Notes for lectures on tidal interactions – caution: may contain typos

BEYOND VACUUM BINARIES

Black holes characterized by mass and spin only, with $T^{\mu\nu}_{\rm BH}=0$
Additional interesting effects arise when $T^{\mu\nu}\neq 0$. Examples:

$$\text{fluid}: \; T^{\rm fluid}_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}, \quad \text{pressure } p, \text{ energy density } \rho, \; 4-\text{velocity } u^\mu$$

$$\text{dust}: \; T^{\rm fluid}_{\mu\nu} \text{ with } p\to 0 \tag{1a}$$

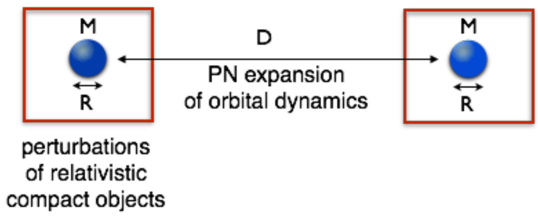
$$\text{complex scalar field}: \; T^{\rm scalar}_{\mu\nu}=\partial_\mu\phi^*\partial_\nu\phi+\partial_\nu\phi^*\partial_\mu\phi-g_{\mu\nu}\left[\partial^\alpha\partial_\alpha\phi+V\left(|\phi|^2\right)\right] \tag{1b}$$

$$\text{Maxwell}: \; T^{\rm em}_{\mu\nu}=\frac{1}{4\pi}\left[F_{\mu\alpha}F_\nu{}^\alpha-\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{1c}$$

Besides Einstein's eq. + Bianchi identity/stress-energy conservation, need additional info to close the system. Richer physics but more difficult to solve than for BHs.

I. LEADING ORDER EFFECT OF INTERNAL STRUCTURE IN A BINARY: TIDAL INTERACTIONS

Historical context: Wilson, Matthews PRL **75**, 4161 (1995) "Instabilities in Close Neutron Star Binaries" + 4 follow-up papers: crushing force of general-relativistic origin which can cause neutron stars to collapse and form individual black holes during the inspiral
Sociological account: D. Kennefick, Social Studies of Science **30**, 5 (2000): "Star Crushing: Theoretical Practice and the Theoretician's Regress"
Rigorous analytical treatment: Flanagan Phys. Rev. D **58**, 124030 (1998):"General-relativistic coupling between orbital motion and internal degrees of freedom for inspiraling binary neutron stars":
Matched asymptotic expansions:

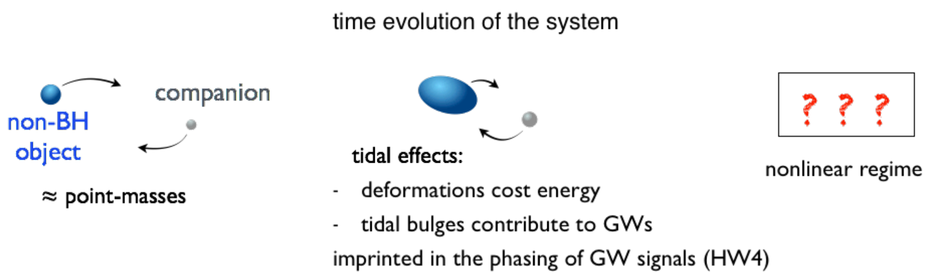


system characterized by the following dimensionless parameters

$$\epsilon=\frac{M}{R}\ (\text{internal gravity}), \qquad \alpha=\frac{R}{D}\ (\text{tidal expansion}), \qquad \epsilon_{\rm orbit}=\frac{M}{D}=\epsilon\alpha\ \ (\text{PN expansion}) \tag{2}$$

To all orders in ϵ but with the assumptions that $D\gg R$ and no pre-existing internal velocity field (but including gravitomagnetic interactions), matched asymptotic expansion shows that

1. there are NO GR non-tidal interactions
2. body-dependence of equations of motion starts at $O(\alpha^5)$ $\quad$ ($\sim$ 5PN)
3. tidal deformation of compact object stars at $O(\alpha^3)$



Plan for the lecture:

1. tidal effects in binaries
2. information needed to determine the motion, characteristic tidal deformability parameters
3. Next time: definition, computation of these parameters from strong-field region
   physics of neutron stars / boson stars imprinted in the GWs, results

A. Newtonian tidal interactions

Application to PN binaries: Damour, Soffel, Xu, 1992 ff. but their work was limited to $\epsilon \ll 1$, i.e. PN throughout
Gravitational potential U sourced by mass-density ρ:

$$\nabla^2 U = 4\pi\rho \tag{3}$$

Green's fn solution (c.f. Jackson, em)

$$U(t,\mathbf{x}) = -\int d^3x' \rho(t,\mathbf{x}') \frac{1}{|\mathbf{x}-\mathbf{x}'|}. \tag{4}$$

Potential outside an isolated body: use that in general, for any field $\phi(\mathbf{x})$ the Taylor expansion around a reference point $\mathbf{z}$ is

$$\begin{aligned}\phi(\mathbf{x}) &= \phi(\mathbf{z}) + (x-z)^i \partial_i \phi(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}} + \frac{1}{2}(x-z)^i (x-z)^j \partial_i\partial_j \phi(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}} + O\left((x-z)^3\right) \\ &= \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x-z)^L \partial_L \phi(\mathbf{x}) \mid_{\mathbf{x}=\mathbf{z}},\end{aligned} \tag{5}$$

where L denotes a string of ℓ indices and the notation is

$$x^L = x^{a_1} x^{a_2} \cdots x^{a_\ell}, \qquad \partial_L = \partial_{a_1}\partial_{a_2}\cdots\partial_{a_\ell}. \tag{6}$$

Apply to gravitational potential of body A: Taylor expand $\mathbf{x}'$ around a reference point $\mathbf{z}_A$, field point $|\mathbf{x}| \gg |\mathbf{x}'|$:

$$U_A = -\int d^3x' \rho \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} (x'-z_A)^L \left(\partial_L \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)_{\mathbf{x}'=\mathbf{z}_A}. \tag{7}$$

1. Aside: Symmetric Trace-Free Tensors

Consider the quantity

$$r = \sqrt{\delta_{ij} x^i x^j} \tag{8}$$

$$\partial_i \frac{1}{r} = -\frac{1}{2}\frac{1}{r^3}\left(2\delta_{bi} x^b\right) = -\frac{x^i}{r^3} \tag{9}$$

$$\partial_j\partial_i \frac{1}{r} = -\frac{\delta_{ij}}{r^3} + \frac{3}{2}\frac{x^i}{r^5}\left(2x^j\right) = \frac{3}{r^3}\left(n^i n^j - \frac{1}{3}\delta_{ij}\right) = \frac{3}{r^3} n^{<ij>} \tag{10}$$

with

$$n_i = \frac{x^i}{r} \quad \text{(unit vector)}, \qquad n^i n_i = 1 \tag{11}$$

and angular brackets around indices to denote the symmetric-trace-free (STF) part of a tensor. For example,

$$x^{<ij>} = x^i x^j - \frac{1}{3}\delta^{ij}|\boldsymbol{x}|^2, \qquad n^{<ijk>} = n^i n^j n^k - \frac{1}{5}\left(\delta^{ij} n^k + \delta^{ik} n^j + \delta^{jk} n^i\right). \tag{12}$$

generalization for derivatives

$$\partial_L \frac{1}{r} = (-1)^\ell (2\ell-1)!! \frac{n^{<L>}}{r^{\ell+1}}. \tag{13}$$

Note that for any contractions $T^L \partial_L r^{-1}$, only the STF part of the tensor T^L contributes, i.e.

$$T_L n^{<L>} = T_{<L>} n^{<L>} \tag{14}$$

Quick explicit check using $T_L = n^i n^j$:

$$n^i n^j n_{<ij>} = 1 - \frac{1}{3} = \frac{2}{3}, \qquad n^{<ij>} n_{<ij>} = 1 - \frac{1}{3} - \frac{1}{3} + \frac{3}{9} = \frac{2}{3}. \tag{15}$$

2. *Back to expansion of potential*

$$\text{Had}: \; U_A = -\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \underbrace{\partial_L \frac{1}{|\mathbf{x}-\mathbf{x}'|} \,|_{x'=z_A}}_{(-1)^\ell (2\ell-1)!! \frac{n_A^{<L>}}{r_A^{\ell+1}}} \underbrace{\int d^3x' \rho (x'-z_A)^L}_{\equiv M_A^{<L>} \text{ multipoles}} \tag{16}$$

$$= -\sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \frac{n_A^{<L>}}{r_A^{\ell+1}} M_A^{<L>}, \tag{17}$$

$$\text{here}: \; n_A^i = \frac{(x-z_A)^i}{r_A}, \qquad r_A = |\mathbf{x} - \mathbf{z}_A| \tag{18}$$

First few mass multipole moments:

$$M_A = \int d^3x \rho \tag{19}$$

$$M_A^i = \int d^3x \rho (x-z_A)^i \quad \text{Can always choose } \mathbf{z}_A \text{ s.t. } M_A^i = 0 \; (\text{center}-\text{of}-\text{mass}) \tag{20}$$

$$M_A^{<ij>} \equiv Q^{ij} = \int d^3x \rho (x-z_A)^{<ij>}. \tag{21}$$

Specifically, $z^i_{A,\mathrm{CM}} = M_A^{-1} \int d^3x \rho x^i$.
In spherical coordinates

$$\mathbf{n}_A = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{22}$$

so can equivalently use a spherical harmonic expansion

$$U_A = -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{r_A^{\ell+1}} Y_{\ell m}(\theta,\phi) M_{\ell m}, \qquad M_{\ell m} = \frac{4\pi}{2\ell+1} \int d^3x \rho r_A^\ell Y^*_{\ell m}(\theta,\phi). \tag{23}$$

3. *Two-body system: tidal moments and equations of motion*

Equations of motion for body A:

$$M_A \ddot{z}_A = -\int_A \rho \partial_i U d^3x, \tag{24}$$

where $U = U_A + U_B$ is the total gravitational potential. Will see that only U_B contributes to A's motion: consider one component of the contribution from U_A. Trick: swap dummy variables $\mathbf{x} \leftrightarrow \mathbf{x}'$:

$$-\int \rho(x) d^3x \partial_i U_A = \int \rho(x) d^3x \int \rho(x') d^3x' \frac{\partial}{\partial x^i} \frac{1}{|\mathbf{x}-\mathbf{x}'|}$$

$$= \int \rho(x') d^3x' \int \rho(x) d^3x \frac{\partial}{\partial x'^i} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{25}$$

$$= 0 \tag{26}$$

since the first and second lines give answers that are the negatives of each other, can be worked out explicitly using $\partial_i |\mathbf{x}-\mathbf{x}'| = (x-x')^i / |\ldots|^3$ which is antisymmetric under $x \leftrightarrow x'$.

To evaluate the acceleration of A for $\alpha = R/D \ll 1$, Taylor expand U_B around z_A:

$$U_B = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (x - z_A)^L \left(\partial_L U_B(\mathbf{x}) \mid_{x=z_A} \right) \tag{27}$$

Equations of motion of body A:

$$M_A \ddot{z}_A^i = -\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\partial_{iL} U_B(\mathbf{x}) \mid_{x=z_A} \right) \int \rho d^3x (x - z_A)^{<L>} \tag{28}$$

$$= -\sum_{\ell=0}^{\infty} \frac{1}{\ell!} M_A^{<L>} \left(\partial_{iL} U_B(\mathbf{x}) \mid_{x=z_A} \right) \tag{29}$$

$$= -M_A \left(\partial_i U_B(\mathbf{x}) \mid_{x=z_A} \right) - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} M_A^{<L>} \mathcal{E}^A_{<iL>} \tag{30}$$

Here, $\mathcal{E}_A^{<L>}$ are tidal moments of body A, with

$$\mathcal{E}_A^{<L>} = \left(\partial_L U_B(\mathbf{x}) \mid_{x=z_A} \right). \tag{31}$$

Suppose B is a point mass (or BH). Its Newtonian potential is

$$U_B = -\frac{M_B}{r_B}. \tag{32}$$

Suppose further that A is a non-BH object that has only a quadrupole:

$$a_A^i = -\frac{M_B n_{AB}^i}{r_{AB}^2} - \frac{1}{2M_A} Q_{jk} \mathcal{E}^{ijk}, \qquad \mathcal{E}_{ijk} = -M_B \partial_{ijk} \frac{1}{r_{AB}} = 15 M_B \frac{n_{AB}^{<ijk>}}{r_{AB}^4} \tag{33}$$

$$a_B^i = -\frac{M_A n_{BA}^i}{r_{BA}^2}. \tag{34}$$

Use relative coordinates (details in Sec. 1.6.7 in Gravity) and introduce the total mass $M = M_A + M_B$, then

$$a^i = -\frac{M n^i}{r^2} - \frac{1}{2\mu} Q_{jk} \mathcal{E}_{ijk} = -\frac{M n^i}{r^2} - \frac{15}{2} Q_{jk} \frac{M}{M_A} \frac{n^{<ijk>}}{r^4} \tag{35}$$

Practical advantage of using multipole and tidal moments: substantially simplified description of the evolution of the entire system. Generically, would need to solve a set of PDEs that follow from the Euler, Poisson and continuity equations.
Use of moments reduces the task to solving a set of ODEs for the center-of-mass worldlines. Need to supplement with information about the time evolution of the multipole moments.
Furthermore, when considering the extension to the situation of a PN description of the orbital dynamics connected via matched asymptotic expansions to the strong-gravity region around the compact objects: don't need all the microscopic details from the strong-field region, will discuss this later and next time.

4. Action principle

The equations of the relative motion can be conveniently summarized in the action:

$$S = S_{\rm orbit} + \int dt \Big[-\frac{1}{2} Q_{ij}\mathcal{E}_{ij} + \mathcal{L}^{\rm int}\Big], \tag{36}$$

where $\mathcal{E}_{ij} = -M_B \partial_i \partial_j r^{-1}$ and $\mathcal{L}^{\rm int}$ describes the internal dynamics of the quadrupole; $L_{\rm orbit} = (\mu/2)\dot{x}^2 + \mu M/r$.

The action principle can straightforwardly promoted to a relativistic result as

$$S_Q = \int d\sigma \, \left[-\frac{z}{2} E_{\mu\nu} Q^{\mu\nu} + z\mathcal{L}^{\rm int}_{\rm rel}\right], \tag{37}$$

Here, σ is a parameter along the worldline of body A, the redshift factor is $z = \sqrt{-u^\mu u_\mu}$ and u^μ is the tangent to the worldline. The 4-dimensional tidal field $E_{\mu\nu}$ is the electric part of the Weyl tensor $C_{\mu\nu\alpha\beta}$ (equivalent to the Riemann tensor here) given by (simpler: in the object's rest frame: R_{0i0j})

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}\frac{u^\alpha u^\beta}{z^2}, \tag{38}$$

with the STF properties $E_{\mu\nu}u^\nu = 0$, $E_{[\mu\nu]} = 0$, and $E^\mu{}_\mu = 0$.
The notation $\mathcal{L}^{\rm int}_{\rm rel}$ means to start from the Newtonian $\mathcal{L}^{\rm int}$, replace all time derivatives by covariant derivatives along A's worldline, and insert appropriate factors of the redshift, e.g. for a frequency $\omega_0 \to z\omega_0$, to make everything covariant and invariant under reparameterizations. Can confirm this using Effective field theory: consider all possible terms that respect the symmetries (general covariance, parity, and time reversal) and re-define variables to eliminate accelerations.

B. Tidally induced multipole moments

Now specify a particular model of the internal dynamics of body A to compute the time evolution of Q_{ij}.

First consider the limit of adiabatic tides. Relevant for $T_A^{\rm int} \sim \rho^{-1/2} \sim \sqrt{R^3/M_A} \ll T_{\rm orb} \sim \sqrt{r^3/M}$ (time variations in $\mathcal{E}_{ij}$) so that the body remains in equilibrium. Linear response:

$$Q^{ij,\,{\rm adiab}}(t) = -\lambda \mathcal{E}^{ij}_A, \tag{39}$$

where the proportionality constants λ (tidal deformability) are related to the body's tidal Love numbers k_2 and radius R by

$$\lambda_\ell = \frac{2}{(2\ell-1)!!} k_\ell R^{2\ell+1}. \tag{40}$$

The quantities λ_ℓ or k_ℓ are measures of the deformation that depend on the details of the body's internal structure and are computed based on an explicit description of the perturbed interior. In the adiabatic limit, the action depends only on the orbital variables and on λ (also HW4):

$$L^{\rm Newt}_{\rm adiab} = L_{\rm orbit} + \frac{3\lambda M_B}{2r^6}. \tag{41}$$

For Neutron star binaries: approximation of the adiabatic limit in a neutron star binary inspiral deteriorates at small separation. More accurate model of Q_{ij}: take into account that it primarily corresponds to the star's fundamental oscillation modes, with

$$\mathcal{L}^{\rm int}_{\rm Newt} = \frac{1}{4\lambda\omega_f^2}\left[\dot{Q}_{ij}\dot{Q}^{ij} - \omega_f^2 Q_{ij}Q^{ij}\right], \qquad \mathcal{L}^{\rm int}_{\rm covar} = \frac{z}{4\lambda z^2\omega_f^2}\left[\left(u^\beta\nabla_\beta Q_{\mu\nu}\right)\left(u^\gamma\nabla_\gamma Q^{\mu\nu}\right) - z^2\omega_f^2 Q_{\mu\nu}Q^{\mu\nu}\right] \tag{42}$$

1. Preview for next time: features of the Love numbers

Observational importance of λ or equivalently k_2 even in the solar system: influences the motion of satellites. Measured for Saturn's moon Titan by tracking the orbit of the Cassini spacecraft as it passed by. Info about Titan's interior: measured large value of k_2 incompatible with composition solely of rock, maybe a subsurface ocean. See Iess et al, Science **337**, 457 (2012).

The Love numbers (Newtonian) are defined by considering the response of body A to an external tidal potential, expanded as

$$U^{\rm tidal} = \sum_{l=2}^{\infty} \frac{1}{\ell!} r^{\ell} n_{<L>} \mathcal{E}_A^L. \tag{43}$$

The total gravitational potential outside body A in this situation is

$$U_{\rm ext} = U_A + U^{\rm tidal} = -\frac{M_A}{r} - \frac{n^{<ij>}}{2}\left[3\frac{Q^{ij}}{r^3} - \mathcal{E}^{ij} r^2\right]. \tag{44a}$$

$$\text{Adiabatic limit}: \; U_{\rm ext}^{\rm adiab} = -\frac{M_A}{r} + \frac{1}{2} r^2 \mathcal{E}^{ij} n^{<ij>}\left[1 + \frac{3\lambda}{r^5}\right] = -\frac{M_A}{r} + \frac{1}{2} r^2 \mathcal{E}^{ij} n^{<ij>}\left[1 + 2k_\ell \left(\frac{R}{r}\right)^{2\ell+1}\right]. \tag{44b}$$

The first set of terms inside the square brackets that grow as r^{ℓ} represent the external potential, and the last set of terms that decay as $r^{-(\ell+1)}$ characterize the body's response. Strategy is to compute the perturbed interior of the object, match this onto the asymptotic external potential, and from the falloff behavior identify the Love numbers.

In GR, must generalize the definitions but a similar strategy can be used. Will discuss this more next time.

II. LECTURE II: TIDAL LOVE NUMBERS AND NEUTRON STAR PHYSICS

Recall main points from last time: setup for near-zone matched asymptotic expansions.
Last time, discussed mainly the PN zone. Key point was that the orbital dynamics require only the multipole moments.

The multipole and tidal moments $\mathcal{E}_{ij}$ and Q_{ij} are defined in the vacuum region $R \ll r \ll \mathcal{R}$, where $\mathcal{R}$ is the radius of curvature of the source of the tidal perturbations that give rise to $\mathcal{E}_{ij}$ and R is the size of the body.

We considered details for Newtonian bodies, then discussed how the results can be promoted to GR. Specifically: potential outside a nearly spherical body (expanded around its CM worldline $\mathbf{z}_A$)

$$U_A(\mathbf{x}) = -\frac{M_A}{r_A} - \frac{3}{2}\frac{Q_{ij}n^{<ij>}}{r_A^3} + \ldots \qquad r_A = |\mathbf{x} - \mathbf{z}_A| \tag{45}$$

$$= -\frac{M_A}{r_A} - \sum_{m=-2}^{2}\frac{Q_{\ell m}Y_{\ell m}}{r_A^3} + \ldots. \tag{46}$$

In a binary, acceleration of A requires expansion of B's potential around z_A^i:

$$M_A\ddot{z}_A^i = -M_A\left(\partial_i U_B\right)_{z_A} - \frac{1}{2}Q_{ij}\mathcal{E}_{ijk} + \ldots \tag{47}$$

Tidal moments:

$$\mathcal{E}_{ij} = \left(\partial_i\partial_j U_B\right)_{z_A} \tag{48}$$

In the binary's CM, for B a point mass, the eoms can be derived from the action

$$S = S_{\rm orbit} + \int dt\left[-\frac{1}{2}Q_{ij}\mathcal{E}_{ij} + \mathcal{L}^{\rm int}(Q_{ij}, \dot{Q}_{ij}, \ldots)\right] \tag{49}$$

mentioned how this can be promoted to GR, e.g. in the body's local rest frame in Fermi normal coordinates

$$\mathcal{E}_{ij} = R_{0i0j} \tag{50}$$

Consider a body that would be spherical in the absence of a companion. Multipole moments in a binary are tidally induced, for small tidal fields and slow variation compared to body's dynamical timescale they are of the form

$$Q_{ij}^{\rm adiab} = -\lambda\mathcal{E}_{ij} \qquad \lambda = \text{tidal deformability} \tag{51}$$

characterizes the internal structure of the body.
Action reduces to

$$S = S_{\rm orbit} + \frac{\lambda}{4}\int dt\mathcal{E}_{ij}\mathcal{E}_{ij}, \tag{52}$$

depends only on the orbital variables (recall $\mathcal{E}_{ij} = -\partial_i\partial_j M_B/r$ for a point-mass companion, i.e. $\mathcal{E}_{ij} = -3M_B n^{<ij>}/r^3$).
To compute λ requires details of the perturbed interior. Need to look at the strong-field region. In this zone, consider a relativistic CO that is spherical and in equilibrium when it is in isolation. Now consider what happens when it is in a generic tidal environment, e.g. due to a distant companion that is not included in the zone.

Newtonian case in this zone, outside the body itself:

$$U_{\rm total} = U_{\rm body} + U_{\rm tidal} \tag{53}$$

$$= -\frac{M_A}{r} - \frac{3}{2}Q_{ij}\frac{n^{<ij>}}{r^3} + \mathcal{O}(r^{-4}) + \frac{1}{2}r^2 n^{<ij>}\mathcal{E}_{ij} + \mathcal{O}(r^3), \quad r = \text{distance from body's CM} \tag{54}$$

$$= -\frac{M_A}{r} + \frac{1}{2}n^{<ij>}r^2\left[-\frac{3Q_{ij}}{r^5} + \mathcal{E}_{ij}\right] \tag{55}$$

$$= -\frac{M_A}{r} + \frac{1}{2}n^{<ij>}r^2\mathcal{E}_{ij}\left[\frac{3\lambda}{r^5} + 1\right] \tag{56}$$

Can see that λ has units of (length)^5, conventionally:

$$\lambda = \frac{2}{3}k_2 R^5, \qquad k_2 = \text{dimensionless Love number} \tag{57}$$

In GR: use g_{tt} in ACMC coordinates instead of U. E.g. Schwarzschild exterior:

$$g_{tt}^{\rm Schw} = -\left(1+2U_{\rm eff}\right), \qquad U_{\rm eff} = -\frac{M}{r} \tag{58}$$

Same definition of λ applies for relativistic definitions of $\mathcal{E}_{ij} = R_{0i0j}$ (local frame) and Q_{ij} (from asymptotic spacetime geometry at large distances).

So far, have only discussed potentials *outside* the body. To determine λ requires details of the perturbed *interior*.

1. *Equilibrium configuration*

First, must solve for the spacetime of the equilibrium configuration.

$$\text{interior}: \;\; G_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad \text{exterior} \;\; G_{\mu\nu} = 0. \tag{59}$$

Solutions must *match* at the surface of the object.

For a spherically symmetric object the metric in Schwarzschild coordinates is

$$ds_0^2 = -e^{\nu(r)}dt^2 + e^{\gamma(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{60}$$

ν and γ have to be determined.

Strategy: pick your favorite stress-energy tensor, e.g. perfect fluid:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{61}$$

where p and ρ are the NS pressure and energy density, u^μ is the fluid's four-velocity given by

$$u^\mu = (u^t, 0, 0, 0) \text{ (rest frame)}, \qquad \to \qquad u_\mu u^\mu = -1 \qquad \Rightarrow u^t = e^{-\nu/2}. \tag{62}$$

Next, solve Einsteins equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ in the interior. Yields the Tolman-Oppenheimer-Volkoff equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho\,, \qquad \frac{d\nu}{dr} = 2\frac{4\pi r^3 p + m}{r(r-2m)}\,, \qquad \frac{dp}{dr} = -\frac{(4\pi r^3 p + m)(\rho + p)}{r(r-2m)}\,, \tag{63}$$

where $m(r)$ is defined by

$$m(r) \equiv \frac{\left[1 - e^{-\gamma(r)}\right] r}{2}\,. \tag{64}$$

To fully solve these Equations requires specifying an EoS, $p = p(\rho)$.

Outside the star, solve $G_{\mu\nu} = 0$. Already matches onto the interior solution, with $m(r)$ becoming the body's constant ADM mass M.

2. *Tidal perturbations*

Work in the body's moving frame, with its origin at the center-of-mass and in the body zone. Consider only the $\ell = 2$ quadrupolar case .

In GR, there are two types of tidal fields: The gravitoelectric field $\mathcal{E}_{ij}$ raises mass multipoles, The gravitomagnetic field $\mathcal{B}_{ij}$ induces current multipoles. These tensors are not determined by the field equations restricted to the body zone. but can be related to components of the Weyl tensor evaluated at the edge of the body zone, info coming from outside the zone. Computed in GR, easiest to work in the limit where the PN approximation applies to the orbital zone. There is an overlap between the body zone and the post-Newtonian zone so the info can be determined.

Will focus on *electric-type*. Need to solve for the body's perturbed configuration and spacetime.

$$\text{metric}: \qquad ds^2 = ds_0^2 + h_{\mu\nu} dx^\mu dx^\nu . \tag{65}$$

The perturbation $h_{\mu\nu}$ has contributions from exterior tidal and body.

$$\text{interior}: \ \ \delta G_{\mu\nu} = 8\pi \delta T_{\mu\nu}, \qquad \text{exterior} \ \ \delta G_{\mu\nu} = 0. \tag{66}$$

Must be continuous across the surface of a material body $r = R$. Matching exterior and interior solutions determines the Love numbers.

The metric perturbation in the Regge-Wheeler gauge can be decomposed into spherical harmonics characterized by the mode integers (ℓ, m) and by a parity π which can be either $(-1)^\ell$ or $(-1)^{\ell+1}$. For small perturbations the (ℓ, m, π) modes are decoupled and the harmonics can be analyzed individually.
Consider even-parity, static (zero-frequency) case appropriate for the $\ell = 2$ adiabatic tidal perturbation:

$$h_{\alpha\beta} dx^\alpha dx^\beta = \sum_{m=-2}^{2} Y_{2m}(\theta,\varphi) \left[e^\nu h_0(r) dt^2 + e^\gamma h_2(r) dr^2 + r^2 k(r)(d\theta^2 + r\sin^2\theta d\varphi^2) \right], \tag{67}$$

Stress-energy tensor will also be linearly perturbed, similar decomposition into spherical harmonics. Another conditions is the normalization of the four-velocity (including perturbations). Next step: substitute into the field equations $\delta G_\mu^\nu = 8\pi \delta T_\mu^\nu$ and the stress-energy conservation equations $\nabla_\mu T^{\mu\nu} = 0$. Combine various components of these equations to eliminate variables. Find that everything can be summarized in the master equation for the perturbation to g_{tt}:

$$\begin{aligned} 0 = {} & \frac{d^2 h_0}{dr^2} + \left\{ \frac{2}{r} + e^\gamma \left[\frac{2M}{r^2} + 4\pi r(p-\rho) \right] \right\} \frac{dh_0}{dr} \\ & + \left\{ e^\gamma \left[-\frac{6}{r^2} + 4\pi(\rho+p)\frac{d\rho}{dp} + 4\pi(5\rho + 9p) \right] - \left(\frac{d\nu}{dr} \right)^2 \right\} h_0 \,. \end{aligned} \tag{68}$$

Need to solve this numerically in the interior of the object. Outside the NS, the metric perturbation reduces to the general form

$$h_0 = a_Q Q_{22}(x) + a_P P_{22}(x)\,, \tag{69}$$

where $x \equiv r/M_A - 1$ and $P_{22}(x)$ and $Q_{22}(x)$ are the normalized associated Legendre functions of the first and second kinds respectively. The normalization is such that for $x \to \infty$ the asymptotic forms are $P_{22}(x) \sim x^2$ and $Q_{22}(x) \sim x^{-3}$. The constants a_P and a_Q are determined by matching the logarithmic derivative of the interior and exterior solutions,

$$y_\ell \equiv \frac{r}{h_0}\frac{dh_0}{dr}\,, \tag{70}$$

at the NS surface. Comparing the asymptotic behavior of h_0 with the definition of Q_{ij} and $\mathcal{E}_{ij}$ in the asymptotic metric and eliminating $\mathcal{E}_{ij}$ by taking the ratio (70) gives $k_2 = k_2(\mathrm{EOS}, \mathrm{mass})$ for microphysical EoS for which $R = R(\mathrm{EoS}, M)$.

BHs: no interior solution. Perturbations should be regular at the event horizon, this requires that $h_{\mu\nu}^{\mathrm{body}} = 0$ so that the Love numbers k_ℓ vanish.

A body also has other types of Love numbers: higher multipoles, magnetic-type (induced current moments by magnetic part of the external Weyl tensor), rotational (rotationally-induced quadrupole by centrifugal potential $= \Omega^2$ in the Newtonian limit), surficial/shape, ...

Shape Love numbers characterize the deformation of the object's surface. In GR the surface displacement is not an invariant quantity: instead, one must consider the perturbation in the surface's intrinsic curvature. Perturbed Ricci scalar of the two-dimensional surface:

$$\mathcal{R} = \frac{1}{R^2}\left(2 + \delta\mathcal{R}\right), \qquad \delta\mathcal{R} = -2\sum_{\ell=2}^{\infty} \frac{\ell+2}{\ell} h_\ell \frac{R^{\ell+1}}{M} \mathcal{E}_L n^L, \tag{71}$$

where h_ℓ, the *surficial Love numbers*, are dimensionless and scale-free measures of the deformation that depend on the details of internal structure. Also, $2/R^2$ is the result for a sphere of radius R. BH case: $\delta\mathcal{R}$ defined by considering horizon and using its null generators to establish a coordinate system – associated Ricci curvature is as above with $R = 2M$. Results for any object

$$h_\ell = \Gamma_1 + 2\Gamma_2\, k_\ell^{\mathrm{el}}, \qquad h_\ell^{\mathrm{Newt}} = 1 + 2k_\ell^{\mathrm{N}} \tag{72}$$

For BHs: $M/R = 1/2$ and $k_\ell^{\rm el} = 0$, and

$$h_\ell^{\rm BH} = \frac{\ell+1}{2(\ell-1)}\frac{\ell!^2}{(2\ell)!}. \tag{73}$$

A. NS basics

1. $>$2000 observed
2. mass $\sim$ sun but radii 10^5 times smaller, enormous gravity, central density up to $\sim 10\rho_0$ *normal nuclear density*, mass density of nucleon matter in heavy atomic nuclei)
3. matter under extreme conditions, far extrapolations from theories of the structure of matter tested in labs
4. must describe huge density range: few $g{\rm cm}^{-3}$ near surface, where the pressure is small, to more than $10^{15} g{\rm cm}^{-3}$ at the center, where the pressure exceeds $10^{36}{\rm dyncm}^{-2}$

B. some historical milestones

1. Landau 1932, speculation on dense stars: maximum mass of WDs, stars with greater mass "possess regions in which the laws of quantum mechanics are violated" – questionable. Also speculated that "the density of matter becomes so great that atomic nuclei come into close contact, forming one gigantic nucleus".
2. Chadwick 1932: discovery of the neutron
3. Baade and Zwicky 1933/34 (prediction): explanation of enormous energy release in SN explosions: "With all reserve we advance the view that SNs represent the transition from ordinary stars to NSs, which in their final stages consist of extremely closely packed neutrons"
4. Oppenheimer-Volkoff (1939): main GR equations of structure (concurrently with Tolman 1939), first calculation of equilibrium configurations in GR for assuming the matter is a noninteracting strongly degenerate relativistic gas of neutrons. Found that stable static neutron stars have a maximum mass of $M_{\rm max} \sim 0.7M_\odot$, lower than the Chandrasekhar mass limit of WDs $\sim 1.4M_\odot$, problematic for the formation of neutron stars from ordinary stars. Understood the simplicity of noninteracting neutrons and discussed a possible repulsive component of neutron-neutron interaction that may stiffen the EOS and increase the maximum mass. Now have direct astrophysical evidence of strong repulsive interaction in dense matter at supranuclear density since NS masses $> 0.7M_\odot$.
5. *EOS of dense matter:* Wheeler and collaborators in the 1950s constructed a model of a neutron star crust and calculated the EOS of neutron star cores composed of free neutrons, protons, and electrons in beta equilibrium.
   Cameron (1959) emphasized the importance of nuclear forces for the neutron star structure, showed that they lead to $M_{\rm max} \sim 2M_\odot$, making the formation of neutron stars in supernova explosions quite realistic.
   Neutron star cores may contain not only neutrons, protons and electrons, but also other particles, such as muons, mesons, and hyperons. Hyperons: Cameron (1959) and Salpeter (1960); in the 1960s some EOSs of hyperonic matter and quark cores, superfluidity
6. *X-ray radiation from NSs.* Expectations at the beginning of the 1960s to discover neutron stars by detecting the thermal radiation from their surfaces (cooling through thermal emission of photons from surfaces and neutrinos from interiors). Seminal paper by Chiu & Salpeter (1964). By 1968, about twenty compact X-ray sources had been observed but their association with neutron stars seemed to be not convincing.
7. 1967, Jocelyn Bell discovered a weak variable radio source that emitted strictly periodic pulses, initial suspicion that the signals were of artificial origin, several other pulsars discovered soon thereafter.
8. Gold (1968): pulsars are rotating magnetized neutron stars (also Pacini 1967): strong dipole magnetic field could transform rotational energy into electromagnetic radiation and accelerate particles to high energies. Most important observational evidence: discovery of the Crab pulsar in 1968. Its pulsation (spin) period appeared to be very short, P = 33 ms. White dwarfs could not sustain such a rapid rotation: they would be destroyed by centrifugal forces.

1. Considerations about the EoS of dense matter

Main-sequence stars resist gravitational collapse mainly through thermal pressure, WDs: electron degeneracy pressure, NSs: degeneracy pressure+nuclear forces.

$$\text{uncertainty principle} \qquad \Delta x \Delta p \gtrsim \hbar \tag{74}$$

If something is localized to a region of size Δx its momentum must be at least $\hbar/\Delta x$.

$$\text{dense environment}: \quad p_F \sim \hbar/\Delta x \qquad \text{momentum due to confinement} \tag{75}$$

associated energy due to the confinement. Increasing this Fermi energy E_F acts as a pressure, sometimes called degeneracy pressure.

Consider particles of rest mass m, number density n. So $\Delta x \sim n^{-1/3}$.
Relativistic regime when $E_F > mc^2$.

$$\text{non} - \text{relativistic particles}: \quad E_F \sim \frac{p_F^2}{2m} \sim \frac{n^{2/3}}{m} \tag{76}$$

$$\text{relativistic particles}: \quad E_F \sim p_F c \sim \frac{n^{1/3}}{m} \tag{77}$$

For electrons, this relativistic crossover happens at $\rho \sim 10^6 g\text{cm}^{-3}$ (fully ionized plasma)
For protons and neutrons, this happens at $\rho \sim 6 \times 10^{15} g\text{cm}^{-3}$, so in a NS ($\rho \lesssim 10^{15}$) the nucleons are mildly relativistic.

Significance: energetic cost of adding another particle is not just mc^2 but rather $mc^2 + E_F$.

Example: electrons at $10^6 g\text{cm}^{-3}$: $E_F \sim m_e c^2 \sim 0.5$MeV. At 10^7, $E_F \sim n^{1/3} \sim 10^{1/3} \times 0.5\text{MeV} \sim 1\text{MeV}$ becomes unfavorable at high density.

Can translate E_F into temperature for ideal gas $T_F \sim E_F/k_B \sim n^{2/3}/m$
$T < T_F$: degenerate matter, rails against Heisenberg (e.g. $p = nk_BT$ for ideal gas, $\Delta p \sim \sqrt{mk_BT}$)

$T_F \sim 10^9$K for electrons in a White Dwarf (WD), generally $T_{\text{WD}} \sim 10^6 - 10^7$K so electrons are degenerate but ions are 2000 times more massive, remain nondegenerate
$m_n \sim 939.6$MeV, $m_p \sim 938$MeV, $m_e \sim 0.5$ MeV

Consideration for NSs:

1. free space neutrons are unstable since $m_p + m_e$ is ~ 1MeV less that m_n, so energetically favorable for n to decay.
2. If $m_p + m_e + E_F > m_n$ (high density) energetically favorable to combine into n, atoms become more neutron rich
3. At $\sim 4 \times 10^{11} g\text{cm}^{-3}$ becomes favorable to have free neutrons ("neutron drip")
4. At even higher densities, neutron Fermi energy is large enough that it's favorable to have other particles appear.
   If new particles appear, the energy "cost" of going to higher densities is less than otherwise expected, since energy goes into the new particles instead of more neutrons. That means that it's easier to compress the star since the energy doesn't rise as much when compressed as would've been expected.