canonical transformation.

   Using reduced (or dimensionless) variables $\mathbf{Q}, \mathbf{P}$ and $\hat{H}_{\text{eff}}$, the EOB Hamiltonian reads

$$\hat{H}_{\text{eff}}(Q,P) = c^2 \sqrt{A(Q) \left[1 + \frac{1}{c^2}\mathbf{P}^2 + \left(\frac{A(Q)}{D(Q)} - 1\right) \frac{1}{c^2} (\mathbf{N}\cdot\mathbf{P})^2\right]}, \tag{2}$$

   where $\mathbf{N} = \mathbf{Q}/Q$ and

$$A(Q) = 1 + \frac{a_1}{c^2\,Q} + \frac{a_2}{c^4\,Q^2} + \frac{a_3}{c^6\,Q^3} + \cdots, \tag{3}$$

$$D(Q) = 1 + \frac{d_1}{c^2\,Q} + \frac{d_2}{c^4\,Q^2} + \cdots, \tag{4}$$

where a_i, d_i are unknown coefficients that will be determined by the mapping to the (reduced) PN Hamiltonian

$$\hat{H}_{\rm real}(q,p) \;=\; \hat{H}_{\rm Newt}(q,p) + \frac{1}{c^2}\,\hat{H}_{\rm 1PN}(q,p) + \cdots\,, \tag{5}$$

$$\hat{H}_{\rm Newt}(q,p) \;=\; \frac{1}{2}\mathbf{p}^2 - \frac{1}{q}\,, \tag{6}$$

$$\hat{H}_{\rm 1PN}(q,p) \;=\; -\frac{1}{8}(1-3\nu)\,\mathbf{p}^4 - \frac{1}{2q}\left[(3+\nu)\,\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\right] + \frac{1}{2q^2}\,, \tag{7}$$

where $\mathbf{q}$ and $\mathbf{p}$ are reduced variables, $\mathbf{n} = \mathbf{q}/q$ and $\nu = m_1\,m_2/(m_1+m_2)^2$, being m_1 and m_2 the black-hole masses. At 1PN order the real and effective Hamiltonians are related as

$$1 + \frac{\hat{H}_{\rm real}(q,p)}{c^2}\left(1 + \alpha_1\,\frac{\hat{H}_{\rm real}(q,p)}{c^2}\right) = \frac{\hat{H}_{\rm eff}(Q(q,p),P(q,p))}{c^2}\,, \tag{8}$$

where α_1 is an unknown coefficient that will be determined by the mapping.

- The canonical transformation at 1PN order is

$$Q^i \;=\; q^i + \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial p_i}\,, \tag{9}$$

$$P_i \;=\; p_i - \frac{1}{c^2}\frac{\partial G_{\rm 1PN}}{\partial q^i}\,, \tag{10}$$

  with

$$G_{\rm 1PN}(\mathbf{q},\mathbf{p}) = (\mathbf{q}\cdot\mathbf{p})\,\left[c_1\mathbf{p}^2 + \frac{c_2}{q}\right]\,, \tag{11}$$

  where c_1, c_2 are unknown coefficients that will be determined by the mapping.
  Insert the canonical transformation given in Eqs. (9) and (10) in Eq. (8) and expand the latter in PN orders through 1PN order. By equating terms with the same structures in $\mathbf{q}$, $\mathbf{p}$, derive the five equations that the five unknown coefficients $a_1, a_2, \alpha_1, c_1, c_2$ must satisfy. In the original paper by Buonanno & Damour (1999), they set $a_2 = 0 = d_1$. In this case you should find that the solutions of the five equations are: $\alpha_1 = \nu/2$, $c_1 = -\nu/2$ and $c_2 = 1 + \nu/2$. [Hint: introduce the parameter $\epsilon^2 \equiv 1/c^2$ and work with the square of Eq. (8), so as to get rid of the square root in Eq. (2). Use Mathematica to carry out the PN expansion. Note that it is sufficient to derive $Q \equiv |\mathbf{Q}| = \sqrt{Q^i\,Q_i}$, $P \equiv |\mathbf{P}| = \sqrt{P^i\,P_i}$ and $\mathbf{N}\cdot\mathbf{P} = N^i\,P_i$ as function of $q \equiv |\mathbf{q}|$, $p \equiv |\mathbf{p}|$ and $\mathbf{n}\cdot\mathbf{p}$ through 1PN order using the canonical transformation given in Eqs. (9) and (10).]
- (*OPTIONAL*:) Extending the previous calculation through 2PN order, Buonanno & Damour (1999) found the following 2PN coefficients for the effective metric: $a_3 = 2\nu$ and $d_2 = -6\nu$. Use the effective Hamiltonian at 2PN order to compute the innermost stable circular orbit (ISCO) radius, frequency, and energy. How do those values compare with the same quantities in Schwarzschild? [Hint: the ISCO can be computed imposing $\partial\hat{H}_{\rm eff}/(\partial Q) = 0 = \partial^2\hat{H}_{\rm eff}/(\partial^2 Q)$ for $\mathbf{N}\cdot\mathbf{P} = 0$.]
- (*OPTIONAL*:) Introduce polar coordinates $(\Phi, R \equiv |\mathbf{Q}|, P_\Phi, P_R \equiv \mathbf{N}\cdot\mathbf{P})$ and use $\mathbf{P}^2 = P_R^2 + P_\Phi^2/R^2$, with P_Φ the orbital angular momentum. The orbital frequency can be derived from the Hamilton equations. It reads:

$$\hat{\Omega} \equiv \dot{\Phi} = \frac{\partial\hat{H}_{\rm eff}}{\partial P_\Phi}\,. \tag{12}$$

  Employ the above equation to compute the frequency during the plunge, i.e., after the ISCO. To do so, set $P_R = 0$ and impose that both the energy and angular momentum are conserved during the plunge. [The latter is a reasonable approximation because the motion is close to a geodesic during the plunge.]

Draw by hand a plot of the plunge frequency versus R. Determine the radius at which the plunge frequency has a peak. The plunge frequency goes to zero at the radius R for which $A = 0$. Why? You should find that in the Schwarzschild limit ($\nu \to 0$) the peak of the plunge frequency is at $R = 3$ (in dimensional variables this is $3M$, where M is the black hole mass), which is the light-ring or photon orbit in Schwarzschild.

3. **Post-Minkowskian scattering and effective-one-body energy mapping** [30 points]

The first post-Minkowskian (1PM) approximation assumes a linear perturbation to flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(G^2), \tag{13}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is the $O(G)$ metric perturbation, using G as a formal expansion parameter (but working to all orders in $1/c$). This approximation is routinely applied to describe the propagation of gravitational waves away from a source, in the far zone, but it can also be applied to the dynamics of the source in the near zone.

For a gravitationally *bound* system, one has $v^2 \sim Gm/r$, so that weak fields imply slow motion, and thus working to all orders in v^2/c^2 but only to linear order in Gm/rc^2 would be (in the strictest sense) superfluous—–e.g., after re-expanding the 1PM approximation in $1/c$, one would obtain the v^4/c^2 and Gmv^2/rc^2 terms in the 1PN Hamiltonian, but one would be missing the G^2m^2/r^2c^2 term, which is of the same order for bound systems.

However, for an *unbound* system, i.e. in a scattering situation, one can access the regime $v^2 \sim c^2 \gg Gm/r$, which is a natural setting for the 1PM approximation.

Let's consider the scattering of two point-masses in the 1PM approximation (with $c = 1$ from now on). Working to linear order in G allows several significant simplifications: To compute the 1PM deflection of a point-mass, say, body 1, deviating only slightly from inertial (straight-line) motion in a background Minkowski spacetime, due to its gravitational interaction with a second point-mass, body 2,

(i) body 1 can be taken to follow a geodesic in the linearized field sourced by body 2—–because corrections to this, from the influence of body 1's own field, are $O(G^2)$—–,

(ii) the field sourced by body 2 can be computed using its zeroth-order (inertial) motion—–as the corrections to its $O(G)$ field due to body 2's own $O(G)$ deflection are also $O(G^2)$—–, and

(iii) the "force" on body 1 can be integrated along its zeroth-order worldline—–because corrections to the $O(G)$ force due to the $O(G)$ deflection are again $O(G^2)$.

The same logic applies with $1 \leftrightarrow 2$.

First consider the zeroth-order state, with $G \to 0$, in which both bodies move inertially in Minkowski spacetime. They have constant zeroth-order 4-momenta $p_1^\mu = m_1 u_1^\mu$ and $p_2^\mu = m_2 u_2^\mu$, with unit 4-velocities each satisfying $u^2 = -1$, and rest masses m_1 and m_2. The bodies' zeroth-order worldlines can be parametrized as

$$x^\mu = z_1^\mu(\tau_1) = z_{10}^\mu + u_1^\mu \tau_1, \tag{14}$$

$$x^\mu = z_2^\mu(\tau_2) = z_{20}^\mu + u_2^\mu \tau_2, \tag{15}$$

where we enforce

$$b \cdot u_1 = b \cdot u_2 = 0, \qquad b^\mu = z_{10}^\mu - z_{20}^\mu, \tag{16}$$

which uniquely define z_{10} and z_{20} as the points of mutual closest approach of the two worldlines, with the vectorial "impact parameter" b^μ, the spacelike separation vector at closest approach, being orthogonal to both worldlines. The relative Lorentz factor between the worldlines is defined as

$$\gamma = -u_1 \cdot u_2. \tag{17}$$

We assume that the worldlines are nonparallel and nonintersecting, also requiring that $b \gg Gm_{1,2}$ for the validity of the 1PM approximation.

The metric perturbation $h_{2\mu\nu}$ sourced by the zeroth-order motion of body 2 can be taken to be

$$h_2^{\mu\nu}(x) = \frac{2Gm_2}{r_2(x)}(2u_2^\mu u_2^\nu + \eta^{\mu\nu}), \tag{18}$$

which is the solution to the harmonic-gauge linearized Einstein equation $\partial_\rho\partial^\rho h^{\mu\nu} = -16\pi G(T^{\mu\nu} - \eta^{\mu\nu}T_\rho{}^\rho/2)$, $\partial_\mu(h^{\mu\nu} - \eta^{\mu\nu}h_\rho{}^\rho/2) = 0$ with the point-mass stress-energy tensor $T^{\mu\nu} = m\int d\tau\, u^\mu u^\nu \delta^4(x - z(\tau))/\sqrt{-g}$ for body 2, with $h \to 0$ at infinity. Here,

$$r_2(x) = \sqrt{(x - z_{20})^2 + (u_2 \cdot (x - z_{20}))^2} \tag{19}$$

is the distance of the field point x from body 2's worldline in body 2's rest frame (as in special relativity).

The linearized geodesic equation for body 1 in the field of body 2 can be written as

$$\frac{du_{1\mu}}{d\tau_1} = u_{1\nu}u_1^\rho \Gamma^\nu{}_{\mu\rho}[h_2] \tag{20}$$

$$= \frac{1}{2}u_1^\nu u_1^\rho \partial_\mu h_{2\nu\rho} + O(G^2), \tag{21}$$

where ∂h_2 is evaluated at $x = z_1(\tau_1)$.

Using the logic of points (i)–(iii), we can compute the net 1PM deflection of body 1 due to its scattering encounter with body 2 by inserting (18) into (20) and integrating along the entire zeroth-order worldline (14):

$$\Delta p_{1\mu} = m_1 \Delta u_{1\mu} = \frac{m_1}{2}u_1^\nu u_1^\rho \int_{-\infty}^{+\infty} d\tau_1\, \partial_\mu h_{2\nu\rho}(x = z_1(\tau_1)) + O(G^2) \tag{22}$$

$$= -\frac{2Gm_1m_2}{b}\frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}\hat{b}_\mu + O(G^2), \tag{23}$$

where $\hat{b}^\mu = b^\mu/b$ and $b = \sqrt{b_\mu b^\mu}$.

[Note that all index-raising and -lowering, contractions, dot products, squares of vectors, etc. below can be done with the Minkowski metric.]

(*OPTIONAL:*) Derive (23) from the preceding equations.

(a) Use the inherent symmetry under interchanging the bodies' identities to find $\Delta p_{2\mu}$. (Note the definition of b^μ.) Show that the scattering process, $p_1^\mu \to p_1^\mu + \Delta p_1^\mu$ and $p_2^\mu \to p_2^\mu + \Delta p_2^\mu$, conserves the system's total 4-momentum

$$P^\mu = p_1^\mu + p_2^\mu, \tag{24}$$

to linear order in G.

The 4-velocity U^μ of the system's center-of-momentum (COM) frame and the system's total energy E in that frame are defined by

$$U^\mu = \frac{P^\mu}{E}, \qquad E = \sqrt{-P_\mu P^\mu}. \tag{25}$$

The individual momenta can be split into parts along and orthogonal to U^μ according to

$$p_1^\mu = m_1 u_1^\mu = E_1 U^\mu + p_\perp^\mu, \tag{26}$$
$$p_2^\mu = m_2 u_2^\mu = E_2 U^\mu - p_\perp^\mu, \tag{27}$$

where $E_{1,2} = -U_\mu p_{1,2}^\mu$ are the individual energies, and $p_\perp^\mu$ is the "relative momentum," which is a spacelike vector orthogonal to U^μ.

(b) Show that E, E_1, E_2, and U^μ are all conserved by the scattering process, to linear order in G.

Thus, $\Delta p_\perp^\mu = \Delta p_1^\mu = -\Delta p_2^\mu$, and

$$\chi = \frac{\Delta p_\perp}{p_\perp} \tag{28}$$

gives the angle (in the small angle approximation) in the COM frame by which both bodies are scattered. Here, $p_\perp$ and $\Delta p_\perp$ are the magnitudes of $p_\perp^\mu$ and $\Delta p_\perp^\mu$.

(c) Express the scattering angle χ in terms of G, m_1, m_2, γ, and L, where $L = b p_\perp$ is the magnitude of the system's total angular momentum in the COM frame.

(d) Express the total energy E in terms of m_1, m_2 and γ.

We have until now considered the "*two-body case*," in which both bodies are deflected by the others' fields, and have expressed the results in terms of quantities defined in the system's COM frame.

Now let's consider the "*test-body case*," in which only one body (the "test body") is dynamical, being scattered by the second body (the "background body") which is stationary, and let's express the results in terms of quantities defined in the rest frame of the background body.

Say that the test body has mass m_t and initial momentum $p_\mathrm{t}^\mu = m_\mathrm{t} u_\mathrm{t}^\mu$, and the background body has mass m_b and velocity u_b^μ. Using coordinates in which the background body is at rest at the spatial origin, the test body's worldline can be parametrized as

$$x^\mu = z_\mathrm{t}^\mu(\tau_\mathrm{t}) = b_\mathrm{t}^\mu + u_\mathrm{t}^\mu \tau_\mathrm{t}, \tag{29}$$

with

$$b_\mathrm{t} \cdot u_\mathrm{b} = b_\mathrm{t} \cdot u_\mathrm{t} = 0, \tag{30}$$

which defines b_t^μ as the vectorial impact parameter. The Lorentz factor of the test body relative to the background is defined by

$$\gamma_\mathrm{t} = -u_\mathrm{b} \cdot u_\mathrm{t}, \tag{31}$$

and its initial momentum can be split into parts along and orthogonal to u_b according to

$$p_\mathrm{t}^\mu = E_\mathrm{t} u_\mathrm{b}^\mu + p_{\mathrm{t}\perp}^\mu, \tag{32}$$

where

$$E_\mathrm{t} = -u_\mathrm{b} \cdot p_\mathrm{t} = m_\mathrm{t} \gamma_\mathrm{t} \tag{33}$$

is the energy of the test body with respect to the background frame, and $p_{\mathrm{t}\perp}^\mu$ is its relative momentum.

(e) Argue that the 1PM deflection of the test body, $\Delta p_{\rm t}^\mu$, is given by a direct adaptation of (23), with $m_1 \to m_{\rm t}$, $m_2 \to m_{\rm b}$, $u_1^\mu \to u_{\rm t}^\mu$, $u_2 \to u_{\rm b}^\mu$, and $b^\mu \to b_{\rm t}^\mu$.

[Note the logic of points (i)–(iii), and that the only difference in our description of the two cases (besides ignoring the deflection of one body in the test-body case) was the reference frames in which they're described, but that the result (23) is fully specially covariant. Also note that b^μ would be invariant under the boost relating the COM frame to the rest frame of body 2, since $b \cdot u_1 = b \cdot u_2 = 0$.]

Express the resultant scattering angle in the background frame,

$$\chi_{\rm t} = \frac{\Delta p_{\rm t}}{p_{{\rm t}\perp}} = \frac{\Delta p_{{\rm t}\perp}}{p_{{\rm t}\perp}}, \tag{34}$$

in terms of G, $m_{\rm b}$, $m_{\rm t}$, $\gamma_{\rm t}$ and $L_{\rm t}$, where $L_{\rm t} = b_{\rm t} p_{{\rm t}\perp}$ is the magnitude of the test body's angular momentum with respect to the background frame.

(f) The scattering angles χ and $\chi_{\rm t}$ for the two cases exhibit a 1PM effective-one-body (EOB) correspondence, first pointed out in arXiv:1609.00354, as follows. Let us map the rest masses between the two cases according to

$$m_{\rm b} = M = m_1 + m_2, \qquad m_{\rm t} = \mu = \frac{m_1 m_2}{M}, \tag{35}$$

which is the mapping of masses from the usual Newtonian EOB mapping. Then, considering the masses fixed, you should find that

$$\chi(E, L) = \chi_{\rm t}(E_t, L_{\rm t}) \qquad \text{when} \qquad L = L_{\rm t} \tag{36}$$

if there is a certain relationship between E and $E_{\rm t}$—note that E and $E_{\rm t}$ can be expressed respectively solely in terms of γ and $\gamma_{\rm t}$ (and the fixed masses). Express the resultant mapping by giving $E_{\rm t}$ as a function of E (and M and μ). You should find that the result matches the "EOB energy map" between the real and effective Hamiltonians from Exercise 1.

(*VERY OPTIONAL*:) By considering how the scattering angle (as a function of energy and angular momentum) in the two cases is related to a canonical Hamiltonian for the 1PM orbital dynamics, show that the EOB energy map correctly produces a 1PM Hamiltonian for a two-body system (to linear order in G but to all orders in $1/c$) from a Hamiltonian for geodesics in the (linearized) Schwarzschild metric.