① • Let $p^\mu \equiv \frac{dx^\mu}{d\lambda}$ be the 4-momentum of a massless particle.

Geodesic eq.

$$p^\alpha \nabla_\alpha p_\mu = 0 = \underbrace{p^\alpha \partial_\alpha p_\mu}_{\frac{d}{d\lambda} p_\mu(\lambda)} - \underbrace{\Gamma^\nu_{\alpha\mu} p_\nu p^\alpha}_{\frac{1}{2} g^{\nu\beta}(\partial_\alpha g_{\beta\mu} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\alpha\mu}) p_\nu p^\alpha =}$$

$$= \frac{1}{2}\left[\cancel{(\partial_\alpha g_{\beta\mu}) p^\alpha p^\beta} + (\partial_\mu g_{\beta\alpha}) p^\alpha p^\beta - \cancel{(\partial_\alpha g_{\beta\mu}) p^\alpha p^\beta}\right] =$$

$$= \frac{1}{2} p^\alpha p^\beta \partial_\mu g_{\alpha\beta}$$

$$\Rightarrow \frac{dp_\mu}{d\lambda} = \frac{1}{2} p^\alpha p^\beta \partial_\mu g_{\alpha\beta}$$

Let $\mu = 0$: $\frac{dp_0}{d\lambda} = \frac{1}{2} p^\alpha p^\beta \underbrace{\partial_0 g_{\alpha\beta}}_{\text{0 by assumption}} = 0 \Rightarrow p_0$ conserved along geodesic

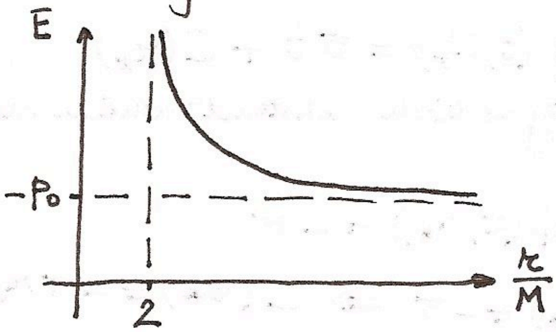
• Observer at rest: $u^\mu_{obs} = (u^t, 0, 0, 0)$

$$u^\mu_{obs} u_\mu^{obs} = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)(u^t)^2 = -1 \Rightarrow u^t = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$$

Let the graviton have 4-momentum p^μ. The observer measures (see 5.100 in Carroll):

$$E = -g_{\mu\nu} u^\mu_{obs} p^\nu = -g_{tt} u^t p^t = -u^t p_0 = -\frac{p_0}{\sqrt{1 - \frac{2M}{r}}} \quad \text{when } p_0 = \text{const.}$$

As the graviton travels to larger r, the measured E is lower and lower, i.e. more and more redshifted.



• Geometrical optics. Radial rays ($d\Omega = 0$). In tortoise coords. the metric reads (see 5.109 in Carroll):

$$ds^2 = \left(1-\frac{2M}{r}\right)\left(-dt^2 + dr_*^2\right) \qquad (\#I)$$

Let

$$\bar{h}_{\mu\nu} = A(t, r_*)\, e^{i\phi(t, r_*)}\, \epsilon_{\mu\nu}$$

when $\epsilon_{\mu\nu}$ is the polarization tensor, s.t. $\epsilon^*_{\mu\nu}\epsilon^{\mu\nu} = 1$. In the eikonal approximation

$$g_{\mu\nu} K^\mu K^\nu = 0 \qquad \text{(Maggiore 1.187)}$$

where $K_\mu \equiv \partial_\mu \phi(t, r_*)$. Then

$$g^{tt}(\partial_t \phi)^2 + g^{r_* r_*}(\partial_{r_*}\phi)^2 = -\frac{1}{1-\frac{2M}{r}}(\partial_t\phi)^2 + \frac{1}{1-\frac{2M}{r}}(\partial_{r_*}\phi)^2 = 0$$

$$\Rightarrow \partial_t \phi(t, r_*) = \pm\, \partial_{r_*}\phi(t, r_*) \qquad (\#II)$$

If $\phi = \phi(t - r_*)$ then eq. (#II) is satisfied since

$$\partial_t \phi(t-r_*) = \partial_{t-r_*}\phi(t-r_*) =$$

$$= -\partial_{r_*-t}\phi(t-r_*) =$$

$$= -\partial_{r_*}\phi(t-r_*)$$

Also K_μ must obey the geodesic eq. (see 1.188 in Maggiore)

$$K^\alpha \bar{D}_\alpha K_\mu = 0$$

Since the metric (#I) does not depend on time $(\partial_0 g_{\mu\nu} = 0)$ from the results of the 1st part we have:

$$K_0 = \text{const.}$$

This means that

$$K_0 = \partial_t \phi = \text{const.} \equiv \sigma$$

$$\Rightarrow \phi(t, r_*) = \sigma t + C(r_*)$$

with $\sigma = $ const. and C an integration constant which depend only on r_*. But from (#II)

$$\partial_{r_*}\phi(t, r_*) = -\partial_t \phi(t, r_*) = -\sigma$$

$$\Rightarrow C'(r_*) = -\sigma \Rightarrow C(r_*) = -\sigma r_* + \delta$$ where δ is a pure constant of integration

$$\Rightarrow \phi(t, r_*) = \sigma(t - r_*) + \delta \checkmark$$

• Then from the 2^{nd} part of the problem, noting that $\hbar k_\mu$ is the 4-momentum of the propagating graviton:

$$E = -\frac{\hbar K_0}{\sqrt{1-\frac{2M}{r}}} = -\frac{\hbar \partial_t \phi}{\sqrt{1-\frac{2M}{r}}} = -\frac{\hbar \sigma}{\sqrt{1-\frac{2M}{r}}}$$ (in geometric units $G=c=1$)

The frequency ω of the graviton is related to the energy E:

$$E = \hbar\omega \Rightarrow \omega = -\frac{\sigma}{\sqrt{1-2M/r}}$$

• The waveform has amplitude $\propto 1/r$ and phase $\phi = \sigma(t-r_*)+\delta$

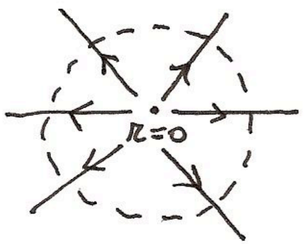
$$h = \mathrm{Re}\left\{\frac{\tilde{A}}{r}\exp[i\phi]\right\}$$ where $\tilde{A}$ is a const.

$$= \frac{\tilde{A}}{r}\cos[\sigma(t-r_*)+\delta]$$

To show that the scalar amplitude A goes like $1/r$, we can use the conservation of graviton flux of geometric optic, which can be written as

$$\mathcal{A} A^2 = \text{const}$$

where $\mathcal{A}$ is the cross sectional area of a bundle of rays. See MTW exercise 22.13. Here we take rays moving radially in all directions, away from $r=0$. Then this bundle of rays has a spherical surface as cross section:

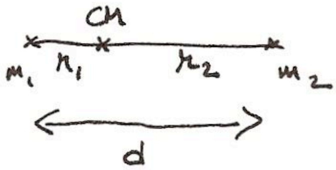


$$\mathcal{A} = 4\pi r^2$$

$$\Rightarrow 4\pi r^2 A^2 = \text{const.}$$

$$\Rightarrow A \propto \frac{1}{r}$$

(4)



$$\omega^2 = \frac{G(m_1+m_2)}{d^3} \quad , \quad m_1 r_1 = m_2 r_2$$

$$d = \frac{m_1+m_2}{m_2} r_1$$

$$V_1 = \omega r_1 = \frac{d m_2 \omega}{m_1+m_2} \quad , \quad V_2 = \frac{d\, m_1 \omega}{m_1+m_2}$$

- Explosion w/out kick $\Rightarrow$ velocities are unaffected.

Only change is $m_2 \longrightarrow m_2 - \Delta \quad (\Delta > 0)$

Escape velocity for the new system for m_1 is:

$$\frac{1}{2} m_1 V_{\infty,1}^2 = \frac{G m_1 (m_2 - \Delta)}{d} \Rightarrow V_{\infty,1}^2 = \frac{2G(m_2-\Delta)}{d}$$

But m_1 still has velocity $v_1 = \dfrac{m_2 d \omega}{m_1+m_2}$

$$v_1^2 = \frac{m_2^2 d^2 \omega^2}{(m_1+m_2)^2} = \frac{m_2^2 \cancel{d^2}}{(m_1+m_2)^{\cancel{2}}} \frac{G\cancel{(m_1+m_2)}}{\cancel{d^3}} =$$

$$= \frac{G m_2^2}{d(m_1+m_2)}$$

The binary disrupts if $V_1 > V_{\infty,1}$. If $m_1 = 1.4\, M_\odot$
$m_2 = 10\, M_\odot$
$\Delta = 8.6\, M_\odot$

then

$$\left.\begin{aligned} V_{\infty,1}^2 &= \frac{2G}{d} \times 1.4\, M_\odot = 2.8\, \frac{GM_\odot}{d} \\ v_1^2 &= \frac{G}{d} \frac{100}{11.4} M_\odot = 8.8\, \frac{GM_\odot}{d} \end{aligned}\right\} V_1 > V_{\infty,1}\text{: the binary disrupts}$$

- Explosion w/ kick: the velocities will be affected. In the CM frame we can work w/ an effective particle $\mu = \dfrac{m_1 m_2}{m_1+m_2}$ orbiting a particle $M = m_1+m_2$ at a distance d, with velocity v_i

Before the SN: $\omega^2 = \dfrac{GM}{d^3}$, $V_i = \omega d$

After the SN: $V_\infty^2 = \dfrac{2G(M-\Delta)}{d}$. If $V_f = |\vec{V}_i + \vec{V}_{kick}| > V_\infty$ then the binary disrupts, otherwise it remains bound on eccentric orbit.

② $(1.4+1.4)M_\odot$ on circular orbit.

- $P = 7.75\,h$. Derive eq. (4) first.

Orbital energy is $E = -\frac{Gm_1m_2}{2R}$ where R is the orbital radius.

Let $\omega = \frac{2\pi}{P}$ be the orbital freq of the binary.

Kepler's law is $\omega^2 = \frac{G(m_1+m_2)}{R^3}$. For a quasi-circular inspiral we can assume that Kepler's law holds throughout the inspiral.

$$\dot R = -\frac{2}{3}\frac{G(m_1+m_2)}{\omega^{2/3}}\frac{\dot\omega}{\omega} = -\frac{2}{3}\frac{R\dot\omega}{\omega}, \qquad \dot\omega = -\frac{2\pi}{P^2}\dot P$$

$$\dot E = \frac{Gm_1m_2}{2R^2}\dot R = -\frac{Gm_1m_2}{3R}\frac{\dot\omega}{\omega} = -\frac{Gm_1m_2}{3}\frac{\omega^{2/3}}{G^{1/3}(m_1+m_2)^{1/3}}\frac{\dot\omega}{\omega} =$$

$$= -\frac{G^{2/3}}{3}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\frac{\dot\omega}{\omega^{1/3}} = -\frac{G^{2/3}}{3}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\left(-\frac{2\pi}{P}\frac{\dot P}{P}\right)\frac{P^{1/3}}{(2\pi)^{1/3}} =$$

$$= +\frac{(2\pi)^{2/3}G^{2/3}}{3}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\frac{\dot P}{P^{5/3}}$$

$$\underbrace{\dot E = -P_{GW}}_{\text{balance eq.}} = -\frac{32}{5}\frac{c^5}{G}\frac{G^{10/3}}{\cancel{2^{10/3}}}\frac{M_c^{10/3}}{c^{10}}(\cancel{2}\omega)^{10/3}, \text{ where } M_c = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

quadrupole formula (Maggiore 4.12)

$$= -\frac{32}{5}\frac{G^{7/3}}{c^5}\frac{(m_1m_2)^2}{(m_1+m_2)^{2/3}}\frac{(2\pi)^{10/3}}{P^{10/3}}$$

Then equating:

$$\frac{\cancel{(2\pi)^{2/3}}\cancel{G^{2/3}}}{3}\frac{\cancel{(m_1m_2)}}{\cancel{(m_1+m_2)^{1/3}}}\dot P/\cancel{P^{5/3}} = -\frac{32}{5}\frac{G^{\cancel{7/3}\,5/3}}{c^5}\frac{(m_1m_2)^{\cancel{2}}}{(m_1+m_2)^{\cancel{2/3}\,1/3}}\frac{(2\pi)^{\cancel{10/3}\,8/3}}{P^{\cancel{10/3}\,5/3}}$$

$$\Rightarrow \boxed{\dot P = -\frac{96}{5}\frac{G^{5/3}}{c^5}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\frac{(2\pi)^{8/3}}{P^{5/3}}} \quad (\#\#)$$

Eq. (4) in the problem is

$$\dot P = -\frac{192\pi}{5}\frac{m_1m_2}{(m_1+m_2)^2}\left(\frac{2\pi G(m_1+m_2)}{c^3P}\right)^{5/3} = -\frac{\overset{96\,(2\pi)}{192\pi}}{5}\frac{m_1m_2}{(m_1+m_2)^2}\frac{(2\pi)^{5/3}G^{5/3}(m_1+m_2)^{5/3}}{c^5P^{5/3}}$$

$$= -\frac{96}{5}\frac{G^{5/3}}{c^5}\frac{m_1m_2}{(m_1+m_2)^{1/3}}\frac{(2\pi)^{8/3}}{P^{5/3}} \quad \text{in agreement with } (\#\#)$$

Let's put $\dot P = -AP^{-5/3}$, where A collects all the factors in front of $P^{-5/3}$. Let $\tau \equiv t_c - t$, where t_c is the time of coalescence; here $P(t_c) = 0$

$$P^{5/3}\, dP = -A\, dt = A\, d\tau$$

$$\frac{P^{8/3}(\tau) - P^{8/3}(0)}{8/3} = A\tau$$

$$\Rightarrow \frac{3}{8A} P^{8/3}(\tau) = \tau$$

$$\boxed{\tau = \frac{3}{8}\,\frac{5}{96}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\,\frac{1}{(2\pi)^{8/3}}\,P^{8/3}} = \quad (*)$$

$$= \frac{5}{256}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\left(\frac{P}{2\pi}\right)^{8/3} =$$

$$= \frac{5}{256}\,\frac{(3\times10^8)^5}{(6.67\times10^{-11})^{5/3}}\,\frac{(2.8\,M_\odot)^{1/3}}{1.4^2\,M_\odot^2}\left(\frac{7.75\times60\times60}{2\pi}\right)^{8/3} s = \quad \text{use } M_\odot = 2\times10^{30}\,kg$$

$$= 5.2\times10^{16}\,s = 1.65\times10^9\,yrs = 1650\,Myr \quad \text{time to coalescence}$$

$$\dot{P} = -\frac{96}{5}\,\frac{(6.67\times10^{-11})^{5/3}}{(3\times10^8)^5}\,\frac{1.4^2\,M_\odot^2}{(2.8\,M_\odot)^{1/3}}\,\frac{(2\pi)^{8/3}}{(7.75\times60\times60)^{5/3}} = \quad \text{it's dimensionless}$$

$$= -2\times10^{-13}\,\frac{s}{s} = -2\times10^{-13}\,\frac{10^6\,\mu s}{\frac{yr}{365\times24\times3600}} =$$

$$= -6.3\,\frac{\mu s}{yr}$$

. Rewrite $(*)$ in terms of R.

$$\left(\frac{2\pi}{P}\right)^2 = \frac{G(m_1+m_2)}{R^3} \Rightarrow P = (2\pi)\,\frac{R^{3/2}}{G^{1/2}(m_1+m_2)^{1/2}}$$

$$\tau = \frac{5}{256}\,\frac{c^5}{G^{5/3}}\,\frac{(m_1+m_2)^{1/3}}{m_1 m_2}\,\frac{R^4}{G^{4/3}(m_1+m_2)^{4/3}} =$$

$$= \frac{5}{256}\,\frac{c^5}{G^3}\,\frac{R^4}{m_1 m_2 (m_1+m_2)} \quad \text{(consistent with 4.26 in Maggiore)}$$

$$\Rightarrow 2R = 2\left[\frac{256}{5}\,\frac{G^3}{c^5}\,m_1 m_2 (m_1+m_2)\,\tau\right]^{1/4} = \quad \text{let } \tau = 10^{10}\,yr$$

$$= 2\left[\frac{256}{5}\,\frac{(6.67\times10^{-11})^3}{(3\times10^8)^5}\,1.4^2 M_\odot^2\;2.8M_\odot\;10^{10}\times365\times24\times3600\right]^{1/4} m =$$

$\cong 6.1\times10^9\,m \Rightarrow$ in order to have coalescence in less than $10^{10}\,yr$, the 2 stars must be closer then $\sim 6.1\times10^9\,m$

• Rewrite (*) in terms of f_{GW}:

$$\left.\begin{aligned} \omega_{GW} &= 2\omega \\ f_{GW} &= \frac{\omega_{GW}}{2\pi} \end{aligned}\right\} \quad \omega = \frac{\omega_{GW}}{2} = \pi f_{GW}$$

$$\tau = \frac{5}{256} \frac{c^5}{G^{5/3}} \frac{(m_1+m_2)^{1/3}}{m_1 m_2} \omega^{-8/3} =$$

$$= \frac{5}{256} \frac{c^5}{G^{5/3}} \frac{(m_1+m_2)^{1/3}}{m_1 m_2} \frac{1}{(\pi f_{GW})^{8/3}} = 4.62 \times 10^5 \, s \left(\frac{1\,Hz}{f_{GW}}\right)^{8/3}$$

$$\tau\Big|_{40\,Hz} \cong 25\,s$$

$$\tau\Big|_{100\,Hz} \cong 2\,s$$

• $r = 1\,\mathrm{Mpc} = 3.1 \times 10^{22}\,\mathrm{m}$

$$M_c = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{1.4}{2^{1/5}} M_\odot$$

From Maggiore 4.20:

$$f_{GW}(\tau) = 134\,\mathrm{Hz} \left(\frac{1.21\,M_\odot}{M_c}\right)^{5/8} \left(\frac{1s}{\tau}\right)^{3/8} =$$

$$\cong 134\,\mathrm{Hz} \left(\frac{1s}{t_c - t}\right)^{3/8}$$

The plot range is

$$\left.\begin{aligned} &100\,\mathrm{Hz} \stackrel{!}{=} 134\,\mathrm{Hz} \left(\frac{1s}{\tau}\right)^{3/8} \Longrightarrow \tau = 2.18\,s \\ &300\,\mathrm{Hz} \stackrel{!}{=} 134\,\mathrm{Hz} \left(\frac{1s}{\tau}\right)^{3/8} \Longrightarrow \tau = 0.12\,s \end{aligned}\right\} \quad 0.12\,s \leqslant \tau \leqslant 2.18\,s$$

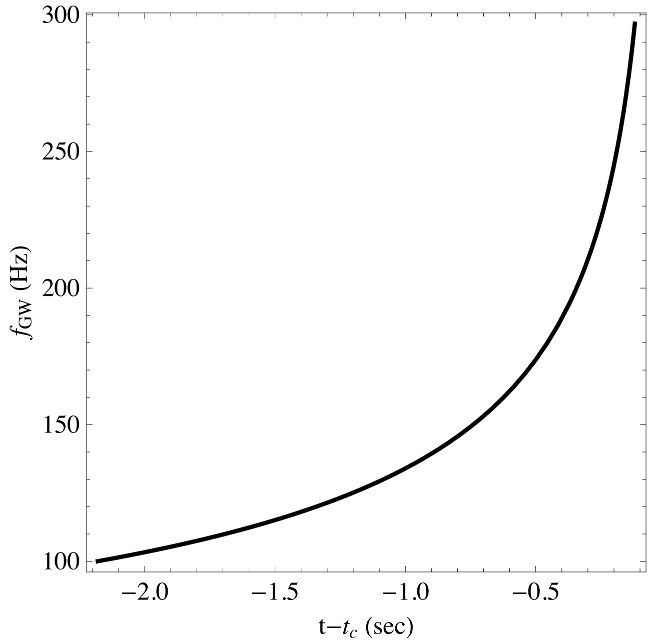
From Maggiore 4.31, using $i = 0$:

$$h_+(\tau) = \frac{1}{r} \left(\frac{GM_c}{c^2}\right)^{5/4} \left(\frac{5}{c\tau}\right)^{1/4} \cos\left[-2\left(\frac{5GM_c}{c^3}\right)^{-5/8} \tau^{5/8} + \Phi_0\right] =$$

$$= 4.3 \times 10^{-21} \left(\frac{1s}{t_c - t}\right)^{1/4} \cos\left[1340\left(\frac{t_c - t}{1s}\right)^{5/8}\right]$$

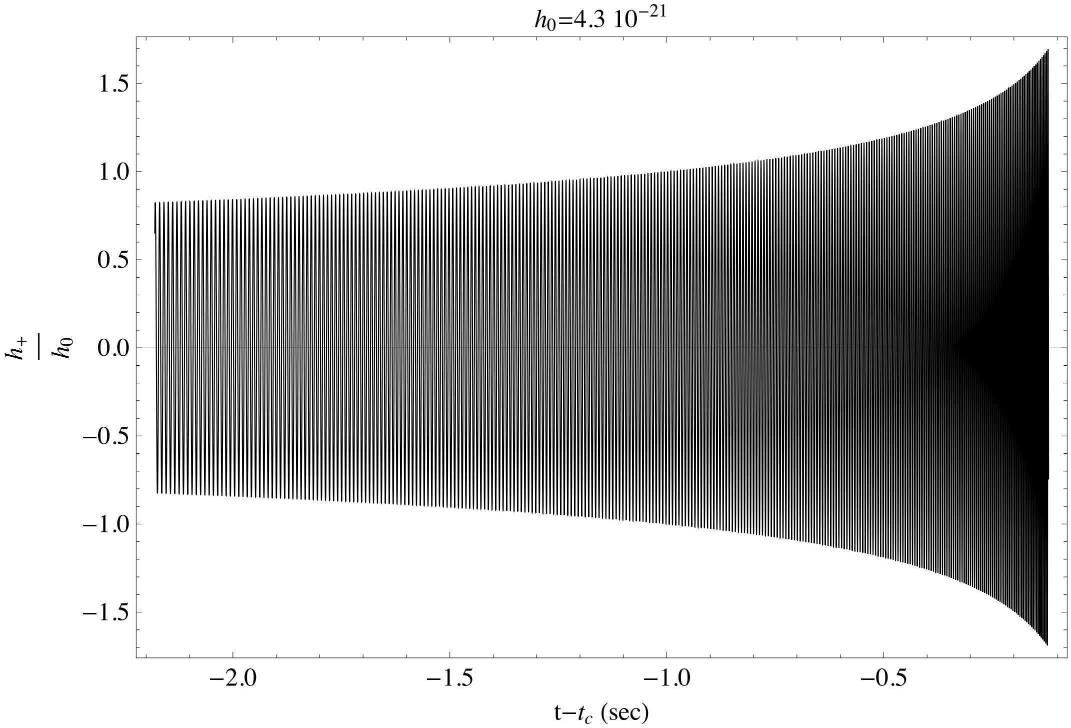
In[544]:=
```
Plot[134 / (-t) ^ (3 / 8), {t, -0.12, -2.18},
  ImageSize → 300, AspectRatio → 1, PlotStyle → {Black, Thick},
  Frame → True, Axes → False, FrameLabel → {"t-tc (sec)", "fGW (Hz)"},
  GridLines → {{}, {0}}, LabelStyle → Medium, PlotRange → All]
```

Out[544]=



In[543]:=
```
Plot[1 / (-t) ^ (1 / 4) Cos[1340 (-t) ^ (5 / 8)], {t, -0.12, -2.18},
  PlotStyle → {Black}, ImageSize → 500, AspectRatio → 2 / 3, Frame → True,
  Axes → False, FrameLabel → {"t-t_c (sec)", "h_+/h_0"}, GridLines → {{}, {0}},
  LabelStyle → Medium, PlotLabel → "h_0=4.3 10^-21", PlotPoints → 1000]
```

Out[543]=



① x^i_{rel} relative coordinate : $x^i_{rel} \equiv (x_1)^i - (x_2)^i$, where $\begin{cases} x^1_{rel} = R\cos\omega t \\ x^2_{rel} = R\cos i \; \sin\omega t \\ x^3_{rel} = R \sin i \; \sin\omega t \end{cases}$

$$(x_{CM})^i \equiv \frac{m_1 (x_1)^i + m_2 (x_2)^i}{m_1 + m_2}$$

$$T^{00} = m_1 c^2 \delta^{(3)}(\vec{x} - \vec{x}_1(t)) + m_2 c^2 \delta^{(3)}(\vec{x} - \vec{x}_2(t))$$

$$M^{klm} = \frac{1}{c^2}\int d^3x \, x^k x^l x^m T^{00} \underset{\vec{x}_{CM}=0}{=} \mu \frac{\delta m}{M} x^k_{rel} x^l_{rel} x^m_{rel}$$

$\mu \equiv m_1 m_2 / M$

where: $\delta m \equiv m_2 - m_1$

$M \equiv m_1 + m_2$

From Maggiore (3.141)

$$(h^{TT}_{ij})_{oct} = \frac{2G}{3c^5 r} \Lambda_{ij,kl}(\hat{n}) \hat{n}_m \dddot{O}^{klm}$$

but we can use M^{klm} interchangably thanks to the contraction with the projectors $\Lambda_{ij,kl} n_m$

$$\Lambda_{ij,kl}(\hat{n}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl} \;, \quad P_{ij} = \delta_{ij} - \hat{n}_i \hat{n}_j$$

Here $\hat{n} = \hat{z} \Rightarrow \hat{n}_i = \delta^3_i$

$$(h_+)_{oct} = (h^{TT}_{xx})_{oct} = (h^{TT}_{11})_{oct}$$

$$(h_\times)_{oct} = (h^{TT}_{xy})_{oct} = (h^{TT}_{12})_{oct}$$

$$(h_+)_{oct} = \frac{2G}{3c^5 r} \Lambda_{11,kl} \delta^3_m \dddot{M}^{klm} = \frac{2G}{3c^5 r} \Lambda_{11,kl} \dddot{M}^{kl3}$$

$$\Lambda_{11,kl} = P_{1k} P_{1l} - \frac{1}{2} \underbrace{P_{11}}_{1} P_{kl} = (\delta_{1k} - \cancel{\delta^3_1 \delta^3_k})(\delta_{1l} - \cancel{\delta^3_1 \delta^3_l}) - \frac{1}{2}(\delta_{kl} - \delta^3_k \delta^3_l)$$

$$\Lambda_{11,kl} \dddot{M}^{kl3} = \left[\delta_{1k} \delta_{1l} - \frac{1}{2}(\delta_{kl} - \delta^3_k \delta^3_l) \right] \dddot{M}^{kl3} =$$

$$= \dddot{M}^{113} - \frac{1}{2}\left(\dddot{M}^{113} + \dddot{M}^{223} + \cancel{\dddot{M}^{333}} - \cancel{\dddot{M}^{333}}\right) =$$

$$= \frac{1}{2}\left(\dddot{M}^{113} - \dddot{M}^{223}\right)$$

$$\dddot{M}^{113} = \frac{d^3}{dt^3}\left[\mu \frac{\delta m}{M} R^3 \cos^2\omega t \sin\omega t \sin i\right] =$$

$$= \mu \frac{\delta m}{M} R^3 \sin i \frac{d^3}{dt^3}\left[\sin\omega t - \sin^3\omega t\right] =$$

$$= \mu \frac{\delta m}{M} R^3 \sin i \left[-\omega^3 \cos\omega t - 6\omega^3 \cos^3\omega t + 21\omega^3 \cos\omega t \sin^2\omega t\right] =$$

$$= -\frac{1}{4}\omega^3 \mu \frac{\delta m}{M} R^3 \sin i \left(\cos\omega t + 27\cos 3\omega t\right)$$

$$\dddot{M}^{223} = \frac{d^3}{dt^3}\left[\mu \frac{\delta m}{M} R^3 \sin i \cos^2 i \sin^3\omega t\right] =$$

$$= \frac{3}{2}\omega^3 \mu \frac{\delta m}{M} R^3 \sin i \cos^2 i \,(9\cos 2\omega t - 5)\cos\omega t$$

$$\dddot{M}^{113} - \dddot{M}^{223} = R^3\frac{\mu}{2}\frac{\delta m}{M}\omega^3 \sin i \Big[-\frac{1}{2}\cos\omega t - \frac{27}{2}\cos 3\omega t - 27\cos^2 i \underbrace{\cos 2\omega t \cos\omega t}_{\frac{1}{2}(\cos\omega t + \cos 3\omega t)}$$

$$+ 15\cos^2 i \cos\omega t\Big] =$$

$$= R^3\frac{\mu}{2}\frac{\delta m}{M}\omega^3 \sin i\Big[-\frac{1}{2}\cos\omega t\,(1 + \overbrace{27\cos^2 i - 30\cos^2 i}^{1-3\cos^2 i}) - \frac{27}{2}\cos 3\omega t$$

$$(1+\cos^2 i)\Big] = \quad \text{use } \cos^2 i = \frac{1}{2}(1+\cos 2i)$$

$$= R^3\frac{\mu}{8}\frac{\delta m}{M}\omega^3 \sin i\,\big[\cos\omega t\,(1+3\cos 2i) - 27\cos 3\omega t\,(3+\cos 2i)\big]$$

$$(h_+)_{oct} = \frac{\cancel{2}}{3}\frac{G}{c^5 r} R^3 \frac{\mu}{\cancel{8}4}\frac{\delta m}{M}\omega^3 \frac{\sin i}{2}\big[\quad\big] =$$

$$= \frac{1}{r}\frac{G R^3 \mu \omega^3}{12 c^5}\frac{\delta m}{M}\sin i\Big[\underbrace{\Big(\frac{1+3\cos 2i}{2}\Big)}_{3\cos^2 i - 1}\cos\omega t - 27\underbrace{\Big(\frac{3+\cos 2i}{2}\Big)}_{1+\cos^2 i}\cos 3\omega t\Big]$$

$$\Lambda_{12,kl} = P_{1k}P_{2l} - \frac{1}{2}P_{12}P_{kl} = (\delta_{1k} - \cancel{\delta^3_1\delta^3_k})(\delta_{2l} - \cancel{\delta^3_2\delta^3_l}) - \underbrace{\frac{1}{2}(\cancel{\delta_{12}} - \cancel{\delta^3_1\delta^3_2})P_{kl}}_{0} =$$

$$= \delta_{1k}\delta_{2l}$$

$$\Lambda_{12,kl}\dddot{M}^{kl3} = \delta_{1k}\delta_{2l}\dddot{M}^{kl3} = \dddot{M}^{123}$$

$$\dddot{M}^{123} = \frac{d^3}{dt^3}\Big[\mu\frac{\delta m}{M}R^3 \sin i \cos i \sin^2\omega t \cos\omega t\Big] =$$

$$= +\frac{1}{4}\mu\frac{\delta m}{M}R^3 \sin i \cos i\, \omega^3 (\sin\omega t - 27\sin 3\omega t)$$

$$(h_\times)_{oct} = \frac{2}{3}\frac{G}{c^5 r}\frac{1}{4}\mu\frac{\delta m}{M}R^3 \sin i \cos i\,\omega^3(\sin\omega t - 27\sin 3\omega t) =$$

$$= \frac{1}{r}\frac{G R^3\mu\omega^3}{6c^5}\frac{\delta m}{M}\sin i\cos i\,(\sin\omega t - 27\sin 3\omega t)$$

The mass-octupole radiation is emitted at $\omega_{GW} = \omega,\ 3\omega$.

② $S^{kl,m} = \int d^3x\, T^{kl} x^m$

$$T^{kl} = m_1 (\dot{x}_1)^k (\dot{x}_1)^l \delta^{(3)}(\vec{x}-\vec{x}_1) + m_2 (\dot{x}_2)^k (\dot{x}_2)^l \delta^{(3)}(\vec{x}-\vec{x}_2)$$

$$\Rightarrow S^{kl,m} = m_1 (\dot{x}_1)^k (\dot{x}_1)^l (x_1)^m + m_2 (\dot{x}_2)^k (\dot{x}_2)^l (x_2)^m = \text{move to C.O.M.}$$

$$= \mu \frac{\delta m}{M} \dot{x}^k_{rel}\, \dot{x}^l_{rel}\, x^m_{rel} \quad \text{where} \quad \begin{cases} \dot{x}^1_{rel} = -\omega R \sin\omega t \\ \dot{x}^2_{rel} = \omega R \cos i \cos\omega t \\ \dot{x}^3_{rel} = \omega R \sin i \cos\omega t \end{cases}$$

$$(h^{TT}_{ij})_{oct+cq} = \frac{1}{r}\frac{4G}{c^5} \Lambda_{ij,kl}(\hat{n})\hat{n}_m \dot{S}^{kl,m} \quad \text{from Maggiore (3.138)}$$

Again $(h_+)_{oct+cq} = (h^{TT}_{11})_{oct+cq}$ and $(h_\times)_{oct+cq} = (h^{TT}_{12})_{oct+cq}$.

Just like in problem 1, the components we need are only $\dot{S}^{11,3}$, $\dot{S}^{22,3}$ and $\dot{S}^{12,3}$

$$\dot{S}^{11,3} = \frac{d}{dt}\left[\mu \frac{\delta m}{M} (\dot{x}^1_{rel})^2 x^3_{rel}\right] = \mu \frac{\delta m}{M} \frac{d}{dt}\left[\omega^2 R^3 \sin^2\omega t \sin\omega t \sin i\right] =$$

$$= 3\mu \frac{\delta m}{M} \omega^3 R^3 \sin^2\omega t \cos\omega t \sin i = \frac{3}{4}\mu \frac{\delta m}{M}\omega^3 R^3 (\cos\omega t - \cos 3\omega t)\sin i$$

$$\dot{S}^{22,3} = \mu \frac{\delta m}{M}\frac{d}{dt}\left[\omega^2 R^3 \cos^2 i \sin i \cos^2\omega t \sin\omega t\right] =$$

$$= \frac{1}{4}\mu \frac{\delta m}{M}\omega^3 R^3 \sin i \cos^2 i (\cos\omega t + 3\cos 3\omega t)$$

$$\dot{S}^{12,3} = \mu \frac{\delta m}{M}\frac{d}{dt}\left[-\omega^2 R^3 \sin i \cos i \sin^2\omega t \cos\omega t\right] =$$

$$= -\frac{1}{2}\mu \frac{\delta m}{M}\omega^3 R^3 \sin i \cos i (\sin\omega t + 3\sin\omega t \cos 2\omega t) =$$

$$= \frac{1}{4}\mu \frac{\delta m}{M}\omega^3 R^3 \sin i \cos i (\sin\omega t - 3\sin 3\omega t)$$

$$(h_+)_{oct+cq} = \frac{1}{r}\frac{4G}{c^5}\frac{1}{2}\left(\dot{S}^{11,3} - \dot{S}^{22,3}\right) =$$

$$= \frac{1}{r}\frac{4G}{c^5}\frac{1}{8}\mu \frac{\delta m}{M}\omega^3 R^3 \sin i\left[3\cos\omega t - 3\cos 3\omega t - \cos^2 i(\cos\omega t + 3\cos 3\omega t)\right] =$$

$$= +\frac{1}{r}\frac{G\mu\omega^3 R^3}{2c^5}\frac{\delta m}{M}\sin i\left[(3-\cos^2 i)\cos\omega t - 3(1+\cos^2 i)\cos 3\omega t\right]$$

$$(h_\times)_{oct+cq} = \frac{1}{r}\frac{4G}{c^5}\dot{S}^{12,3} = \frac{1}{r}\frac{\cancel{4}G}{c^5}\frac{1}{\cancel{4}}\mu\frac{\delta m}{M}\omega^3 R^3 \sin i \cos i (\sin\omega t - 3\sin 3\omega t) =$$

$$= \frac{1}{r}\frac{G\mu\omega^3 R^3}{c^5}\frac{\delta m}{M}\sin i \cos i (\sin\omega t - 3\sin 3\omega t)$$

Finally

$$(h_+)_{cq} = (h_+)_{oct+cq} - (h_+)_{oct} =$$

$$= \frac{1}{r}\frac{G}{2c^5}\mu\frac{\delta m}{M}(R\omega)^3 \sin i \Big[\cos\omega t\left(3 - \cos^2 i - \frac{1}{2}\cos^2 i + \frac{1}{6}\right)$$

$$- 3\cos 3\omega t\left(1 + \cos^2 i - \frac{3}{2} - \frac{3}{2}\cos^2 i\right)\Big] =$$

$$= \frac{1}{r}\frac{G}{24c^5}\mu\frac{\delta m}{M}(R\omega)^3 \sin i \Big[(29 - 9\cos 2i)\cos\omega t + 9(3 + \cos 2i)$$

$$\cos 3\omega t\Big]$$

$$(h_\times)_{cq} = (h_\times)_{oct+cq} - (h_\times)_{oct} =$$

$$= \frac{1}{r}\frac{G}{c^5}\mu\frac{\delta m}{M}(R\omega)^3 \sin i \cos i \left[\sin\omega t - 3\sin 3\omega t - \frac{1}{6}(\sin\omega t - 27\sin 3\omega t)\right]$$

$$= \frac{1}{r}\frac{G}{6c^5}\mu\frac{\delta m}{M}(R\omega)^3 \sin i \cos i\,(5\sin\omega t + 9\sin 3\omega t)$$

$$③\quad P_{oct+cq} = \frac{r^2 c^3}{8G}\int_0^{\pi} di\,\sin i\;\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\,\left[(\dot h_+)^2_{oct+cq} + (\dot h_\times)^2_{oct+cq}\right]$$

Let $h_0 \equiv \dfrac{1}{r}\,\dfrac{G\mu R^3\omega^3\delta m}{2c^5 M}$

$$(h_+)_{oct+cq} = h_0 \sin i\left[(3-\cos^2 i)\cos\omega t - 3(1+\cos^2 i)\cos 3\omega t\right]$$

$$(h_\times)_{oct+cq} = h_0 \sin 2i\left[\sin\omega t - 3\sin 3\omega t\right]$$

$$(\dot h_+)_{oct+cq} = \omega h_0 \sin i\left[-(3-\cos^2 i)\sin\omega t + 9(1+\cos^2 i)\sin 3\omega t\right]$$

$$(\dot h_\times)_{oct+cq} = \omega h_0 \sin 2i\left[\cos\omega t - 9\cos 3\omega t\right]$$

$$(\dot h_+)^2_{oct+cq} + (\dot h_\times)^2_{oct+cq} = \omega^2 h_0^2 \sin^2 i\left[(3-\cos^2 i)^2\sin^2\omega t + \right.$$

$$\left. + 81(1+\cos^2 i)^2\sin^2 3\omega t - 18(3-\cos^2 i)(1+\cos^2 i)\sin\omega t\sin 3\omega t\right] +$$

$$+ \omega^2 h_0^2 \underbrace{\sin^2 2i}_{4\sin^2 i\cos^2 i}\left[\cos^2\omega t + 81\cos^2 3\omega t - 18\cos\omega t\cos 3\omega t\right] =$$

$$= \omega^2 h_0^2 \sin^2 i\left\{(3-\cos^2 i)^2\,\frac{1-\cos 2\omega t}{2} + 81(1+\cos^2 i)^2\,\frac{1-\cos 6\omega t}{2}\right.$$

$$-18(3-\cos^2 i)(1+\cos^2 i)\sin\omega t\sin 3\omega t\; +$$

$$\left. + 4\cos^2 i\left[\frac{1+\cos 2\omega t}{2} + 81\,\frac{1+\cos 6\omega t}{2} - 18\cos\omega t\cos 3\omega t\right]\right\} =$$

$$= \omega^2 h_0^2 \sin^2 i\left\{\frac{1}{2}(3-\cos^2 i)^2 + \frac{81}{2}(1+\cos^2 i)^2 - \frac{1}{2}\cos 2\omega t\left[(3-\cos^2 i)^2 - 4\cos^2 i\right]\right.$$

$$-\frac{1}{2}\cos 6\omega t\left[81(1+\cos^2 i)^2 - 81\cdot 4\cos^2 i\right] + 164\cos^2 i$$

$$\left. -18(3-\cos^2 i)(1+\cos^2 i)\sin\omega t\sin 3\omega t - 18\cdot 4\cos^2 i\cos\omega t\cos 3\omega t\right\}$$

$$\langle\ldots\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\,(\ldots)$$

$\langle\cos 2\omega t\rangle = 0$, $\langle\sin\omega t\sin 3\omega t\rangle = 0$ only the terms w/out

$\langle\cos 6\omega t\rangle = 0$, $\langle\cos\omega t\cos 3\omega t\rangle = 0$ $\Rightarrow$ t-dependence survive the $\langle\ldots\rangle$

$$\langle(\dot h_+)^2_{oct+cq} + (\dot h_\times)^2_{oct+cq}\rangle = \frac{1}{2}\omega^2 h_0^2\sin^2 i\left[(3-\cos^2 i)^2 + 81(1+\cos^2 i)^2 + 328\cos^2 i\right]$$

$$\int_{-1}^{1} d(\cos i)\,\langle(\dot h_+)^2_{oct+cq} + (\dot h_\times)^2_{oct+cq}\rangle = \frac{\omega^2 h_0^2}{2}\underbrace{\int_{-1}^{1} dx\,(1-x^2)\left[(3-x^2)^2 + 81(1+x^2)^2 + 328x^2\right]}_{\frac{27136}{105}}$$

$$P_{oct+cq} = \frac{r^2 c^3}{8G}\,\frac{\omega^2 h_0^2}{2}\,\frac{27136}{35} = \frac{\cancel{r^2} c^3}{\cancel{G}}\,\frac{1696}{105}\,\omega^2\,\frac{1}{\cancel{r^2}}\,\frac{G^{\cancel{2}}\mu^2 R^6\omega^6(\delta m)^2}{4c^{10}M^2} =$$

$$= \frac{424}{105}\frac{G}{c^7}\mu^2\left(\frac{\delta m}{M}\right)^2 R^6\omega^8$$

The fraction of P emitted at freq. ω is given by the terms:

$$h_+\big|_\omega = h_0 \sin i\,(3-\cos^2 i)\cos\omega t$$

$$h_\times\big|_\omega = h_0 \sin 2i \sin\omega t$$

$$\dot h_+\big|_\omega = -\omega h_0 \sin i\,(3-\cos^2 i)\sin\omega t$$

$$\dot h_+\big|_\omega = \omega h_0 \sin 2i\cos\omega t$$

$$(\dot h_+|_\omega)^2 + (\dot h_+|_\omega)^2 = \omega^2 h_0^2\left[\sin^2 i\,(3-\cos^2 i)^2 \overbrace{\frac{1-\cos 2\omega t}{2}}^{\sin^2\omega t} + \sin^2 2i\,\overbrace{\frac{1+\cos 2\omega t}{2}}^{\cos^2\omega t}\right]$$

Take the $\langle\ldots\rangle$ and the terms that survive are:

$$\frac{1}{2}\omega^2 h_0^2\left[\sin^2 i\,(3-\cos^2 i)^2 + \sin^2 2i\right] = \frac{1}{2}\omega^2 h_0^2 \sin^2 i\left[(3-\cos^2 i)^2 + 4\cos^2 i\right] =$$

$$= \frac{1}{2}\omega^2 h_0^2 (1-\cos^2 i)\left[(3-\cos^2 i)^2 + 4\cos^2 i\right]$$

$$\int_{-1}^{1} dx\,(1-x^2)\left[(3-x^2)^2 + 4x^2\right] = \frac{1216}{105}$$

$$P_{oct+cq}\big|_\omega = \frac{r^2 c^3}{8G}\frac{1216}{105}\frac{1}{2}\omega^2 h_0^2 = \frac{19}{105}\frac{G}{c^7}\mu^2\left(\frac{\delta m}{M}\right)^2 R^6\omega^8$$

$$\Rightarrow \frac{P_{oct+cq}(\omega)}{P_{quad}(2\omega)} = \frac{\frac{19}{105}\frac{\cancel{G}}{c^7}\cancel{\mu^2}\left(\frac{\delta m}{M}\right)^2 R^6\omega^8}{\frac{32}{5}\frac{\cancel{G}}{c^5}\cancel{\mu^2} R^4\omega^6} = \frac{19}{672}\left(\frac{\delta m}{M}\right)^2\frac{1}{c^2}R^2\omega^2$$

Use $\omega^2 R^2 = \frac{GM}{R} = v^2$ then:

$$\frac{P_{oct+cq}(\omega)}{P_{quad}(2\omega)} = \frac{19}{672}\left(\frac{\delta m}{M}\right)^2\left(\frac{v}{c}\right)^2$$

The fraction of P_{oct+cq} emitted at freq. 3ω is given by the terms:

$$h_+\big|_{3\omega} = h_0\sin i\left[-3(1+\cos^2 i)\cos 3\omega t\right] \Rightarrow \dot h_+\big|_{3\omega} = \omega 9 h_0 \sin i\,(1+\cos^2 i)\sin 3\omega t$$

$$h_\times\big|_{3\omega} = -3h_0 \sin 2i\,\sin 3\omega t \Rightarrow \dot h_\times\big|_{3\omega} = -9\omega h_0 \sin 2i\cos 3\omega t$$

$$(\dot h_+|_{3\omega})^2 + (\dot h_\times|_{3\omega})^2 = \omega^2 h_0^2\, 81\left[\sin^2 i\,(1+\cos^2 i)^2\,\frac{1-\cos 6\omega t}{2} + \sin^2 2i\,\frac{1+\cos 6\omega t}{2}\right.$$

Take $\langle \cdots \rangle$ and we're left with

$$\frac{81}{2} \omega^2 h_0^2 \sin^2 i \left[(1+\cos^2 i)^2 + 4\cos^2 i \right] =$$

$$= \frac{81}{2} \omega^2 h_0^2 (1-\cos^2 i) \left[(1+\cos^2 i)^2 + 4\cos^2 i \right]$$

$$\int_{-1}^{1} dx\, (1-x^2)\left[(1+x^2)^2 + 4x^2 \right] = \frac{64}{21}$$

$$P_{oct+cq}\Big|_{3\omega} = \frac{r^2 c^3}{8G} \frac{64}{21} \frac{81}{2} \omega^2 h_0^2 = \frac{27}{7} \frac{G}{c^7} \mu^2 \left(\frac{\delta m}{M}\right)^2 R^6 \omega^8$$

$$\Rightarrow \frac{P_{oct+cq}(3\omega)}{P_{quad}(2\omega)} = \frac{\frac{27}{7} \frac{\cancel{G}}{c^7} \cancel{\mu^2} \left(\frac{\delta m}{M}\right)^2 R^6 \omega^8}{\frac{32}{5} \frac{\cancel{G}}{c^5} \cancel{\mu^2} R^4 \omega^6} = \frac{135}{224} \left(\frac{\delta m}{M}\right)^2 \frac{R^2 \omega^2}{c^2} =$$

$$= \frac{135}{224} \left(\frac{\delta m}{M}\right)^2 \left(\frac{v}{c}\right)^2$$

Let $v/c = 10^{-2}$, then

$$\frac{P_{oct+cq}(\omega)}{P_{quad}(2\omega)} \cong 2.8 \times 10^{-6} \left(\frac{\delta m}{M}\right)^2$$

$$\frac{P_{oct+cq}(3\omega)}{P_{quad}(2\omega)} \cong 61 \times 10^{-6} \left(\frac{\delta m}{M}\right)^2$$