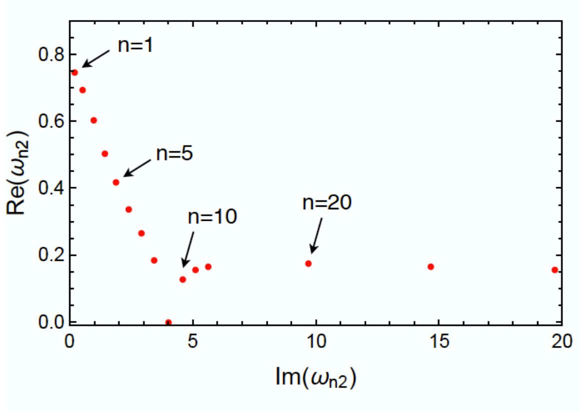

FIG. 1: Real vs. imaginary parts of the $\ell = 2$ quasinormal mode frequencies for a Schwarzschild black hole for different overtones (red dots).

1. Black-hole quasi-normal modes

The plot of the results for the real and imaginary parts of the $\ell = 2$ quasinormal mode frequencies for a nonspinning black hole for different overtones looks as shown in Fig. 1.

When interpreting $\mathrm{Re}(\omega_{n\ell})$ as an oscillation frequency and $\mathrm{Im}(\omega_{n\ell})$ as a decay rate, the features exhibited in this plot seem counterintuitive based on expectations for the oscillation modes of a string or an elastic body, for which both the oscillation frequency and the decay rate increase with increasing overtone number n, i.e. with an increasing number of nodes in the wavefunction. The QNM plot, however, shows that $\mathrm{Re}(\omega_{n\ell})$ is first decreasing with n, then has a zero, and then increases to an asymptotically constant value for large n.

This behavior can seem more natural when considering a re-interpretation of $\mathrm{Re}(\omega_{n\ell})$ and $\mathrm{Im}(\omega_{n\ell})$. To this end, we consider a simple damped oscillator with amplitude $\psi(t)$, oscillation frequency ω_0, and linear damping γ_0, obeying the equation of motion

$$\ddot{\psi} + \gamma_0 \dot{\psi} + \omega_0^2 \psi = 0. \tag{1}$$

The general solution is of the form

$$\psi(t) = a_1 e^{i\omega_+ t} + a_2 e^{i\omega_- t}, \tag{2}$$

where a_1 and a_2 are constants determined by the initial conditions and

$$\omega_\pm = \pm\sqrt{\omega_0^2 - (\gamma_0/2)^2} + i\frac{\gamma_0}{2}. \tag{3}$$

We see that the solutions are of the form $\exp[(i\omega_{\rm R}?\omega_{\rm I})t]$, with

$$\omega_{\rm R} = \sqrt{\omega_0^2 - (\gamma_0/2)^2}, \qquad \omega_{\rm I} = \frac{\gamma_0}{2}. \tag{4}$$

Inverting this to solve for the parameters of the oscillator ω_0 and γ_0 in terms of the oscillation modes of the solution leads to

$$\omega_0 = \sqrt{\omega_{\rm R}^2 + \omega_{\rm I}^2}, \qquad \gamma_0 = 2\omega_{\rm I}. \tag{5}$$

Note that only in the limit $\gamma_0/2 \ll \omega_0$ corresponding to very long-lived modes we get the identification $\omega_0 \approx \omega_{\rm R}$. However, when modeling the quasinormal modes of black holes as arising from

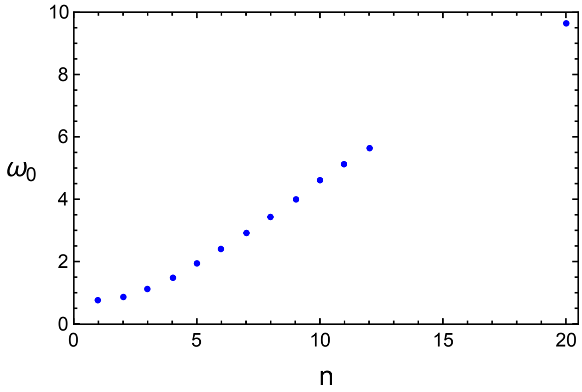

FIG. 2: Frequency of the equivalent oscillator degrees of freedom for quasinormal modes of a Schwazschild black hole (blue dots) computed from the relation (5).

oscillator degrees of freedom analogous to those in Eq. (1), the opposite limit applies. This is seen in Fig. 1, where for most modes $\omega_{\rm I} \gg \omega_{\rm R}$. In this limit, the frequency of the oscillator degree of freedom is $\omega_0 \approx \omega_{\rm I}$.

Taking into account the identification (5) between the frequency and damping of the oscillators and the real and imaginary parts of the frequency of the solution leads to the version of Fig. 1 shown in Fig. 2.

We observe that in terms of ω_0 the structure of the black hole frequency spectrum becomes similar to expectations for generic oscillators. The frequency ω_0 increases monotonically with the overtone number n, and since the damping coefficient $\gamma_0 = 2\omega_{\rm I}$, the damping also increases monotonically with n. Thus, in terms of the equivalent harmonic oscillators, the least damped mode ($n = 1$) also has the lowest value of ω_0, and with increasing ω_0 the lifetime of the excitation becomes shorter.

ANDREA TARACCHINI — Week 7

- $\mathcal{L} = \mathcal{L}_0 + \frac{1}{c^2}\mathcal{L}_2$

$$\mathcal{L}_0 = \frac{m_1}{2}\vec{v}_1^{\,2} + \frac{m_2}{2}\vec{v}_2^{\,2} + \frac{Gm_1m_2}{r}$$

$$\mathcal{L}_2 = \frac{m_1}{8}(\vec{v}_1^{\,2})^2 + \frac{m_2}{8}(\vec{v}_2^{\,2})^2 + \frac{Gm_1m_2}{2r}\left[3(\vec{v}_1^{\,2}+\vec{v}_2^{\,2}) - 7\,\vec{v}_1\cdot\vec{v}_2 - (\vec{v}_1\cdot\hat{r})(\vec{v}_2\cdot\hat{r}) - \frac{G(m_1+m_2)}{r}\right]$$

$$\vec{p}_1 = \frac{\partial\mathcal{L}}{\partial\vec{v}_1} = m_1\vec{v}_1 + \frac{1}{c^2}\left\{\frac{m_1}{2}(\vec{v}_1^{\,2})\vec{v}_1 + \frac{Gm_1m_2}{2r}\left[6\vec{v}_1 - 7\vec{v}_2 - \hat{r}(\vec{v}_2\cdot\hat{r})\right]\right\}$$

$$\vec{p}_2 = \frac{\partial\mathcal{L}}{\partial\vec{v}_2} = m_2\vec{v}_2 + \frac{1}{c^2}\left\{\frac{m_2}{2}(\vec{v}_2^{\,2})\vec{v}_2 + \frac{Gm_1m_2}{2r}\left[6\vec{v}_2 - 7\vec{v}_1 - \hat{r}(\vec{v}_1\cdot\hat{r})\right]\right\}$$

Let $\vec{r}_1 = \frac{1}{2}(\vec{R}-\vec{r})$, $\vec{r}_2 = \frac{1}{2}(\vec{R}+\vec{r})$, $\vec{p}_1 = \vec{P} - \vec{p}$, $\vec{p}_2 = \vec{P} + \vec{p}$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{p}_1+\vec{p}_2) \underset{\text{Euler-Lagrange eqs.}}{=} \frac{\partial\mathcal{L}}{\partial\vec{r}_1} + \frac{\partial\mathcal{L}}{\partial\vec{r}_2} = \frac{\partial R^i}{\partial\vec{r}_1}\frac{\partial\mathcal{L}}{\partial R^i} + \frac{\partial r^i}{\partial\vec{r}_1}\frac{\partial\mathcal{L}}{\partial r^i} + \frac{\partial R^i}{\partial\vec{r}_2}\frac{\partial\mathcal{L}}{\partial R^i} + \frac{\partial r^i}{\partial\vec{r}_2}\frac{\partial\mathcal{L}}{\partial r^i} =$$

$$= \frac{\partial\mathcal{L}}{\partial\vec{R}} - \cancel{\frac{\partial\mathcal{L}}{\partial\vec{r}}} + \frac{\partial\mathcal{L}}{\partial\vec{R}} + \cancel{\frac{\partial\mathcal{L}}{\partial\vec{r}}} = 2\frac{\partial\mathcal{L}}{\partial\vec{R}}$$

Note that $\mathcal{L} = \mathcal{L}(r, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$, then it does NOT depend on $\vec{R}$:

$$\frac{\partial\mathcal{L}}{\partial\vec{R}} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} \text{ is conserved}$$

- $H = H_0 + \frac{1}{c^2}H_2 = \vec{p}_1\cdot\dot{\vec{r}}_1 + \vec{p}_2\cdot\dot{\vec{r}}_2 - \mathcal{L}$

At 0PN order we have
$$\vec{p}_1 = m_1\vec{v}_1 + \mathcal{O}(c^{-2})$$
$$\vec{p}_2 = m_2\vec{v}_2 + \mathcal{O}(c^{-2})$$

then

$$H_0 = \vec{p}_1\cdot\frac{\vec{p}_1}{m_1} + \vec{p}_2\cdot\frac{\vec{p}_2}{m_2} - \mathcal{L}_0 = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{Gm_1m_2}{r}$$

At 1PN order I can replace $\vec{v}_{1,2}$ with $\vec{p}_{1,2}/m_{1,2}$ inside $\mathcal{L}_2$:

$$\mathcal{L}_2 = \frac{1}{8}\frac{(\vec{p}_1^{\,2})^2}{m_1^3} + \frac{1}{8}\frac{(\vec{p}_2^{\,2})^2}{m_2^3} + \frac{Gm_1m_2}{2r}\left[3\left(\frac{\vec{p}_1^{\,2}}{m_1^2} + \frac{\vec{p}_2^{\,2}}{m_2^2}\right) - \frac{7\,\vec{p}_1\cdot\vec{p}_2}{m_1m_2} - \frac{(\vec{p}_1\cdot\hat{r})(\vec{p}_2\cdot\hat{r})}{m_1m_2} - \frac{G(m_1+m_2)}{r}\right]$$

$H_2 = -\mathcal{L}_2$ at 1PN

Move to a frame where $\vec{P} = 0 \Rightarrow \vec{p}_1 = -\vec{p}_2 = -\vec{p}$. Use μ and M:

$$H_0 = \frac{\vec{p}^{\,2}}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - \frac{G\mu M}{r} = \frac{\vec{p}^{\,2}}{2}\frac{m_1+m_2}{m_1m_2} - \frac{G\mu M}{r} = \boxed{\frac{\vec{p}^{\,2}}{2\mu} - \frac{G\mu M}{r}}$$

$$H_2 = -\frac{1}{8}(\vec{p}^{\,2})^2 \underbrace{\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)} - \frac{G\mu M}{2r}\left[\vec{p}^{\,2}\underbrace{\left(\frac{3}{m_1^2}+\frac{3}{m_2^2}+\frac{7}{m_1 m_2}\right)} + \frac{(\vec{p}\cdot\hat{r})^2}{\underbrace{m_1 m_2}_{\mu M}} - \frac{GM}{r}\right]$$

$$= \frac{m_1^3+m_2^3}{(\mu M)^3} = \frac{M}{(\mu M)^3}(m_1^2+m_2^2-m_1 m_2) =$$

$$= \frac{1}{\mu^3 M^2}(M^2-3\mu M) = \frac{1}{\mu^3}\left(1-3\,\mu/M\right) =$$

$$= \frac{1}{\mu^3}(1-3\nu)$$

$$= \frac{3(m_1^2+m_2^2)}{(\mu M)^2} + \frac{7}{\mu M} = \frac{3(M^2-2\mu M)+7\mu M}{(\mu M)^2} =$$

$$= \frac{1}{\mu^2}\left[3\left(1-2\frac{\mu}{M}\right)+\frac{7\mu}{M}\right] = \frac{1}{\mu^2}(3+\nu)$$

Then

$$\boxed{H_2 = \frac{1}{8\mu^3}(3\nu-1)(\vec{p}^{\,2})^2 - \frac{G\mu M}{2r}\left[(3+\nu)\left(\frac{\vec{p}}{\mu}\right)^2 + \frac{(\vec{p}\cdot\hat{r})^2}{\mu M} - \frac{GM}{r}\right]}$$

We can get rid of μ completely: $\mu = \nu M$.

- For a <u>circular orbit</u>:

$\dot{r} = 0$, $\vec{p}\cdot\hat{r} = \frac{d}{dt}(\vec{p}\cdot\hat{r}) = 0 \Rightarrow \vec{p}_{circ} = p_\varphi^{circ}/r\;\hat{\varphi}$ (where I introduced polar coords $\{r,\varphi\}$ in the orbital plane)

Note that the angular momentum is

$$\vec{L} = \vec{r}\times\vec{p} = (r\,\hat{r})\times\left(\frac{p_\varphi}{r}\hat{\varphi}\right) = p_\varphi(\hat{r}\times\hat{\varphi}) = p_\varphi\hat{L}$$ where $\hat{L}$ is $\perp$ orbital plane

Then $\vec{p}^{\,2}_{circ} = L^2_{circ}/r^2$. In general $\vec{p}^{\,2} = p_r^2 + L^2/r^2$, where $p_r \equiv \vec{p}\cdot\hat{r}$.

$$E \equiv H\big|_{circ} = \frac{L^2_{circ}}{2\mu r^2} - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{(3\nu-1)}{8\mu^3}\frac{L^4_{circ}}{r^4} - \frac{G\mu M}{2r}\left[\frac{(3+\nu)}{\mu^2}\frac{L^2_{circ}}{r^2} - \frac{GM}{r}\right]\right.$$

Note that: $\dot{r} = \frac{\partial H}{\partial p_r}$ is automatically 0 if $p_r = 0$. Therefore the only eqn. left to exploit is: $\dot{p}_r = -\frac{\partial H}{\partial r} \stackrel{!}{=} 0$.

$$\left.\frac{\partial H_0}{\partial r}\right|_{p_r=0} = \frac{G\mu M}{r^2} - \frac{1}{2\mu}\left(\frac{+2L^2}{r^3}\right)$$

(note that this gives us the correct Newtonian result for L^2_{circ}, that is $G\mu^2 M r$, so that $E = -\frac{1}{2}\frac{G\mu M}{r}$)

$$\left.\frac{\partial H_2}{\partial r}\right|_{p_r=0} = \frac{3\nu-1}{8\mu^3}\,2\left(\frac{L^2}{r^2}\right)\left(\frac{-2L^2}{r^3}\right) + \frac{G\mu M}{2r^2}\left[\frac{3+\nu}{\mu^2}\frac{L^2}{r^2} - \frac{GM}{r}\right] - \frac{G\mu M}{2r}\left[\frac{3+\nu}{\mu^2}\left(\frac{-2L^2}{r^3}\right) + \frac{GM}{r^2}\right]$$

$$= -\frac{3\nu-1}{2\mu^3}\frac{L^4}{r^5} + \frac{G\mu M}{2r^3}\left[\frac{3+\nu}{\mu^2}\frac{L^2}{r} - GM + \frac{3+\nu}{\mu^2}\frac{2L^2}{r} - GM\right]$$

$$= -\frac{3\nu-1}{2\mu^3}\frac{L^4}{r^5} + \frac{G\mu M}{2r^3}\left[\frac{9+3\nu}{\mu^2}\frac{L^2}{r} - 2GM\right]$$

$L_{circ} = L_0 + \frac{1}{c^2} L_2$ where $L_0 = \mu\sqrt{GMr}$

$\left(\frac{\partial H_0}{\partial r} + \frac{1}{c^2}\frac{\partial H_2}{\partial r}\right)_{p_r=0} = 0$ at 1PN I can replace $L \to L_0$ in $\frac{\partial H_2}{\partial r}$

and get:

$$\left.\frac{\partial H_2}{\partial r}\right|_{p_r=0} = -\frac{3\nu-1}{2\cancel{\mu^3}}\,\cancel{\mu^4}\frac{G^2M^2\cancel{r^2}}{\cancel{r^5}\,3} + \frac{G\mu M}{2r^3}\left[\frac{9+3\nu}{\cancel{\mu^2}}\,\frac{\cancel{\mu^2}GM\cancel{r}}{\cancel{r}} - 2GM\right] =$$

$$= \frac{4\mu G^2M^2}{r^3}$$

Then

$$\frac{G\mu M}{r^2} - \frac{1}{\mu r^3}\left(L_0 + \frac{1}{c^2}L_2\right)^2 + \frac{1}{c^2}\,\frac{4\mu G^2M^2}{r^3} =$$

$$\simeq \underbrace{\left(\frac{G\mu M}{r^2} - \frac{L_0^2}{\mu r^3}\right)}_{0} + \frac{1}{c^2}\left(-\frac{2L_0L_2}{\mu r^3} + \frac{4\mu G^2M^2}{r^3}\right) \stackrel{!}{=} 0$$

$\Rightarrow L_2 = \frac{2G^2M^2\mu}{\sqrt{GMr}}$, so at 1PN we have:

$$\Rightarrow L_{circ} = \mu\sqrt{GMr} + \frac{1}{c^2}\,\frac{2G^2M^2\mu}{\sqrt{GMr}} =$$

$$= \mu\sqrt{GMr}\left[1 + \frac{1}{c^2}\,\frac{2G^2M^2}{GMr}\right] = \boxed{\mu\sqrt{GMr}\left[1 + \frac{1}{c^2}\frac{2GM}{r}\right]}$$

$\omega = \left.\frac{\partial H}{\partial L}\right|_{L=L_{circ},\,p_r=0}$ is the orb. freq. for circ. orbits

$$\left.\frac{\partial H_0}{\partial L}\right|_{L=L_{circ},\,p_r=0} = \frac{L_0 + 1/c^2\,L_2}{\mu r^2}$$

$$\left.\frac{\partial H_2}{\partial L}\right|_{L=L_0,\,p_r=0} = -\frac{GM(3+\nu)L_0}{r^3\mu} + \frac{(3\nu-1)L_0^3}{2r^4\mu^3}$$ where I neglected the L_2 piece at 1PN

Substituting the explicit expression for L_0, L_2 and simplifying:

$$\omega = \frac{\sqrt{GMr}}{2r^3}\left[2r + \frac{1}{c^2}GM(\nu-3)\right] \Rightarrow \boxed{\omega = \frac{\sqrt{GM}}{r^{3/2}}\left[1 + \frac{1}{c^2}\frac{GM(\nu-3)}{2r}\right]}$$

Squaring it

$$\omega^2 = \frac{GM}{r^3}\left[1+\frac{1}{c^2}\frac{GM}{r}(\nu-3)\right]$$

Note: $x \equiv \frac{v^2}{c^2} = \frac{(GM\omega)^{2/3}}{c^2}$ is dimensionless

Also: $\frac{GM}{c^2 r}$ is dimensionless

$$v^2 = (GM\omega)^{2/3} = \frac{(GM)^{2/3}(GM)^{1/3}}{r}\left[1+\frac{1}{c^2}\frac{GM}{r}\left(\frac{\nu}{3}-1\right)\right]$$

$$= \frac{GM}{r}\left[1+\frac{1}{c^2}\frac{GM}{r}\left(\frac{\nu}{3}-1\right)\right]$$

$r = r_0 + \frac{1}{c^2} r_2$ where $r_0 = \frac{GM}{v^2}$

$$\frac{GM}{r_0}\left(1-\frac{1}{c^2}\frac{r_2}{r_0}\right)\left[1+\frac{1}{c^2}\frac{GM}{r_0}\left(\frac{\nu}{3}-1\right)\right] = v^2$$

$$\underbrace{\frac{GM}{r_0}}_{v^2} + \frac{1}{c^2}\left[\frac{GM}{r_0}\left(\frac{\nu}{3}-1\right) - \frac{r_2}{r_0}\right] = \cancel{v^2} \Rightarrow \frac{GM}{r_0}\left(\frac{\nu}{3}-1\right) - \frac{r_2}{r_0} = 0$$

$$\Rightarrow r_2 = GM\left(\frac{\nu}{3}-1\right)$$

$$\Rightarrow \boxed{r = \frac{GM}{v^2}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{3}-1\right)\right]}$$

$$L_{circ} = \mu\sqrt{GM}\,\frac{\sqrt{GM}}{v}\left[1+\left(\frac{v}{c}\right)^2\frac{1}{2}\left(\frac{\nu}{3}-1\right)\right]\left[1+\frac{1}{c^2}\frac{2\cancel{GM}}{\cancel{GM}}v^2\right]$$

$$= \frac{\mu MG}{v}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{6}-\frac{1}{2}+2\right)\right] =$$

$$= \boxed{\frac{\mu MG}{v}\left[1+\left(\frac{v}{c}\right)^2\left(\frac{\nu}{6}+\frac{3}{2}\right)\right]}$$

$$E_{circ} = \frac{L_0^2 + 2L_0L_2/c^2}{2\mu r^2} - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{(3\nu-1)}{8\mu^3}\frac{L_0^4}{r^4} - \frac{G\mu M}{2r}\left[\frac{(3+\nu)}{\mu^2}\frac{L_0^2}{r^2} - \frac{GM}{r}\right]\right.$$

= substitute L_0, L_2 and simplify

$$= -\frac{G\mu M}{8r^2}\left[+4r + \frac{GM}{c^2}(\nu-7)\right] = -\frac{G\mu M}{2r}\left[1+\frac{1}{c^2}\frac{GM}{4r}(\nu-7)\right] =$$

$$= -\frac{G\mu M}{2r_0}\left[1-\frac{1}{c^2}\frac{r_2}{r_0}\right]\left[1+\frac{1}{c^2}\frac{GM}{4r_0}(\nu-7)\right] =$$

$$= \boxed{-\frac{\mu v^2}{2}\left[1-\left(\frac{v}{c}\right)^2\frac{1}{12}(\nu+9)\right]}$$

• Start from H (which is already in the frame where $\vec{P}=0$)

$$\mathcal{L}_0 = \frac{1}{2}\mu\vec{v}^2 + \frac{G\mu M}{r} \quad \text{(just Newtonian)}$$

$$\mathcal{L}_2 = (-H_2)\Big|_{\vec{p}=\mu\vec{v}} = \frac{1}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\Big[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 - \frac{GM}{r}\Big]\Big\}$$

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \mu\vec{v} + \frac{1}{c^2}\Big\{\frac{1}{2}\mu(1-3\nu)\vec{v}^3 + \frac{G\mu M}{r}\big[(3+\nu)\vec{v} + \nu\hat{r}(\hat{r}\cdot\vec{v})\big]\Big\}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow E = \text{const.}$$

$$E = \vec{p}\cdot\vec{v} - \mathcal{L} = \mu\vec{v}^2 + \frac{1}{c^2}\Big\{\underline{\underline{\frac{1}{2}\mu(1-3\nu)\vec{v}^4}} + \frac{G\mu M}{r}\big[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2\big] - \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2}\Big\{\underline{\underline{-\frac{1}{8}(1-3\nu)\mu\vec{v}^4}} - \frac{G\mu M}{2r}\Big[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 - \frac{GM}{r}\Big]\Big\} =$$

$$= \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2}\Big\{\frac{3}{8}\mu(1-3\nu)\vec{v}^4 + \frac{G\mu M}{2r}\Big[(3+\nu)\vec{v}^2 + \nu(\hat{r}\cdot\vec{v})^2 + \frac{GM}{r}\Big]\Big\}$$ energy for generic orbit ✓

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow L = \text{const.} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

Use $\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} \Rightarrow \vec{v}^2 = \dot{r}^2 + r^2\dot{\varphi}^2$, then

$$L = \mu r^2\dot{\varphi} + \frac{1}{c^2}\Big\{\frac{1}{2}\mu(1-3\nu)\vec{v}^2 r^2\dot{\varphi} + G\mu M(3+\nu)r\dot{\varphi}\Big\}$$

Find $\dot{\varphi}$ to substitute into E. We want to invert $L = L(r, \dot{r}, \dot{\varphi})$

At 0PN

$$L = \mu r^2\dot{\varphi}_0 \Rightarrow \dot{\varphi}_0 = \frac{L}{\mu r^2}$$

At 1PN

$$L = \mu r^2\dot{\varphi}_0 + \frac{1}{c^2}\Big\{\mu r^2\dot{\varphi}_2 + \frac{1}{2}\mu(1-3\nu)\vec{v}_0^2 r^2\dot{\varphi}_0 + G\mu M(3+\nu)r\dot{\varphi}_0\Big\}$$

$$\Rightarrow \dot{\varphi}_2 = -\frac{1}{\mu r^2}\Big[\frac{1}{2}\mu(1-3\nu)\vec{v}_0^2 r^2\dot{\varphi}_0 + G\mu M(3+\nu)r\dot{\varphi}_0\Big]$$

where $\vec{v}_0^2 = \dot{r}^2 + r^2\dot{\varphi}_0^2$

Replace $\dot\varphi = \dot\varphi_0 + \frac{1}{c^2}\dot\varphi_2$ into E:

$$E = \frac{1}{2}\mu\left[\dot r^2 + r^2\left(\dot\varphi_0 + \frac{1}{c^2}\dot\varphi_2\right)^2\right] - \frac{G\mu M}{r} + \frac{1}{c^2}\left\{\frac{3}{8}\mu(1-3\nu)(\dot r^2 + r^2\dot\varphi_0^2)^2 + \right.$$

$$\left. + \frac{G\mu M}{2r}\left[(3+\nu)(r^2+r^2\dot\varphi_0^2) + \nu\dot r^2 + \frac{GM}{r}\right]\right\} =$$

$$= \frac{1}{2}\mu(\dot r^2 + r^2\dot\varphi_0^2) + \frac{1}{c^2}\left\{\mu r^2\dot\varphi_0\dot\varphi_2 + (\text{all the 1PN terms above})\right\} \Rightarrow$$

$$\frac{E}{\mu} = \dot r^4\,\frac{1}{c^2}\left(\frac{3}{8} - \frac{9}{8}\nu\right) + \dot r^2\left[\overbrace{\frac{1}{2}}^{\text{Newtonian}} + \frac{1}{c^2}\left(\frac{3GM}{2r} + \frac{L^2}{4M^2r^2\nu^2} - \frac{3L^2}{4M^2r^2\nu} + \frac{G\nu M}{r}\right)\right]$$

$$+\left[\underbrace{-\frac{GM}{r} + \frac{L^2}{2M^2r^2\nu^2}}_{\text{Newtonian}} + \frac{1}{c^2}\left(\frac{G^2M^2}{2r^2} - \frac{GL^2}{2Mr^3\nu^2}(3+\nu) + \frac{L^4}{8M^4r^4\nu^4}(3\nu-1)\right)\right]$$

Neglect the term $\dot r^4$. Solve for $\dot r^2$

$$\dot r^2 = 2\left(\frac{E}{\mu} + \frac{GM}{r} - \frac{L^2}{2M^2r^2\nu^2}\right) + \frac{1}{c^2}\left\{-\frac{2GM}{r}\frac{E}{\mu}(3+2\nu) + \frac{GL^2}{M\nu^2r^3}(5+6\nu)\right.$$

$$\left. + \frac{3L^4}{4M^4\nu^4r^4}(1-3\nu) + \frac{1}{r^2}\left[\frac{L^2}{M^2\nu^2}\frac{E}{\mu}(3\nu-1) - G^2M^2(7+4\nu)\right]\right\} \quad (**)$$

Note that we have the following structure:

$$\frac{1}{2}\dot r^2 + V(r) = (\text{constants})$$

Therefore radial perturbation will have a freq

$$\Omega_r^2 = \left.\frac{\partial^2 V}{\partial r^2}\right|_{circ}$$

I use $-\frac{1}{2}$ of the RHS of eq. (**) as my radial potential.
I don't have to worry that I'm including constants into V, as they are killed by ∂_r^2

I also replace E, L, r with the circ. orbit values at 1PN:

$$\frac{E}{\mu} \to -\frac{v^2}{2}\left[1 - \left(\frac{v}{c}\right)^2\left(\frac{\nu}{12} + \frac{9}{12}\right)\right], \quad r \to \frac{GM}{v^2}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{3} - 1\right)\right]$$

$$L \to \frac{\nu M^2 G}{v}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{6} + \frac{3}{2}\right)\right]$$

Then at 1PN I get

$$\Omega_r^2 = \frac{v^6}{G^2M^2}\left[1 - 6\frac{v^2}{c^2}\right]$$

$$GM\Omega = v^3 \Rightarrow \Omega_r^2 = \Omega^2\left(1 - 6\frac{v^2}{c^2}\right) \Rightarrow \left(\frac{\Omega_r}{\Omega}\right)^2 = 1 - 6\frac{v^2}{c^2}$$

$$\Rightarrow K = \frac{\Omega}{\Omega_r} = 1 + 3\frac{v^2}{c^2}$$

• In polar coords.

$$H_0 = \frac{1}{2\mu}\left(p_R^2 + \frac{p_\varphi^2}{R^2}\right) - \frac{G\mu M}{R}$$

$$H_2 = \frac{1}{8\mu^3}(3\nu-1)\left(p_R^2 + p_\varphi^2/R^2\right) - \frac{G\mu M}{2R}\left[\frac{3+\nu}{\mu^2}\left(p_R^2 + \frac{p_\varphi^2}{R^2}\right) + \frac{p_R^2}{\mu M} - \frac{GM}{R}\right]$$

Let's use for simplicity $G=1$ and $\mu=1$ (see 5.8 in gr-qc/0209089)

$$\frac{H}{\mu} = \frac{\vec{P}^2}{2} - \frac{M}{R} + \frac{1}{c^2}\left\{\frac{3\nu-1}{8}(\vec{P}^2)^2 - \frac{M}{R}\left[\frac{3+\nu}{2}\vec{P}^2 + \frac{\nu P_R^2}{2}\right] + \frac{M^2}{2R^2}\right\}$$

with $\vec{P}^2 = P_R^2 + \frac{P_\varphi^2}{R^2}$, in polar form. Let $H \equiv H^{ADM}/\mu$:

$$\dot{R} = \frac{\partial H}{\partial P_R}, \quad \dot{P}_R = -\frac{\partial H}{\partial R} \qquad \text{note that } H = H(R, P_R, P_\varphi)$$

$$\Omega = \frac{\partial H}{\partial P_\varphi}, \quad \dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

For a circular orbit $R=R_0$, $P_R = 0$, $\underbrace{\left.\frac{\partial H}{\partial R}\right|_{R=R_0, P_R=0}}_{(\#)} = 0 \Leftrightarrow \dot{P}_R = 0$

Let $R = R_0 + \delta R$

$P_R = \delta P_R$

$\Omega = \Omega_0 + \delta\Omega$

$P_\varphi = P_{\varphi 0} + \delta P_\varphi$

At linear order:

$$\delta\dot{R} = \frac{\partial H}{\partial P_R}(R_0+\delta R, \delta P_R, P_{\varphi 0}+\delta P_\varphi) \overset{\text{Taylor}}{=} \overbrace{\frac{\partial H}{\partial P_R}(R_0, 0, P_{\varphi 0})}^{\dot{R}|_{circ}=0} + \overbrace{\frac{\partial^2 H}{\partial P_R \partial R}(R_0, 0, P_{\varphi 0})\,\delta R}^{0 \text{ by } (\#)} +$$

$$+ \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi 0})\,\delta P_R + \underbrace{\frac{\partial^2 H}{\partial P_R \partial P_\varphi}(R_0, 0, P_{\varphi 0})}_{\frac{\partial}{\partial P_\varphi}\left(\frac{\partial H}{\partial P_R}\right)_{R=R_0,\, P_R=0,\, P_\varphi=P_{\varphi 0}}}\delta P_\varphi$$

$$= C_0\, \delta P_R$$

↳ this is linear in P_R, so it vanishes for $P_R = 0$

Where

$$C_0 \equiv \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi 0}) = 1 + \frac{1}{c^2}\left\{\frac{3\nu-1}{2}\left(\frac{P_{\varphi 0}}{R_0}\right)^2 - (3+2\nu)\frac{M}{R_0}\right\}$$

$\delta\dot{P}_\varphi = 0$ since $\dot{P}_\varphi = 0$

$$\delta\dot{P}_R = -\frac{\partial H}{\partial R}(R_0+\delta R, \delta P_R, P_{\varphi 0}+\delta P_\varphi) \overset{\text{Taylor}}{=} -\cancel{\frac{\partial H}{\partial R}(R_0, 0, P_{\varphi 0})} - \frac{\partial^2 H}{\partial R^2}(R_0, 0, P_{\varphi 0})\,\delta R$$

0 by virtue of (#)

$$- \underbrace{\cancel{\frac{\partial^2 H}{\partial R \partial P_R}(R_0, 0, P_{\varphi 0})\, \delta P_R}} - \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta P_\varphi =$$

again, we have a function linear in P_R which vanishes when $P_R = 0$

$$= - A_0\, \delta R - B_0\, \delta P_\varphi$$

where

$$A_0 \equiv + \frac{\partial^2 H}{\partial R^2}(R_0, 0, P_{\varphi 0}) = \frac{3 P_{\varphi 0}^2}{R_0^4} - \frac{2M}{R_0^3} + \frac{1}{c^2}\left\{ \frac{3M^2}{R_0^4} - 6M(3+\nu)\frac{P_{\varphi 0}^2}{R_0^5} + \frac{5}{2}(3\nu - 1)\frac{P_{\varphi 0}^4}{R_0^6} \right\}$$

$$B_0 \equiv + \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0}) = - 2\frac{P_{\varphi 0}}{R_0^3} + \frac{1}{c^2}\left\{ 3M(3+\nu)\frac{P_{\varphi 0}}{R_0^4} - 2(3\nu - 1)\frac{P_{\varphi 0}^3}{R_0^5} \right\}$$

$$\Omega_0 + \delta\Omega = \frac{\partial H}{\partial P_\varphi}(R_0 + \delta R, \delta P_R, P_{\varphi 0} + \delta P_\varphi) = \underbrace{\frac{\partial H}{\partial P_\varphi}(R_0, 0, P_{\varphi 0})}_{\Omega_0} + \frac{\partial^2 H}{\partial R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta R +$$

$$+ \underbrace{\cancel{\frac{\partial^2 H}{\partial P_R \partial P_\varphi}(R_0, 0, P_{\varphi 0})\, \delta P_R}} + \frac{\partial^2 H}{\partial P_\varphi^2}(R_0, 0, P_{\varphi 0})\, \delta P_\varphi$$

again $\frac{\partial H}{\partial P_R}$ is linear in P_R

$$\Rightarrow \delta\Omega = B_0\, \delta R + D_0\, \delta P_\varphi$$

where

$B_0 =$ (same as above)

$$D_0 \equiv \frac{\partial^2 H}{\partial P_\varphi^2}(R_0, 0, P_{\varphi 0}) = \frac{1}{R_0^2} + \frac{1}{c^2}\left\{ -\frac{M}{R_0^3}(3+\nu) + \frac{3}{2}(3\nu - 1)\frac{P_{\varphi 0}^2}{R_0^4} \right\}$$

Look for solution $\sim e^{i\sigma t}$

$$\delta P_R = \frac{\delta \dot{R}}{C_0} \Rightarrow \delta \dot{P}_R = \frac{\delta \ddot{R}}{C_0} = - A_0\, \delta R - B_0\, \delta P_\varphi$$

$\delta \dot{P}_\varphi = 0 \Rightarrow \delta P_\varphi = \text{const.}$, without loss of generality put $\delta P_\varphi = 0$

Then

$$\delta \ddot{R} = - A_0 C_0\, \delta R \qquad \text{use } \delta R \sim e^{i\sigma t}$$

$$-\sigma^2 = - A_0 C_0 \Rightarrow \sigma = \pm\sqrt{+A_0 C_0}$$

In order to have stability, the perturbation δR must be oscillatory and not divergent as $t \to \infty$, so σ must be real, thus the condition is:

$$\boxed{A_0 C_0 > 0}$$

Compute $A_0 C_0$ and replace $R_0 \rightsquigarrow r_{circ} = \frac{M}{v^2}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{3} - 1\right)\right]$

$P_{\varphi 0} \rightsquigarrow l_{circ} = \frac{M}{v}\left[1 + \left(\frac{v}{c}\right)^2\left(\frac{\nu}{6} + \frac{3}{2}\right)\right]$

and keep only up to 1PN:

$$A_0 C_0 = \frac{v^6}{M^2}\left[1 - 6\frac{v^2}{c^2}\right] \underset{\underset{M\Omega = v^3}{\uparrow}}{=} \boxed{\Omega^2\left(1 - 6\, v^2/c^2\right)}$$

$A_0 C_0 = 0 \iff \frac{v^2}{c^2} = \frac{1}{6}$ that is $R = 6M$ in geom. units, the Schw. ISCO

From previous results on periastron

$$\Omega_r^2 = \Omega^2\left(1 - 6\, v^2/c^2\right) = A_0 C_0$$

So at the ISCO also $\Omega_r = 0$. This means that a perturbation of the ISCO has ∞ radial period, i.e. it's a divergent perturbation, and means that we plunge.

The ISCO is the last stable circ. orbit.

3. Post-Minkowskian scattering and effective-one-body energy mapping

First consider the zeroth-order state, with $G \to 0$, in which both bodies move inertially in Minkowski spacetime. They have constant zeroth-order 4-momenta $p_1^\mu = m_1 u_1^\mu$ and $p_2^\mu = m_2 u_2^\mu$, with unit 4-velocities each satisfying $u^2 = -1$, and rest masses m_1 and m_2. The bodies' zeroth-order worldlines can be parametrized as

$$x^\mu = z_1^\mu(\tau_1) = z_{10}^\mu + u_1^\mu \tau_1, \tag{6}$$
$$x^\mu = z_2^\mu(\tau_2) = z_{20}^\mu + u_2^\mu \tau_2, \tag{7}$$

where we enforce

$$b \cdot u_1 = b \cdot u_2 = 0, \qquad b^\mu = z_{10}^\mu - z_{20}^\mu, \tag{8}$$

which uniquely define z_{10} and z_{20} as the points of mutual closest approach of the two worldlines, with the vectorial "impact parameter" b^μ, the spacelike separation vector at closest approach, being orthogonal to both worldlines. The relative Lorentz factor between the worldlines is defined as

$$\gamma = -u_1 \cdot u_2. \tag{9}$$

We assume that the worldlines are nonparallel and nonintersecting, also requiring that $b \gg Gm_{1,2}$ for the validity of the 1PM approximation.

The metric perturbation $h_{2\mu\nu}$ sourced by the zeroth-order motion of body 2 can be taken to be

$$h_2^{\mu\nu}(x) = \frac{2Gm_2}{r_2(x)}(2u_2^\mu u_2^\nu + \eta^{\mu\nu}), \tag{10}$$

which is the solution to the harmonic-gauge linearized Einstein equation $\partial_\rho \partial^\rho h^{\mu\nu} = -16\pi G(T^{\mu\nu} - \eta^{\mu\nu} T_\rho{}^\rho/2)$, $\partial_\mu(h^{\mu\nu} - \eta^{\mu\nu} h_\rho{}^\rho/2) = 0$ with the point-mass stress-energy tensor $T^{\mu\nu} = m \int d\tau\, u^\mu u^\nu \delta^4(x - z(\tau))/\sqrt{-g}$ for body 2, with $h \to 0$ at infinity. Here,

$$r_2(x) = \sqrt{(x - z_{20})^2 + (u_2 \cdot (x - z_{20}))^2} \tag{11}$$

is the distance of the field point x from body 2's worldline in body 2's rest frame (as in special relativity).

The linearized geodesic equation for body 1 in the field of body 2 can be written as

$$\frac{du_{1\mu}}{d\tau_1} = u_{1\nu} u_1^\rho \Gamma^\nu{}_{\mu\rho}[h_2] \tag{12}$$
$$= \frac{1}{2} u_1^\nu u_1^\rho \partial_\mu h_{2\nu\rho} + O(G^2), \tag{13}$$

where ∂h_2 is evaluated at $x = z_1(\tau_1)$.

Using the logic of points (i)–(iii), we can compute the net 1PM deflection of body 1 due to its scattering encounter with body 2 by inserting (10) into (12) and integrating along the entire zeroth-order worldline (6):

$$\Delta p_{1\mu} = m_1 \Delta u_{1\mu} = \frac{m_1}{2} u_1^\nu u_1^\rho \int_{-\infty}^{+\infty} d\tau_1\, \partial_\mu h_{2\nu\rho}(x = z_1(\tau_1)) + O(G^2) \tag{14}$$
$$= -\frac{2Gm_1 m_2}{b} \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} \hat{b}_\mu + O(G^2), \tag{15}$$

where $\hat{b}^\mu = b^\mu/b$ and $b = \sqrt{b_\mu b^\mu}$.

[Note that all index-raising and -lowering, contractions, dot products, squares of vectors, etc. below can be done with the Minkowski metric.]

(*OPTIONAL:*) Derive (15) from the preceding equations.

Solution:
Substituting (11) into (10), and substituting that into (14), treating u_1^μ and u_2^μ as constant for the zeroth-order motion, using (9), yields

$$\begin{aligned}
\Delta p_{1\mu} &= Gm_1m_2(2\gamma^2-1)\int_{-\infty}^{+\infty} d\tau_1 \left[\partial_\mu \frac{1}{[(x-z_{20})^2+(u_2\cdot(x-z_{20}))^2]^{1/2}}\right](x=z_1(\tau_1)) \qquad (16)\\
&= -Gm_1m_2(2\gamma^2-1)\int_{-\infty}^{+\infty} d\tau_1 \left[\frac{(x-z_{20})_\mu + u_{2\mu}(u_2\cdot(x-z_{20}))}{[(x-z_{20})^2+(u_2\cdot(x-z_{20}))^2]^{3/2}}\right](x=z_1(\tau_1))\\
&= -Gm_1m_2(2\gamma^2-1)\int_{-\infty}^{+\infty} d\tau_1 \, \frac{b_\mu - \gamma\tau_1 u_{2\mu}}{[b^2+(\gamma^2-1)\tau_1^2]^{3/2}}\\
&= -Gm_1m_2(2\gamma^2-1)\left[b_\mu\left(\int_{-\infty}^{+\infty} d\tau_1\, \frac{1}{[b^2+(\gamma^2-1)\tau_1^2]^{3/2}} = \frac{2}{b^2\sqrt{\gamma^2-1}}\right)\right.\\
&\qquad\left. -\gamma u_{2\mu}\left(\int_{-\infty}^{+\infty} d\tau_1\, \frac{\tau_1}{[b^2+(\gamma^2-1)\tau_1^2]^{3/2}} = 0\right)\right]
\end{aligned}$$

where the third line has inserted (6) and used (8)–(9). This gives (15).

(a) Use the inherent symmetry under interchanging the the bodies' identities to find $\Delta p_{2\mu}$. (Note the definition of b^μ.) Show that the scattering process, $p_1^\mu \to p_1^\mu + \Delta p_1^\mu$ and $p_2^\mu \to p_2^\mu + \Delta p_2^\mu$, conserves the system's total 4-momentum

$$P^\mu = p_1^\mu + p_2^\mu, \qquad (17)$$

to linear order in G.

Solution:

Everything in (15) is invariant under $1 \leftrightarrow 2$ except for $\hat{b}_\mu$, which flips sign, when we exchange $1 \leftrightarrow 2$ in (8). Thus, $\Delta p_{2\mu} = -\Delta p_{1\mu}$. Thus, $\Delta P_\mu = 0$.

The 4-velocity U^μ of the system's center-of-momentum (COM) frame and the system's total energy E in that frame are defined by

$$U^\mu = \frac{P^\mu}{E}, \qquad E = \sqrt{-P_\mu P^\mu}. \qquad (18)$$

The individual momenta can be split into parts along and orthogonal to U^μ according to

$$p_1^\mu = m_1 u_1^\mu = E_1 U^\mu + p_\perp^\mu, \qquad (19)$$
$$p_2^\mu = m_2 u_2^\mu = E_2 U^\mu - p_\perp^\mu, \qquad (20)$$

where $E_{1,2} = -U_\mu p_{1,2}^\mu$ are the individual energies, and $p_\perp^\mu$ is the "relative momentum," which is a spacelike vector orthogonal to U^μ.

(b) Show that E, E_1, E_2, and U^μ are all conserved by the scattering process, to linear order in G.

Solution:

E and U^μ are conserved because they are the magnitude and direction of P^μ, which is conserved. We then have

$$\Delta E_1 = \Delta(-U \cdot p_1) = -U \cdot \Delta p_1 \propto U \cdot b = 0, \tag{21}$$

where the last equality follows from (8) and the fact that U is a linear combination of u_1 and u_2. The same goes for E_2.

Thus, $\Delta p_\perp^\mu = \Delta p_1^\mu = -\Delta p_2^\mu$, and

$$\chi = \frac{\Delta p_\perp}{p_\perp} \tag{22}$$

gives the angle (in the small angle approximation) in the COM frame by which both bodies are scattered. Here, $p_\perp$ and $\Delta p_\perp$ are the magnitudes of $p_\perp^\mu$ and $\Delta p_\perp^\mu$.

(c) Express the scattering angle χ in terms of G, m_1, m_2, γ, and L, where $L = bp_\perp$ is the magnitude of the system's total angular momentum in the COM frame.

Solution:

Simple substitution yields

$$\chi = \frac{2Gm_1m_2}{L}\frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}. \tag{23}$$

(d) Express the total energy E in terms of m_1, m_2 and γ.

Solution:

Squaring (17) and using (18)–(20) gives

$$E^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma. \tag{24}$$

We have until now considered the "*two-body case*," in which both bodies are deflected by the others' fields, and have expressed the results in terms of quantities defined in the system's COM frame.

Now let's consider the "*test-body case*," in which only one body (the "test body") is dynamical, being scattered by the second body (the "background body") which is stationary, and let's express the results in terms of quantities defined in the rest frame of the background body.

Say that the test body has mass m_t and initial momentum $p_\mathrm{t}^\mu = m_\mathrm{t} u_\mathrm{t}^\mu$, and the background body has mass m_b and velocity u_b^μ. Using coordinates in which the background body is at rest at the spatial origin, the test body's worldline can be parametrized as

$$x^\mu = z_\mathrm{t}^\mu(\tau_\mathrm{t}) = b_\mathrm{t}^\mu + u_\mathrm{t}^\mu \tau_\mathrm{t}, \tag{25}$$

with

$$b_\mathrm{t} \cdot u_\mathrm{b} = b_\mathrm{t} \cdot u_\mathrm{t} = 0, \tag{26}$$

which defines b_t^μ as the vectorial impact parameter. The Lorentz factor of the test body relative to the background is defined by

$$\gamma_\mathrm{t} = -u_\mathrm{b} \cdot u_\mathrm{t}, \tag{27}$$

and its initial momentum can be split into parts along and orthogonal to u_b according to

$$p_\mathrm{t}^\mu = E_\mathrm{t} u_\mathrm{b}^\mu + p_{\mathrm{t}\perp}^\mu, \tag{28}$$

where

$$E_\mathrm{t} = -u_\mathrm{b} \cdot p_\mathrm{t} = m_\mathrm{t}\gamma_\mathrm{t} \tag{29}$$

is the energy of the test body with respect to the background frame, and $p_{\mathrm{t}\perp}^\mu$ is its relative momentum.

(e) Argue that the 1PM deflection of the test body, Δp_t^μ, is given by a direct adaptation of (15), with $m_1 \to m_\mathrm{t}$, $m_2 \to m_\mathrm{b}$, $u_1^\mu \to u_\mathrm{t}^\mu$, $u_2 \to u_\mathrm{b}^\mu$, and $b^\mu \to b_\mathrm{t}^\mu$.

[Note the logic of points (i)–(iii), and that the only difference in our description of the two cases (besides ignoring the deflection of one body in the test-body case) was the reference frames in which they're described, but that the result (15) is fully specially covariant. Also note that b^μ would be invariant under the boost relating the COM frame to the rest frame of body 2, since $b \cdot u_1 = b \cdot u_2 = 0$.]

Express the resultant scattering angle in the background frame,

$$\chi_\mathrm{t} = \frac{\Delta p_\mathrm{t}}{p_{\mathrm{t}\perp}} = \frac{\Delta p_{\mathrm{t}\perp}}{p_{\mathrm{t}\perp}}, \tag{30}$$

in terms of G, m_b, m_t, γ_t and L_t, where $L_\mathrm{t} = b_\mathrm{t} p_{\mathrm{t}\perp}$ is the magnitude of the test body's angular momentum with respect to the background frame.

Solution:

The argument is basically given here, and making the suggested replacements, entailing $\gamma \to \gamma_\mathrm{t}$, yields

$$\chi_\mathrm{t} = \frac{2Gm_\mathrm{b}m_\mathrm{t}}{L_\mathrm{t}} \frac{2\gamma_\mathrm{t}^2 - 1}{\sqrt{\gamma_\mathrm{t}^2 - 1}}. \tag{31}$$

(f) The scattering angles χ and χ_t for the two cases exhibit a 1PM effective-one-body (EOB) correspondence, first pointed out in arXiv:1609.00354, as follows. Let us map the rest masses between the two cases according to

$$m_\mathrm{b} = M = m_1 + m_2, \qquad m_\mathrm{t} = \mu = \frac{m_1 m_2}{M}, \tag{32}$$

which is the mapping of masses from the usual Newtonian EOB mapping. Then, considering the masses fixed, you should find that

$$\chi(E, L) = \chi_\mathrm{t}(E_t, L_\mathrm{t}) \qquad \text{when} \qquad L = L_\mathrm{t} \tag{33}$$

if there is a certain relationship between E and E_t——note that E and E_t can be expressed respectively solely in terms of γ and γ_t (and the fixed masses). Express the resultant mapping by giving E_t as a function of E (and M and μ). You should find that the result matches the "EOB energy map" between the real and effective Hamiltonians from Exercise 1.

Solution:

Setting (23) equal to (31), using $m_1 m_2 = M\mu = m_\mathrm{b} m_\mathrm{t}$ from (32), we have

$$\frac{2GM\mu}{L} \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} = \chi = \chi_\mathrm{t} = \frac{2GM\mu}{L_\mathrm{t}} \frac{2\gamma_\mathrm{t}^2 - 1}{\sqrt{\gamma_\mathrm{t}^2 - 1}}, \tag{34}$$

which will be true when $L = L_\mathrm{t}$ if

$$\gamma = \gamma_\mathrm{t}. \tag{35}$$

Using (32) in (24) yields

$$E^2 = M^2 + 2M\mu(\gamma - 1) \qquad \Leftrightarrow \qquad \mu\gamma = \mu + \frac{E^2 - M^2}{2M}, \tag{36}$$

while $\mu\gamma_\mathrm{t} = E_\mathrm{t}$ from (29) and (32), and so $\gamma = \gamma_\mathrm{t}$ implies

$$E_\mathrm{t} = \mu + \frac{E^2 - M^2}{2M}. \tag{37}$$

or, restoring factors of c by dimensional analysis,

$$E_\mathrm{t} = \mu c^2 + \frac{E^2 - M^2c^4}{2Mc^2}. \tag{38}$$

To connect with Problem 2., we must identify

$$E = Mc^2 + \mu\hat{H}_\mathrm{real}, \qquad E_\mathrm{t} = \mu\hat{H}_\mathrm{eff}. \tag{39}$$

Using these and $\nu = \mu/M$ in (38)/μc^2 gives

$$\frac{\hat{H}_\mathrm{eff}}{c^2} = 1 + \frac{\hat{H}_\mathrm{real}}{c^2}\left(1 + \frac{\nu}{2}\frac{\hat{H}_\mathrm{real}}{c^2}\right), \tag{40}$$

which matches (8) in Problem 2. with $\alpha_1 = \nu/2$.

(*VERY OPTIONAL*:) By considering how the scattering angle (as a function of energy and angular momentum) in the two cases is related to a canonical Hamiltonian for the 1PM orbital dynamics, show that the EOB energy map correctly produces a 1PM Hamiltonian for a two-body system (to linear order in G but to all orders in $1/c$) from a Hamiltonian for geodesics in the (linearized) Schwarzschild metric.

Solution:
We'll first show how two Hamiltonians which lead to the correct scattering angles, one for the test-body case and one for the two-body case, can be deduced from the scattering angles. Then we'll show that one is the EOB energy map applied to the other. (The test-body Hamiltonian can also be directly obtained from geodesics in Schwarzschild at 1PM order.) We'll then briefly discuss the issue of this being sufficient to ensure that these Hamiltonians are more generally correct.

For either the two- or test-body case, say we have a Hamiltonian $H(\boldsymbol{R}, \boldsymbol{P})$ depending on a (3-vector) relative position variable $\boldsymbol{R}(t)$ and its canonically conjugate momentum $\boldsymbol{P}(t)$, which determines the equations of motion via

$$\dot{\boldsymbol{R}} = \frac{\partial H}{\partial \boldsymbol{P}}, \qquad \dot{\boldsymbol{P}} = -\boldsymbol{\nabla} H, \tag{41}$$

where $\boldsymbol{\nabla} = \partial/\partial\boldsymbol{R}$.

At zeroth order in G, we want H to give free motion, which means that it can only depend on $\boldsymbol{P}^2$. Then a general ansatz for H through linear order in G is

$$H = \alpha(\boldsymbol{P}^2) - G\beta(\boldsymbol{R}, \boldsymbol{P}) \tag{42}$$

where α and β will also depend on rest masses, but nothing else. The equations of motion now read

$$\boldsymbol{V} \equiv \dot{\boldsymbol{R}} = \frac{\partial H}{\partial \boldsymbol{P}} = 2\alpha' \boldsymbol{P} - G\frac{\partial \beta}{\partial \boldsymbol{P}}, \tag{43}$$

$$\dot{\boldsymbol{P}} = -\boldsymbol{\nabla} H = G\boldsymbol{\nabla}\beta. \tag{44}$$

At zeroth order in G, we have

$$\Big(\boldsymbol{V} = 2\alpha'\boldsymbol{P} = \text{const.}\Big) + O(G), \tag{45}$$

and

$$\boldsymbol{R} = \boldsymbol{B} + \boldsymbol{V}t + O(G), \tag{46}$$

where $\boldsymbol{B}$ is a constant vector, which we can take to satisfy $\boldsymbol{B}\cdot\boldsymbol{V} = 0$ without loss of generality. This makes $\boldsymbol{B}$ the vectorial impact parameter, i.e. $\boldsymbol{R}$ at closest approach (here, at $t = 0$).

Taking a time derivative of (43), using (44), writing things in terms of both $\boldsymbol{V}$ and $\boldsymbol{P}$ with (45) implicit, yields

$$\begin{aligned}\dot{\boldsymbol{V}} &= 2\alpha'\dot{\boldsymbol{P}} + 4\alpha''(\boldsymbol{P}\cdot\dot{\boldsymbol{P}})\boldsymbol{P} - G\left(\boldsymbol{V}\cdot\boldsymbol{\nabla} + \dot{\boldsymbol{P}}\cdot\frac{\partial}{\partial\boldsymbol{P}}\right)\frac{\partial\beta}{\partial\boldsymbol{P}}\\ &= 2G\alpha'\boldsymbol{\nabla}\beta + G(\boldsymbol{V}\cdot\boldsymbol{\nabla})\left[\left(\frac{2\alpha''}{\alpha'}\boldsymbol{P} - \frac{\partial}{\partial\boldsymbol{P}}\right)\beta\right] + O(G^2).\end{aligned} \tag{47}$$

We can find the net $O(G)$ change in $\boldsymbol{V}$ by integrating (47) over the entire zeroth-order worldline (46). When we do this [treating $\boldsymbol{P}$ or $\boldsymbol{V}$ as constants in the integrand, valid to linear order in G, because the whole RHS of (47) is $O(G)$], the second group of terms, because it is $(\boldsymbol{V}\cdot\boldsymbol{\nabla})$(something), which is equivalent to (d/dt)(something) here, will drop out, under the assumption that β and its derivatives vanish at infinity. We thus have the simple result

$$\Delta\boldsymbol{V} = 2G\alpha'\int_{-\infty}^{+\infty} dt\,\big[\boldsymbol{\nabla}\beta\big]_{\boldsymbol{R}=\boldsymbol{B}+\boldsymbol{V}t} + O(G^2). \tag{48}$$

If (with some foresight) we now specialize our ansatz for H so that

$$\beta(\boldsymbol{R},\boldsymbol{P}) = \frac{\beta_0(\boldsymbol{P}^2)}{|\boldsymbol{R}|}, \tag{49}$$

then we have [dropping the $+O(G^2)$]

$$\Delta\boldsymbol{V} = 2G\alpha'\beta_0\int_{-\infty}^{+\infty} dt\,\left[\boldsymbol{\nabla}\frac{1}{|\boldsymbol{R}|}\right]_{\boldsymbol{R}=\boldsymbol{B}+\boldsymbol{V}t} \tag{50}$$

$$= -2G\alpha'\beta_0\int_{-\infty}^{+\infty} dt\,\frac{\boldsymbol{B}+\boldsymbol{V}t}{|\boldsymbol{B}+\boldsymbol{V}t|^3} \tag{51}$$

$$= -\frac{4G\alpha'\beta_0}{VB^2}\boldsymbol{B}, \tag{52}$$

where $V = |\boldsymbol{V}|$, sim. B, etc. The scattering angle in the small angle approximation is

$$\chi = \frac{|\Delta\boldsymbol{V}|}{V} = \frac{4G\alpha'\beta_0}{V^2B}. \tag{53}$$

The angular momentum $\boldsymbol{L}$ which is conserved by Hamilton's equations (41) [as a consequence of rotation invariance] is $\boldsymbol{L} = \boldsymbol{R} \times \boldsymbol{P}$. Evaluating this in the zeroth-order state gives

$$\boldsymbol{L} = \boldsymbol{B} \times \boldsymbol{P} \qquad \Rightarrow \qquad L = BP = \frac{VB}{2\alpha'}, \tag{54}$$

having used (45) and $\boldsymbol{B} \cdot \boldsymbol{V} = 0$. Thus,

$$\chi = \frac{2G\beta_0}{VL}, \tag{55}$$

noting that V and β_0 are functions only of $\boldsymbol{P}^2$, which can be related to the zeroth-order energy/Hamiltonian $H = \alpha(\boldsymbol{P}^2) + O(G)$. We will identify the L here with the L's from above, for each case.

Now, all of that can be applied to both the test- and two-body cases. First let's establish a Hamiltonian for the test-body case. Note that this is by no means unique, but it will suffice for our purposes to find one good Hamiltonian.

Adding "t" to everything for the test body, let us take its zeroth-order Hamiltonian, i.e. $\alpha_{\mathrm{t}}(\boldsymbol{P}_{\mathrm{t}}^2)$, to be its actual energy as a function of the spatial components of its actual 4-momentum, identifying these with its canonical momentum for the Hamiltonian; as in SR,

$$p_{\mathrm{t}}^{\mu} = (E_{\mathrm{t}}, \boldsymbol{P}_{\mathrm{t}}), \qquad p_{\mathrm{t}}^2 = -E_{\mathrm{t}}^2 + \boldsymbol{P}_{\mathrm{t}}^2 = -m_{\mathrm{t}}^2, \tag{56}$$

$$\Rightarrow \qquad E_t = \sqrt{m_{\mathrm{t}}^2 + \boldsymbol{P}_{\mathrm{t}}^2} = \alpha_{\mathrm{t}}(\boldsymbol{P}_{\mathrm{t}}^2) \qquad \Rightarrow \qquad \alpha'_{\mathrm{t}} = \frac{1}{2\alpha_{\mathrm{t}}}. \tag{57}$$

Using $E_{\mathrm{t}} = m_{\mathrm{t}}\gamma_{\mathrm{t}}$ as above, we have

$$\alpha_{\mathrm{t}} = m_{\mathrm{t}}\gamma_{\mathrm{t}}, \qquad \gamma_{\mathrm{t}} = \sqrt{1 + \boldsymbol{P}_{\mathrm{t}}^2/m_{\mathrm{t}}^2}, \qquad \boldsymbol{P}_{\mathrm{t}} = \frac{\boldsymbol{V}_{\mathrm{t}}}{2\alpha'_{\mathrm{t}}} = m_{\mathrm{t}}\gamma_{\mathrm{t}}\boldsymbol{V}_{\mathrm{t}}, \tag{58}$$

and thus,

$$P_{\mathrm{t}} = m_{\mathrm{t}}\sqrt{\gamma_{\mathrm{t}}^2 - 1} = m_{\mathrm{t}}\gamma_{\mathrm{t}}V_{\mathrm{t}}, \qquad V_{\mathrm{t}} = \frac{\sqrt{\gamma_{\mathrm{t}}^2 - 1}}{\gamma_{\mathrm{t}}}. \tag{59}$$

Using this and comparing (55) with "t"s to (31),

$$\chi_{\mathrm{t}} = \frac{2G\beta_{0\mathrm{t}}}{V_{\mathrm{t}}L_{\mathrm{t}}} = \frac{2G\gamma_{\mathrm{t}}\beta_{0\mathrm{t}}}{L_{\mathrm{t}}\sqrt{\gamma_{\mathrm{t}}^2 - 1}} = \frac{Gm_{\mathrm{b}}m_{\mathrm{t}}}{L_{\mathrm{t}}}\frac{2\gamma_{\mathrm{t}}^2 - 1}{\sqrt{\gamma_{\mathrm{t}}^2 - 1}}, \tag{60}$$

we conclude that

$$\beta_{0\mathrm{t}} = m_{\mathrm{b}}m_{\mathrm{t}}\frac{2\gamma_{\mathrm{t}}^2 - 1}{\gamma_{\mathrm{t}}}, \tag{61}$$

and thus, the Hamiltonian for the test body is

$$H_{\mathrm{t}}(\boldsymbol{R}_{\mathrm{t}}, \boldsymbol{P}_{\mathrm{t}}) = m_{\mathrm{t}}\gamma_{\mathrm{t}} - \frac{Gm_{\mathrm{b}}m_{\mathrm{t}}}{|\boldsymbol{R}_{\mathrm{t}}|}\frac{2\gamma_{\mathrm{t}}^2 - 1}{\gamma_{\mathrm{t}}}, \qquad \gamma_{\mathrm{t}} = \sqrt{1 + \frac{\boldsymbol{P}_{\mathrm{t}}^2}{m_{\mathrm{t}}^2}}. \tag{62}$$

One can check that this matches the linear-in-G part of the canonical Hamiltonian for geodesics in Schwarzschild with mass m_{b} (in harmonic or isotropic coordinates).

Now we do the same thing for the two-body system, again using (42)–(55), now without "t"s, but we need an ansatz for $\alpha(\boldsymbol{P}^2)$. Motivated by the EOB equivalence for the scattering angles, with the energy map (37), we can try taking α for the two-body system (which should coincide with the system's total energy at zeroth order) to be related to α_t in the same way that E is related to E_t, while identifying $\boldsymbol{P} = \boldsymbol{P}_\mathrm{t}$, and using the usual rest mass maps (32):

$$\alpha(\boldsymbol{P}^2) = \sqrt{M^2 + 2M\mu(\gamma - 1)}, \qquad \gamma = \sqrt{1 + \frac{\boldsymbol{P}^2}{\mu^2}}, \tag{63}$$

which is as in (36) with $E \to \alpha$. Differentiating with respect to $\boldsymbol{P}^2$ and then using the magnitude of (45) and the previous equation here gives

$$\alpha' = \frac{M}{2\gamma\mu\alpha}, \qquad V = 2\alpha' P = \frac{MP}{\gamma\mu\alpha} = \frac{M}{\alpha}\frac{\sqrt{\gamma^2 - 1}}{\gamma} \tag{64}$$

Using this and comparing (55) to (23),

$$\chi = \frac{2G\beta_0}{VL} = \frac{2G\alpha\gamma\beta_0}{ML\sqrt{\gamma^2 - 1}} = \frac{GM\mu}{L}\frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \tag{65}$$

we conclude that

$$\beta_0 = \frac{M}{\alpha} M\mu \frac{2\gamma^2 - 1}{\gamma}, \tag{66}$$

and thus, the Hamiltonian for the two-body system is

$$H(\boldsymbol{R}, \boldsymbol{P}) = \alpha - \frac{M}{\alpha}\frac{GM\mu}{|\boldsymbol{R}|}\frac{2\gamma^2 - 1}{\gamma}, \tag{67}$$

with α and γ as in (63). To linear order in G, this is the same as

$$H = \sqrt{M^2 + 2M(H_\mathrm{t} - \mu)}, \qquad H_\mathrm{t} = \gamma - \frac{GM\mu}{|\boldsymbol{R}|}\frac{2\gamma^2 - 1}{\gamma}, \qquad \gamma = \sqrt{1 + \frac{\boldsymbol{P}^2}{\mu^2}}, \tag{68}$$

which is the energy map applied to the test-body Hamiltonian.

So, we derived two 1PM Hamiltonians H and H_t which produce the correct scattering angles, and we've seen that H is the EOB energy map applied to H_t. Does this mean that they are the correct 1PM Hamiltonians more generally, even for bound motion? We made a few arbitrary choices, and our Hamiltonians are in a specific "gauge". Would everything we did be the same if our Hamiltonians were in a different gauge, i.e. if subjected to a canonical transformation?

The second question can be easily answered for a canonical transformation at linear order in G. For a canonical transformation with a generating function $G\mathcal{G}(\boldsymbol{R}, \boldsymbol{P})$, with $\boldsymbol{\nabla} = \partial/\partial\boldsymbol{R}$ again,

$$\boldsymbol{R} \to \boldsymbol{R} + G\frac{\partial\mathcal{G}}{\partial\boldsymbol{P}}, \qquad \boldsymbol{P} \to \boldsymbol{P} - G\boldsymbol{\nabla}\mathcal{G}, \tag{69}$$

the Hamiltonian undergoes

$$H \to H + \Delta H, \qquad \Delta H = G\boldsymbol{\nabla}H \cdot \frac{\partial\mathcal{G}}{\partial\boldsymbol{P}} - G\frac{\partial H}{\partial\boldsymbol{P}} \cdot \boldsymbol{\nabla}\mathcal{G}. \tag{70}$$

With the G's out front, we need only use the zeroth order Hamiltonian, i.e. $H = \alpha(\boldsymbol{P}^2) + O(G)$, in ΔH, and thus, to linear order in G,

$$\Delta H = G\frac{\partial\alpha}{\partial\boldsymbol{P}} \cdot \boldsymbol{\nabla}\mathcal{G} = 2G\alpha'\boldsymbol{P} \cdot \boldsymbol{\nabla}\mathcal{G} = G\boldsymbol{V} \cdot \boldsymbol{\nabla}\mathcal{G}. \tag{71}$$

With this contributing as an addition to $-G\beta$ in (48), it will drop out for the same reasons as discussed above (48). Thus, the scattering angle is invariant under canonical transformations at linear order in G.

This would show that the Hamiltonians deduced from the scattering angles above are the correct Hamiltonians, even for bound motion, if one could show that any Hamiltonian can be put into the form of our ansatz above, $H(\boldsymbol{R},\boldsymbol{P}) = \alpha(\boldsymbol{P}^2) - G\beta_0(\boldsymbol{P}^2)/|\boldsymbol{R}|$. We will not give a complete proof of this here, but we can note that the rather general form

$$H = \alpha(P^2) - \frac{GM\mu}{|\boldsymbol{R}|}\Big(c_1 + c_2\boldsymbol{N}\cdot\boldsymbol{P} + c_3\boldsymbol{P}^2 + c_4(\boldsymbol{N}\cdot\boldsymbol{P})^2 + \dots\Big), \tag{72}$$

with a polynomial in $\boldsymbol{P}^2$ and $\boldsymbol{N}\cdot\boldsymbol{P}$, with $\boldsymbol{N} = \boldsymbol{R}/|\boldsymbol{R}|$, can be changed via an $O(G)$ canonical transformation into a form with no $\boldsymbol{N}\cdot\boldsymbol{P}$ terms. The c_2 term e.g. is eliminated with $\mathcal{G} \propto \ln|\boldsymbol{R}|$, and the c_4 was seen in Problem 2, where we saw that it can be transformed into the c_3 term and an $O(G^2)$ term.