HW#2 —Phys879—Spring 2017
Due before class, Monday, Feb. 27, 2017
Course webpage: http://www.aei.mpg.de/2000472

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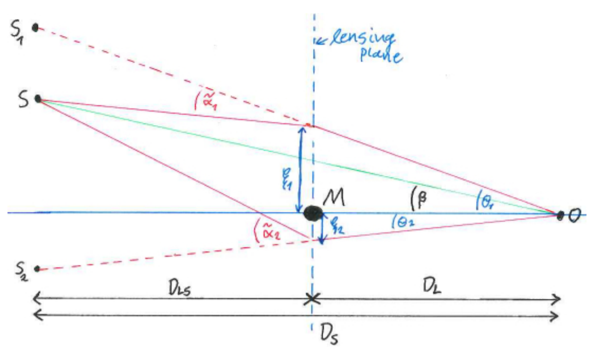
Recommended readings:

1. "Gravitational lensing in Astronomy", J. Wambsganss, Living Review in Relativity (1998) `http://link.springer.com/article/10.12942/lrr-1998-12`.
2. "Binary Black Hole Mergers in the First Advanced LIGO Observing Run", B. P. Abbott et al. (2016) `https://arxiv.org/pdf/1606.04856.pdf` (see Sec. VI).
3. "Implications of the gravitational-wave event GW150914", C. Miller (2016) `http://www.astro.umd.edu/ miller/reprints/miller16d.pdf` (see Sec. 4).

Exercises (prepared by A. Buonanno, T. Hinderer, C. Miller, J. Steinhoff & J. Vines):

1. **Gravitational lensing of gravitational waves** [10 pts.]



We work in natural units ($G = 1 = c$). Let us consider a gravitational wave in the geometric optics approximation. As it travels from the source S to the observer O on Earth, the gravitational-wave rays (red) are deflected, or "lensed", by a heavy (point-)mass M. Calculate the delay in travel times by two rays taking opposite paths around the mass. Assume that the rays and the mass M are in the same plane, and that the distances $D_{\rm L}$, $D_{\rm LS}$ are large compared to ξ_1, ξ_2 (i.e., the deflection is confined to a "lensing plane".) You can use the classical result of Einstein for the deflection angles $\tilde{\alpha}_i = 4M/\xi_i$.

The cosmological redshift can be neglected here, but the Shapiro delay should be taken into account. The approximate line element reads

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 + \frac{2M}{r}\right) dx^i dx^i + O\left(\frac{M^2}{r^2}\right), \tag{1}$$

where $r^2 = x^i x^i$.

What is the difference in arrival times between the two rays for a gravitational signal similar to GW150914 (distance $D_S = 1$ Gly) passing a galaxy of 10^{12} solar masses in the middle of its way and at an angle of $\beta = 2$ arsec? Compare this to the GW150914 duration of ~ 200 ms observed in the detector. What are the magnification factors of the two rays?

2. **Energy-momentum tensor of a plane gravitational wave** [5 pts.]

Calculate the effective energy-momentum tensor of a plane gravitational wave in the TT gauge from

$$T^{\rm GW}_{\mu\nu} = \frac{1}{32\pi} < \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} >, \tag{2}$$

$$h^{\mu\nu} = \left[(e^\mu_x e^\nu_x - e^\mu_y e^\nu_y) A_+ + (e^\mu_x e^\nu_y + e^\mu_y e^\nu_x) A_\times\right] e^{i k_\alpha x^\alpha} + c.c. \tag{3}$$

It holds $e^\mu_x e_{y\mu} = e^\mu_x k_\mu = e^\mu_y k_\mu = k^\mu k_\mu = 0$, $e^\mu_x e_{x\mu} = e^\mu_y e_{y\mu} = 1$, and $A_+, A_\times \in \mathbb{C}$.

3. **Resonant mass detectors** [10 pts.]

   The first gravitational-wave detector was a resonant mass detector or bar detector. It was built at the University of Maryland by Joseph Weber in the late 60's. It was a large, heavy, metal bar. The bar would absorb an impinging gravitational wave and be set into oscillations. Hopefully, those oscillations would be detectable.

   The simplest way to model a bar detector is with a damped spring. Let us assume that we have two masses m_1 and m_2, along the x-axis, connected by a spring with spring constant k and subjected to a dissipative force $F_{\rm diss} = -b\, dx/dt$. In equilibrium the masses are separated by a length L. Let us assume that a plane gravitational wave with frequency ω arrives along the z-axis and it is polarized only along the x-axis, i.e., $h_+ = h \cos\omega\, t, h_\times = 0$.

   - Assuming that $L \ll \lambda_{\rm GW}$, with $\lambda = 2\pi c/\omega$, derive the equation of motion for the displacement of the masses $x(t)$ with respect to the equilibrium position. [Hint: follow the description in the local inertial frame.]
   - The solution of the equation of motion in the previous item can be written as $x(t) = A\,\cos(\omega\, t+\delta)$. Derive A and δ, and the maximum of the oscillation at resonance.
   - Derive the kinetic energy of the oscillations, the potential energy of the oscillator, the work done on the oscillator by the gravitational wave and the rate of energy dissipated. Compute those quantities averaging over a cycle of oscillations.
   - Assume that $h = 10^{-21}$, $L = 1\,$m, reduced mass $\mu = m_1\, m_2/(m_1 + m_2) = 1000\,$kg, quality factor $Q = 10^{-6}$ and $f = \omega_0/2/\pi = 1\,$kHz, compute the maximum of oscillations at resonance and the total energy in the oscillations, i.e., kinetic energy and potential energy, averaging over a cycle of oscillations. When comparing the averaged total energy to the thermal energy at room temperature, what do you conclude?

4. **Attenuation of gravitational waves** [10 pts.]

   Assume that a gravitational wave encounters a viscous fluid, which is initially at rest with fluid four-velocity given by $u^\mu = (1, 0, 0, 0)$.

   - The shearing of the fluid is described by the shear tensor

   $$\sigma_{\mu\nu} = \frac{1}{2}\nabla_\mu u_\nu + \frac{1}{2}\nabla_\nu u_\mu + \frac{1}{2}u_\mu u^\alpha \nabla_\alpha u_\nu + \frac{1}{2}u_\nu u^\alpha \nabla_\alpha u_\mu - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)\nabla_\alpha u^\alpha\,. \tag{4}$$

     Show that the shear has purely spatial components when the gravitational-wave is expressed in the transverse-traceless (TT) gauge, and that

   $$\sigma_{ij} = \frac{1}{2}\frac{\partial}{\partial t}h_{ij}^{\rm TT}\,. \tag{5}$$

   - The shearing of the viscous fluid generates a contribution to the stress-energy tensor of the form

   $$T_{\mu\nu} = -2\eta\,\sigma_{\mu\nu}\,, \tag{6}$$

     where we indicate with η the coefficient of viscosity of the fluid. What is the linearized field equation for the gravitational wave in the TT gauge in presence of the viscous fluid? Show that a plane wave travelling along the z-axis is attentuated by the fluid by an amount $e^{-z/l}$ where l is the attenuation length scale $l = c^3/(8\pi G\eta)$.
   - Chocolate has a coefficient of viscosity of $\eta = 25\,{\rm kg/(m\,s)}$. Calculate the distance L that a gravitational wave must travel through chocolate before it is attentuated by a factor $1/e$.

5. **Estimating binary–black-hole merger rates** [10 pts.]

Make your own estimate of the rate per volume of BH-BH mergers (expressed in number per $\mathrm{Gpc}^{-3}\ \mathrm{yr}^{-1}$), including the 90% credible interval, based on the three events reported thus far (for these purposes we assume that LVT151012 was a real event). The first Advanced LIGO run had 49 total days in which both detectors were taking data, so that will be our baseline time. Potentially relevant numbers are: GW150914 was at a distance of 420 Mpc (we'll ignore the uncertainties for simplicity) and had a signal to noise ratio of 23.7; GW151226 was at a distance of 440 Mpc and had a signal to noise ratio of 13.0; LVT151012 was at a distance of 1 Gpc and had a signal to noise ratio of 9.7. Suppose that the threshold for announcing a detection is a signal to noise ratio of 12.0 (recall that LVT151012 was a marginal detection), and remember that for a given event the distance scales as the reciprocal of the signal to noise ratio.

- With no other information, what would be your best estimate for the rate per volume based on each of the events individually (i.e., without combining them or estimating uncertainties)?
- How should you estimate the uncertainties for each event individually? More specifically, how would you calculate the 90% credible interval for the rate based on each event individually?
- How should you combine the information from the three events? Do this without, then with, the uncertainties included.
- Suppose now that you are given the information that one of the events (pick any of them) was in a direction to which Advanced LIGO was unusually sensitive. What effect, if any, would this have on your best estimate of the rate based on that event (i.e., would it decrease your best estimate, increase your best estimate, or leave it unchanged)?
- Same question as above, but with regard to the orientation: suppose that one of the events was known to have its binary orbital axis pointed nearly towards us, which means that we see a high amplitude compared to the orientation-averaged amplitude. What effect would this have on your best estimate of the rate from that event alone?