1. **Basic physics of GW150914:**

[1] refers to arXiv:1608.01940.

[Note that unlike [1], we use the Einstein summation convention: repeated indices are summed; e.g., there is a sum over the index k implied in (2), and sums over both i and j in (11).]

(a) *Power emitted in GWs:*

If the 3-vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the displacements of the two masses m_1 and m_2 from the system's center of mass (CoM), and if we define the relative displacement $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, then, by the definition of the CoM,

$$m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2 = 0, \qquad \Rightarrow \qquad \boldsymbol{x}_1 = \frac{m_2}{M}\boldsymbol{x}, \qquad \boldsymbol{x}_2 = -\frac{m_1}{M}\boldsymbol{x}, \tag{1}$$

where $M = m_1 + m_2$. We use this in the definition [1](24) of the quadrupole tensor to get the second line here,

$$\begin{aligned} Q^{ij} &= \sum_{A=1,2} m_A\Big(x_A^i x_A^j - \frac{1}{3}\delta^{ij}x_A^k x_A^k\Big), &(2)\\ &= \Big(x^i x^j - \frac{1}{3}\delta^{ij}x^k x^k\Big)\left[m_1\frac{m_2^2}{M^2} + m_2\frac{m_1^2}{M^2} = \frac{m_1 m_2}{M} = \mu\right] &(3)\\ &= \mu r^2\Big(n^i n^j - \frac{1}{3}\delta^{ij}\Big). &(4)\end{aligned}$$

In the third line, we have used $\boldsymbol{x} = r\boldsymbol{n}$, with $r = |\boldsymbol{x}|$ and the unit vector $\boldsymbol{n} = \boldsymbol{x}/r$.

For a circular orbit in the x-y plane, we have $\boldsymbol{n} = (\cos(\omega t), \sin(\omega t), 0)$, and r is constant.

We need to take repeated time derivatives of $Q_{ij}(t)$. We could proceed in terms of components in Cartesian coordinates, but it's rather convenient to note that

$$\dot{\boldsymbol{n}} = \omega\boldsymbol{\phi}, \qquad \dot{\boldsymbol{\phi}} = -\omega\boldsymbol{n}, \qquad \boldsymbol{n}\cdot\boldsymbol{\phi} = 0, \qquad \boldsymbol{\phi}^2 = 1, \tag{5}$$

where $\boldsymbol{\phi} = (-\sin(\omega t), \cos(\omega t), 0)$ is the unit vector in the direction of the relative velocity $\dot{\boldsymbol{x}}$. We then have

$$\begin{aligned} \dot{Q}^{ij} &= \mu r^2\omega(n^i\phi^j + \phi^i n^j) = 2\mu r^2\omega n^{(i}\phi^{j)}, &(6)\\ \ddot{Q}^{ij} &= 2\mu r^2\omega^2(\phi^i\phi^j - n^i n^j), &(7)\\ \dddot{Q}^{ij} &= -8\mu r^2\omega^3 n^{(i}\phi^{j)}. &(8)\end{aligned}$$

Then, from (7) and [1](4), the GW strain tensor is

$$\begin{aligned} h_{ij} = \frac{2G}{c^4 d_L}\ddot{Q}_{ij} &= \frac{4G\mu r^2\omega^2}{c^4 d_L}(\phi_i\phi_j - n_i n_j) &(9)\\ &= \frac{4G\mu r^2\omega^2}{c^4 d_L}\Big((\hat{y}_i\hat{y}_j - \hat{x}_i\hat{x}_j)\cos(2\omega t) - 2\hat{x}_{(i}\hat{y}_{j)}\sin(2\omega t)\Big), &(10)\end{aligned}$$

where the second line has substituted $\boldsymbol{n} = \cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}}$ and $\boldsymbol{\phi} = -\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}$ and used the double angle formulas.

From (8) and [1](5), using $n_{(i}\phi_{j)}n_{(i}\phi_{j)} = (1/2)(n_i\phi_j + n_j\phi_i)n_i\phi_j = 1/2$, the emitted power is

$$\dot{E}_{\rm GW} = -\frac{G}{5c^5}\dddot{Q}_{ij}\dddot{Q}_{ij} = -\frac{32G\mu^2 r^4\omega^6}{5c^5}. \tag{11}$$

(b) *Frequency evolution and estimating the mass scale:*

We equate $\dot{E}_{\rm GW}$ with $\dot{E}_{\rm orb}$, where $E_{\rm orb}$ = kinetic + potential $= \mu(r\omega)^2/2 - GM\mu/r = -GM\mu/2r$ is the Newtonian energy of the circular orbit, obeying Kepler's third law "K3": centripetal acceleration $= r\omega^2 = GM/r^2 =$ gravitational acceleration. The calculation is fully outlined between [1](27) and [1](28), and only algebra remains to obtain [1](28), or the rearrangement [1](7). One obtains [1](8) by integrating [1](28). Recall $\omega = 2\pi f_{\rm orb} = \pi f_{\rm GW}$.

A chirp mass estimate is found from [1](7),

$$\mathcal{M} = (\mu^3 M^2)^{1/5} = \frac{c^3}{G}\left[\frac{5^3}{96^3\pi^8}\Big(f_{\rm GW} \sim 64\,{\rm Hz}\Big)^{-11}\Big(\dot{f}_{\rm GW} \sim \frac{200\,{\rm Hz}}{0.1\,{\rm s}}\Big)^3\right]^{1/5} \sim 50\,M_\odot, \tag{12}$$

where these numbers are just from someone eyeballing the slope ($\dot{f}_{\rm GW}$) of the tangent to the curve of [1] Fig. 2 at 64 Hz. This is consistent with the $30\,M_\odot$ number quoted in the text.

For a 1:1 mass ratio, $m_1 = m_2 = M/2$, we have that $\mu = M/4$, and so $M = 2^{6/5}\mathcal{M}$. We will henceforth use $M \approx 70\,M_\odot$ as in [1].

We should assume that Newtonian physics is (approximately) applicable if the timescale $t_{\rm GW} \sim E/\dot{E}$ over which GWs are taking energy from the system is longer than the orbital timescale $t_{\rm orb} \sim \omega^{-1}$. The GW emission is then "slow". Over short times, we can then ignore the GWs and consider Newtonian physics to accurately describe the short-time physics. This is a form of "adiabatic approximation".

The validity of this approximation is measured by the smallness of

$$\frac{t_{\rm orb}}{t_{\rm GW}} \sim \frac{\dot{E}_{\rm GW}}{\omega E_{\rm orb}} \sim \frac{GM^2r^4\omega^6/c^5}{\omega\, GM^2/r} = \left(\frac{v}{c} = \frac{r\omega}{c} = \frac{(GM\omega)^{1/3}}{c}\right)^5 \sim 0.05\left(\frac{M}{70\,M_\odot}\,\frac{f_{\rm GW}}{150\,{\rm Hz}}\right)^{5/3}, \tag{13}$$

so it seems to be a pretty good approximation even at the end of the signal.

——However, note that the first relativistic corrections to Newtonian gravity, the so-called "first post-Newtonian" or "1PN" corrections (which have nothing to do with GWs), actually scale as $(v/c)^2$, and this is $\sim .3$ at 150 Hz for a $70\,M_\odot$ system, so the Newtonian approximation is not to be trusted in great detail at the end of the signal.

(c) *Drag force instead of GWs?:*

If there is a drag force of magnitude $F = bv^p$ opposing the velocity, then the rate at which energy is dissipated is $\dot{E}_{\rm drag} = -Fv = -bv^{p+1}$. This now replaces (11). Just as for GWs, we equate this to the rate change of $E_{\rm orb} \propto -1/r$, and we use K3, $r \propto \omega^{-2/3}$, along with $v = r\omega$,

$$\begin{aligned}\dot{E}_{\rm drag} &\propto -v^{p+1} \propto -(r\omega)^{p+1} \propto -\omega^{(p+1)/3} \\ &\propto \dot{E}_{\rm orb} \propto \tfrac{d}{dt}(-r^{-1} \propto -\omega^{2/3}) \propto -\omega^{-1/3}\dot{\omega},\end{aligned} \tag{14}$$

and thus,

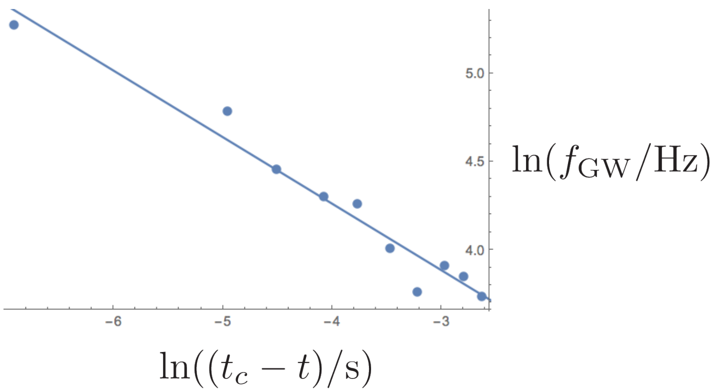
$$\dot{\omega} \propto \omega^{(p+2)/3} \qquad \Rightarrow \qquad \frac{d\omega}{\omega^{(p+2)/3}} \propto dt \qquad \Rightarrow \qquad \omega^{-(p-1)/3} \propto -t + {\rm const.}, \tag{15}$$

i.e. $\alpha = (p+2)/3$ and $\beta = -(p-1)/3$, assuming here that $p > 1$. [If $p = 1$, then $\alpha = 1$ but there is no β, as $\omega \propto e^t$. If $p < 1$, the formulae (15) still hold, but with $-t \to +t$.] The values for GW emission are $\alpha = 11/3$ and $\beta = -8/3$, which would correspond to $p = 9$. This is very different from a drag force with $p = 1$ for laminar flow or $p = 2$ for turbulent flow.

Below is a log-log plot of the data from [1] Fig. 2, for which the slope should be $1/\beta$. [We have omitted the final and third to final data points from [1] Fig. 2, the only two points whose frequencies are not labelled on the right.] Here, we have taken the "time of coalescence" t_c, the integration constant in (15), the time when the frequency goes to infinity according to the Newtonian approximation, to be $t_c = 0.421$ s. This choice was based on estimating the zero crossing in [1] Fig. 2 (which assumes the -8/3 exponent, which makes this admittedly circular logic). The fitting procedure is fairly sensitive to the choice of t_c,

$$t_c = \{0.421, 0.422, 0.423\}\,\text{Hz} \qquad \Rightarrow \qquad 1/\beta = \{-0.38, -0.44, -0.49\}, \tag{16}$$

but one obtains the best fitting statistic for 0.421, the choice plotted below. This closely matches the expected $1/\beta = -3/8 = -.375$, and any of the three values above are far from $1/\beta = -3$ for a turbulent drag force (or undefined β for laminar drag).



(d) *Orbital separation; why BHs?:*

The radial separation follows directly from K3, $r^3\omega^2 = GM$, with $\omega = \pi f_{\text{GW}}$,

$$r = \frac{(GM)^{1/3}}{(\pi f_{GW})^{2/3}} = (350\,\text{km})\left(\frac{M}{70\,M_\odot}\right)^{1/3}\left(\frac{150\,\text{Hz}}{f_{\text{GW}}}\right)^{2/3}, \tag{17}$$

while we get 10^3 km at $f_{\text{GW}} = 30$ Hz at the beginning of the signal. The argument, as discussed extensively in Sec. 3 of [1], is that we have two objects each with mass $\sim 35\,M_\odot$, which seem to be freely orbiting until they are less than 350 km apart, which means that they must have radii less than 175 km. The implied density is far beyond normal stellar densities, at least at neutron star density or higher, but neutron stars are known to be unstable above 5 $M_\odot$ (to be very conservative). The only plausible candidate objects according to known physics are black holes.

(e) *Distance to the source:*

If we simply ignore the tensorial factor (made of unit vectors), the "magnitude" of (9) gives us the strain

$$h \sim \frac{4G\mu r^2\omega^2}{c^4 d_L} = \frac{(GM)^{5/3}(\pi f_{GW})^{2/3}}{c^4 d_L}, \tag{18}$$

where the second equality has used K3, $\mu = M/4$, and $\omega = \pi f_{\text{GW}}$. Using $h \sim 10^{-21}$ at the peak $f_{\text{GW}} = 150$ Hz, this formula yields $d_L \sim 3 \times 10^{25}\,\text{m} \sim 1\,\text{Gpc}$. The estimate in [1] is a third of this, $d_L \sim 300\,\text{Mpc}$. Their analysis of the total luminosity is probably better than just dropping the tensorial factors——however, even the most sophisticated analyses leave roughly a factor of 2 uncertainty in the source distance for GW150914.

(2) $h \sim \frac{G}{c^4}\frac{\ddot{I}_2}{r} \sim \frac{G}{c^2}\frac{E_{kin}/c^2}{r}$

- Meteorite

$R = 10^3 m$ $\quad v = 25 Km/s = 25 \times 10^3 m/s$

$\rho \sim 4000 \, kg/m^3$ (typical density from meteorites. wustl.edu)

$\Rightarrow M = \frac{4}{3}\pi R^3 \rho \cong 1.7 \times 10^{13} kg$

$I_2 \sim \int \rho(\vec{x}) x^2 d^3x \sim \int M \delta(x - vt)\delta(y)\delta(z)\, x^2 d^3x$

$= Mv^2t^2$ for a trajectory $\vec{x}_o(t) = vt\hat{x}$

$\ddot{I}_2 \sim 2Mv^2 \sim E_{kin}$

$\boxed{r \sim \lambda\!\!\!^{-}} = \frac{c}{2\pi f} = \frac{cT}{2\pi}$ we must be at least $\lambda\!\!\!^{-}$ away from the source to be in the wave-zone

$T \sim \frac{R}{v} = 0.04 s \Rightarrow \lambda\!\!\!^{-} \sim 2 \times 10^6 m$

$h \sim \frac{G}{c^2}\frac{E_{kin}/c^2}{\lambda\!\!\!^{-}} = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}} \frac{1}{2 \times 10^6} \times \frac{2 \times 1.7 \times 10^{13} \times (25 \times 10^3)^2}{9 \times 10^{16}} =$

$= 8.7 \times 10^{-29} \sim 10^{-30}$ ✓

- Piezoelectric

$f = 10^8 Hz$

$\vec{x}_o(t) = A\cos(2\pi f t)\hat{x}$, $\dot{\vec{x}}_o(t) = -2\pi f A \sin(2\pi f t)\hat{x}$

$\ddot{I}_2 \sim E_{kin} \simeq \frac{1}{2} M |\dot{\vec{x}}_o(t)|^2 \sim \frac{1}{2} M (2\pi f A)^2$

A large piezoelectric crystal I found advertised online was

$M \sim 10 \, Kg$ (www.virtualacquisitionshowcase.com/document/1193/brochure)

with motions of the order of a few mm $\Rightarrow A \sim 1 mm = 10^{-3} m$

$\Rightarrow \frac{E_{kin}}{c^2} \sim \frac{1}{2} \times 10 \times 4\pi^2 \times 10^{16} \times 10^{-6} \frac{1}{9 \times 10^{16}} kg = 2 \times 10^{-5} kg$

$\lambda\!\!\!^{-} = \frac{c T_{GW}}{2\pi}$

$\ddot{I}_2 = \frac{d^2}{dt^2}\left(MA^2\cos^2(2\pi f t)\right) = -8A^2 f^2 \pi^2 M \cos(\underbrace{4\pi f}_{2\pi/T_{GW}} t) \Rightarrow$

$T_{GW} = \frac{2\pi}{4\pi f} \Rightarrow \lambda\!\!\!^{-} = \frac{c}{\cancel{2\pi}}\frac{\cancel{2\pi}}{4\pi f} \cong 0.2 m$

$$h \sim \frac{G}{c^2}\,\frac{E_{kin}/c^2}{r} = \frac{6.67\times10^{-11}}{9\times10^{16}}\;\frac{2\times10^{-5}}{2\times10^{-1}} = 7\times10^{-32}$$

① $h = A_0 \frac{G}{c^2}\frac{I_0}{r} + A_1 \frac{G}{c^3}\frac{\dot{I}_1}{r} + A_2 \frac{G}{c^4}\frac{\ddot{I}_2}{r} + \dots +$

$+ B_1 \frac{G}{c^4}\frac{\dot{J}_1}{r} + B_2 \frac{G}{c^5}\frac{\ddot{J}_2}{r} \dots =$

$$= \frac{G}{c^2 r}\sum_{l=0}^{\infty}\frac{A_l}{c^l}\frac{d^l I_l}{dt^l} + \frac{G}{c^4 r}\sum_{l=1}^{\infty}\frac{B_l}{c^{l-1}}\frac{d^l J_l}{dt^l} \qquad (\#\#)$$

where A_l, B_l are real numbers.

$$\left[\frac{G}{c^2 r}\right] = \frac{M\cancel{L}}{\cancel{T}^2}\frac{\cancel{L}^2}{M^2}\left(\frac{\cancel{L}}{\cancel{T}}\right)^{-2}\frac{1}{\cancel{L}} = \frac{1}{M}$$

$$\left[\frac{G}{c^4 r}\right] = [c^{-2}]\left[\frac{G}{c^2 r}\right] = \frac{T^2}{L^2 M}$$

$$[I_l] \sim \left[\int x^l \rho\, d^3x\right] = ML^l$$

$$[J_l] \sim \left[\int x^l v\rho\, d^3x\right] = ML^l\frac{L}{T}$$

$$[h] = \left[\frac{G}{r c^2}\right]\sum_{l=0}^{\infty}\left[\frac{A_l}{c^l}\frac{d^l I_l}{dt^l}\right] + \left[\frac{G}{rc^4}\right]\sum_{l=1}^{\infty}\left[\frac{B_l}{c^{l-1}}\frac{d^l J_l}{dt^l}\right] =$$

$$= \underbrace{\frac{1}{\cancel{M}}\sum_{l=0}^{\infty}\frac{\cancel{T^l}}{\cancel{L^l}}\frac{1}{\cancel{T^l}}\cancel{ML^l}}_{\text{dimensionless}} + \underbrace{\frac{T^2}{L^2\cancel{M}}\sum_{l=1}^{\infty}\frac{\cancel{T^{l-1}}}{\cancel{L^{l-1}}}\frac{1}{T\cancel{^l}}\cancel{M}L^{\cancel{l}}\frac{L}{T}}_{=\frac{\cancel{T^2}}{\cancel{L^2}}\sum_{l=1}^{\infty}\frac{\cancel{L^2}}{\cancel{T^2}} \to \text{dimensionless}}$$

$\Rightarrow [h] = 1$ dimensionless

Consider the l-th term in the expansion; call it h_l, so that $h = \sum_l h_l$. Let's assume a generic term only depends on:

$$\left[\frac{G}{r}\right] = \frac{L^2}{MT^2}$$

$$\left[\frac{d^\alpha}{dt^\alpha}\right] = \frac{1}{T^\alpha}$$

$$[I_\gamma] = ML^\gamma \quad \text{or} \quad [J_\gamma] = \frac{ML^\gamma}{T}$$

$$[c^\beta] = \frac{L^\beta}{T^\beta} \quad , \quad \text{where } \alpha \geq 0 \,,\, \gamma \geq 0.$$

So the dimensionality of the term is:

$$1)\ \left[\frac{G}{r}\right]\left[\frac{d^\alpha}{dt^\alpha}\right][I_\gamma][c^\beta] = \frac{L^2}{\cancel{M}T^2}\,\frac{1}{T^\alpha}\,\cancel{M}L^\gamma\,\frac{L^\beta}{T^\beta} = L^{\beta+\gamma+2}\,T^{-(\beta+\alpha+2)}$$

or

$$2)\ \left[\frac{G}{r}\right]\left[\frac{d^\alpha}{dt^\alpha}\right][J_\gamma][c^\beta] = \frac{L^2}{\cancel{M}T^2}\,\frac{1}{T^\alpha}\,\cancel{M}L^\gamma\,\frac{L}{T}\,\frac{L^\beta}{T^\beta} = L^{\beta+\gamma+3}\,T^{-(\beta+\alpha+3)}$$

Since h is adimensional, each term must be adimensional as well, which implies:

$$1)\ \begin{cases}\beta+\gamma+2=0\\ \beta+\alpha+2=0\end{cases}$$

$$\gamma-\alpha=0 \Rightarrow \begin{cases}\alpha=\gamma\\ \beta=-\gamma-2\end{cases}$$

So the term is: $\dfrac{G}{r}\,\dfrac{d^\alpha}{dt^\alpha}\,I_\alpha\,\dfrac{1}{c^{\alpha+2}}$ with $\alpha \geq 0$

or

$$2)\ \begin{cases}\beta+\gamma+3=0\\ \beta+\alpha+3=0\end{cases}$$

$\alpha=\gamma$ and $\beta=-\gamma-3$

so the term is: $\dfrac{G}{r}\,\dfrac{d^\alpha}{dt^\alpha}\,J_\alpha\,\dfrac{1}{c^{\alpha+3}}$ with $\alpha \geq 0$

Replace now $\alpha \leftrightarrow l$, then:

$$h = \sum_{l=0}^{\infty}\left[\frac{A_l}{c^2}\,\frac{G}{r}\,\frac{d^l I_l}{dt^l}\,\frac{1}{c^l} + \frac{B_l}{c^3}\,\frac{G}{r}\,\frac{d^l J_l}{dt^l}\,\frac{1}{c^l}\right]$$

with A_l, B_l dimensionless.

This is exactly (##); so we showed that dimensional analysis requires unambiguously form (##) for h.